Contents

Every section throughout the book also includes "Exploration" and "Extending the Ideas" features which follow the exercises.

CHAPTER 10 Parametric, Vector, and Polar Functions 530

APPENDIX

Chapter 1

Prerequisites for Calculus

Exponential functions are used to model situations in which growth or decay change dramatically. Such situations are found in nuclear power plants, which contain rods of plutonium-239; an extremely toxic radioactive isotope.

Operating at full capacity for one year, a 1,000 megawatt power plant discharges about 435 lb of plutonium-239. With a half-life of 24,400 years, how much of the isotope will remain after 1,000 years? This question can be answered with the mathematics covered in Section 1.3.

Chapter 1 Overview

This chapter reviews the most important things you need to know to start learning calculus. It also introduces the use of a graphing utility as a tool to investigate mathematical ideas, to support analytic work, and to solve problems with numerical and graphical methods. The emphasis is on functions and graphs, the main building blocks of calculus.

Functions and parametric equations are the major tools for describing the real world in mathematical terms, from temperature variations to planetary motions, from brain waves to business cycles, and from heartbeat patterns to population growth. Many functions have particular importance because of the behavior they describe. Trigonometric functions describe cyclic, repetitive activity; exponential, logarithmic, and logistic functions describe growth and decay; and polynomial functions can approximate these and most other functions.

1.1 Lines

What you'll learn about

- Increments
- Slope of a Line
- Parallel and Perpendicular Lines
- Equations of Lines
- Applications

. . . and why

Linear equations are used extensively in business and economic applications.

Increments

One reason calculus has proved to be so useful is that it is the right mathematics for relating the rate of change of a quantity to the graph of the quantity. Explaining that relationship is one goal of this book. It all begins with the slopes of lines.

When a particle in the plane moves from one point to another, the net changes or *increments* in its coordinates are found by subtracting the coordinates of its starting point from the coordinates of its stopping point.

DEFINITION Increments

If a particle moves from the point (x_1, y_1) to the point (x_2, y_2), the **increments** in its coordinates are

$$\Delta x = x_2 - x_1 \quad \text{and} \quad \Delta y = y_2 - y_1.$$

The symbols Δx and Δy are read "delta x" and "delta y." The letter Δ is a Greek capital d for "difference." Neither Δx nor Δy denotes multiplication; Δx is not "delta times x" nor is Δy "delta times y."

Increments can be positive, negative, or zero, as shown in Example 1.

EXAMPLE 1 Finding Increments

The coordinate increments from $(4, -3)$ to $(2, 5)$ are

$$\Delta x = 2 - 4 = -2, \qquad \Delta y = 5 - (-3) = 8.$$

From $(5, 6)$ to $(5, 1)$, the increments are

$$\Delta x = 5 - 5 = 0, \qquad \Delta y = 1 - 6 = -5.$$

Now try Exercise 1.

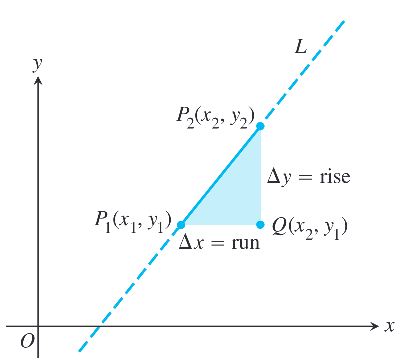


Figure 1.1 The slope of line L is $m = \dfrac{\text{rise}}{\text{run}} = \dfrac{\Delta y}{\Delta x}$.

Slope of a Line

Each nonvertical line has a *slope,* which we can calculate from increments in coordinates.

Let L be a nonvertical line in the plane and $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ two points on L (Figure 1.1). We call $\Delta y = y_2 - y_1$ the **rise** from P_1 to P_2 and $\Delta x = x_2 - x_1$ the **run** from

P_1 to P_2. Since L is not vertical, $\Delta x \neq 0$ and we define the slope of L to be the amount of rise per unit of run. It is conventional to denote the slope by the letter m.

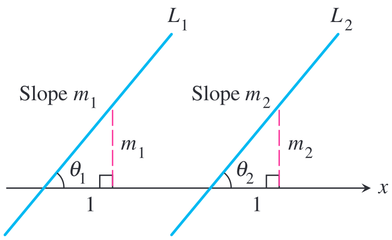


Figure 1.2 If $L_1 \parallel L_2$, then $\theta_1 = \theta_2$ and $m_1 = m_2$. Conversely, if $m_1 = m_2$, then $\theta_1 = \theta_2$ and $L_1 \parallel L_2$.

> **DEFINITION Slope**
>
> Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be points on a nonvertical line, L. The **slope** of L is
>
> $$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

A line that goes uphill as x increases has a positive slope. A line that goes downhill as x increases has a negative slope. A horizontal line has slope zero since all of its points have the same y-coordinate, making $\Delta y = 0$. For vertical lines, $\Delta x = 0$ and the ratio $\Delta y/\Delta x$ is undefined. We express this by saying that vertical lines *have no slope.*

Parallel and Perpendicular Lines

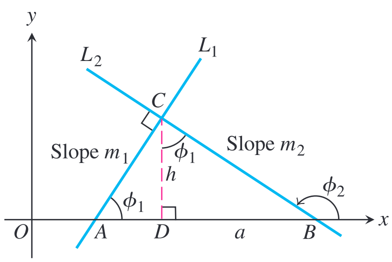


Figure 1.3 $\triangle ADC$ is similar to $\triangle CDB$. Hence ϕ_1 is also the upper angle in $\triangle CDB$, where $\tan \phi_1 = a/h$.

Parallel lines form equal angles with the x-axis (Figure 1.2). Hence, nonvertical parallel lines have the same slope. Conversely, lines with equal slopes form equal angles with the x-axis and are therefore parallel.

If two nonvertical lines L_1 and L_2 are perpendicular, their slopes m_1 and m_2 satisfy $m_1 m_2 = -1$, so each slope is the *negative reciprocal* of the other:

$$m_1 = -\frac{1}{m_2}, \qquad m_2 = -\frac{1}{m_1}.$$

The argument goes like this: In the notation of Figure 1.3, $m_1 = \tan \phi_1 = a/h$, while $m_2 = \tan \phi_2 = -h/a$. Hence, $m_1 m_2 = (a/h)(-h/a) = -1$.

Equations of Lines

The vertical line through the point (a, b) has equation $x = a$ since every x-coordinate on the line has the value a. Similarly, the horizontal line through (a, b) has equation $y = b$.

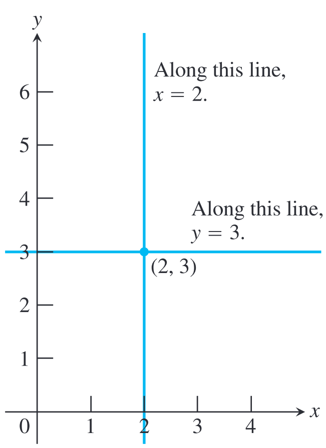


Figure 1.4 The standard equations for the vertical and horizontal lines through the point (2, 3) are $x = 2$ and $y = 3$. (Example 2)

EXAMPLE 2 Finding Equations of Vertical and Horizontal Lines

The vertical and horizontal lines through the point (2, 3) have equations $x = 2$ and $y = 3$, respectively (Figure 1.4). ***Now try Exercise 9.***

We can write an equation for any nonvertical line L if we know its slope m and the coordinates of one point $P_1(x_1, y_1)$ on it. If $P(x, y)$ is *any* other point on L, then

$$\frac{y - y_1}{x - x_1} = m,$$

so that

$$y - y_1 = m(x - x_1) \quad \text{or} \quad y = m(x - x_1) + y_1.$$

> **DEFINITION Point-Slope Equation**
>
> The equation
>
> $$y = m(x - x_1) + y_1$$
>
> is the **point-slope equation** of the line through the point (x_1, y_1) with slope m.

EXAMPLE 3 Using the Point-Slope Equation

Write the point-slope equation for the line through the point (2, 3) with slope $-3/2$.

SOLUTION

We substitute $x_1 = 2$, $y_1 = 3$, and $m = -3/2$ into the point-slope equation and obtain

$$y = -\frac{3}{2}(x - 2) + 3 \quad \text{or} \quad y = -\frac{3}{2}x + 6.$$

Now try Exercise 13.

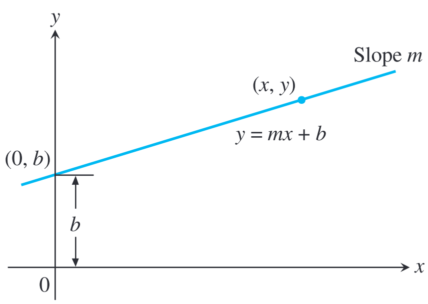


Figure 1.5 A line with slope m and y-intercept b.

The y-coordinate of the point where a nonvertical line intersects the y-axis is the **y-intercept** of the line. Similarly, the x-coordinate of the point where a nonhorizontal line intersects the x-axis is the **x-intercept** of the line. A line with slope m and y-intercept b passes through $(0, b)$ (Figure 1.5), so

$$y = m(x - 0) + b, \quad \text{or, more simply,} \quad y = mx + b.$$

DEFINITION Slope-Intercept Equation

The equation

$$y = mx + b$$

is the **slope-intercept equation** of the line with slope m and y-intercept b.

EXAMPLE 4 Writing the Slope-Intercept Equation

Write the slope-intercept equation for the line through $(-2, -1)$ and $(3, 4)$.

SOLUTION

The line's slope is

$$m = \frac{4 - (-1)}{3 - (-2)} = \frac{5}{5} = 1.$$

We can use this slope with either of the two given points in the point-slope equation. For $(x_1, y_1) = (-2, -1)$, we obtain

$$y = 1 \cdot (x - (-2)) + (-1)$$
$$y = x + 2 + (-1)$$
$$y = x + 1.$$

Now try Exercise 17.

If A and B are not both zero, the graph of the equation $Ax + By = C$ is a line. Every line has an equation in this form, even lines with undefined slopes.

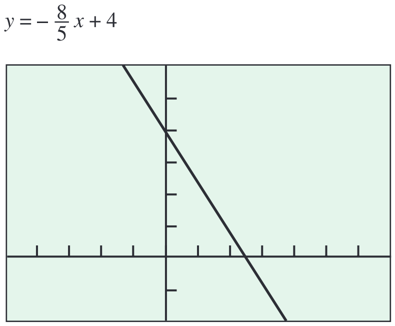
DEFINITION General Linear Equation

The equation

$$Ax + By = C \quad (A \text{ and } B \text{ not both } 0)$$

is a **general linear equation** in x and y.

Although the general linear form helps in the quick identification of lines, the slope-intercept form is the one to enter into a calculator for graphing.



[–5, 7] by [–2, 6]

Figure 1.6 The line $8x + 5y = 20$. (Example 5)

EXAMPLE 5 Analyzing and Graphing a General Linear Equation

Find the slope and y-intercept of the line $8x + 5y = 20$. Graph the line.

SOLUTION

Solve the equation for y to put the equation in slope-intercept form:

$$8x + 5y = 20$$
$$5y = -8x + 20$$
$$y = -\frac{8}{5}x + 4$$

This form reveals the slope ($m = -8/5$) and y-intercept ($b = 4$), and puts the equation in a form suitable for graphing (Figure 1.6).

Now try Exercise 27.

EXAMPLE 6 Writing Equations for Lines

Write an equation for the line through the point $(-1, 2)$ that is **(a)** parallel, and **(b)** perpendicular to the line L: $y = 3x - 4$.

SOLUTION

The line L, $y = 3x - 4$, has slope 3.

(a) The line $y = 3(x + 1) + 2$, or $y = 3x + 5$, passes through the point $(-1, 2)$, and is parallel to L because it has slope 3.

(b) The line $y = (-1/3)(x + 1) + 2$, or $y = (-1/3)x + 5/3$, passes through the point $(-1, 2)$, and is perpendicular to L because it has slope $-1/3$.

Now try Exercise 31.

EXAMPLE 7 Determining a Function

The following table gives values for the linear function $f(x) = mx + b$. Determine m and b.

x	$f(x)$
-1	$14/3$
1	$-4/3$
2	$-13/3$

SOLUTION

The graph of f is a line. From the table we know that the following points are on the line: $(-1, 14/3)$, $(1, -4/3)$, $(2, -13/3)$.

Using the first two points, the slope m is

$$m = \frac{-4/3 - (14/3)}{1} - (-1) = \frac{-6}{2} = -3.$$

So $f(x) = -3x + b$. Because $f(-1) = 14/3$, we have

$$f(-1) = -3(-1) + b$$
$$14/3 = 3 + b$$
$$b = 5/3.$$

continued

Thus, $m = -3$, $b = 5/3$, and $f(x) = -3x + 5/3$.
We can use either of the other two points determined by the table to check our work.

Now try Exercise 35.

Applications

Many important variables are related by linear equations. For example, the relationship between Fahrenheit temperature and Celsius temperature is linear, a fact we use to advantage in the next example.

EXAMPLE 8 Temperature Conversion

Find the relationship between Fahrenheit and Celsius temperature. Then find the Celsius equivalent of 90ºF and the Fahrenheit equivalent of −5ºC.

SOLUTION

Because the relationship between the two temperature scales is linear, it has the form $F = mC + b$. The freezing point of water is $F = 32°$ or $C = 0°$, while the boiling point is $F = 212°$ or $C = 100°$. Thus,

$$32 = m \cdot 0 + b \quad \text{and} \quad 212 = m \cdot 100 + b,$$

so $b = 32$ and $m = (212 - 32)/100 = 9/5$. Therefore,

$$F = \frac{9}{5}C + 32, \quad \text{or} \quad C = \frac{5}{9}(F - 32).$$

These relationships let us find equivalent temperatures. The Celsius equivalent of 90ºF is

$$C = \frac{5}{9}(90 - 32) \approx 32.2°.$$

The Fahrenheit equivalent of −5ºC is

$$F = \frac{9}{5}(-5) + 32 = 23°.$$

Now try Exercise 43.

It can be difficult to see patterns or trends in lists of paired numbers. For this reason, we sometimes begin by plotting the pairs (such a plot is called a **scatter plot**) to see whether the corresponding points lie close to a curve of some kind. If they do, and if we can find an equation $y = f(x)$ for the curve, then we have a formula that

1. summarizes the data with a simple expression, and
2. lets us predict values of y for other values of x.

The process of finding a curve to fit data is called **regression analysis** and the curve is called a **regression curve.**

There are many useful types of regression curves—power, exponential, logarithmic, sinusoidal, and so on. In the next example, we use the calculator's linear regression feature to fit the data in Table 1.1 with a line.

Some graphing utilities have a feature that enables them to approximate the relationship between variables with a linear equation. We use this feature in Example 9.

Table 1.1 World Population

Year	Population (millions)
1980	4454
1985	4853
1990	5285
1995	5696
2003	6305
2004	6378
2005	6450

Source: U.S. Bureau of the Census, *Statistical Abstract of the United States, 2004–2005.*

EXAMPLE 9 Regression Analysis—Predicting World Population

Starting with the data in Table 1.1, build a linear model for the growth of the world population. Use the model to predict the world population in the year 2010, and compare this prediction with the Statistical Abstract prediction of 6812 million.

continued

Why Not Round the Decimals in Equation 1 Even More?

If we do, our final calculation will be way off. Using $y = 80x - 153{,}849$, for instance, gives $y = 6951$ when $x = 2010$, as compared to $y = 6865$, an increase of 86 million. The rule is: *Retain all decimal places while working a problem. Round only at the end.* We rounded the coefficients in Equation 1 enough to make it readable, but not enough to hurt the outcome. However, we knew how much we could safely round *only from first having done the entire calculation with numbers unrounded.*

Rounding Rule

Round your answer as appropriate, but do not round the numbers in the calculations that lead to it.

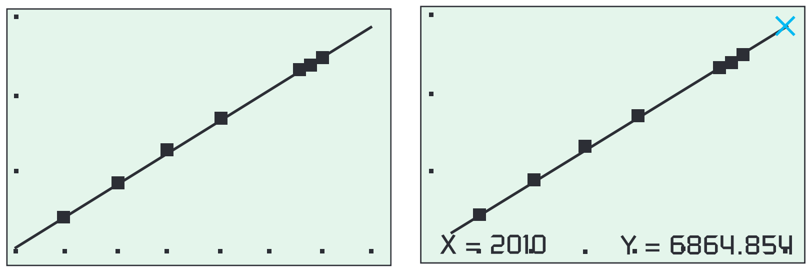
SOLUTION

Model Upon entering the data into the grapher, we find the regression equation to be approximately

$$y = 79.957x - 153848.716, \tag{1}$$

where x represents the year and y the population *in millions.*

Figure 1.7a shows the scatter plot for Table 1.1 together with a graph of the regression line just found. You can see how well the line fits the data.



[1975, 2010] by [4000, 7000]

(a)

[1975, 2010] by [4000, 7000]

(b)

Figure 1.7 (Example 9)

Solve Graphically Our goal is to predict the population in the year 2010. Reading from the graph in Figure 1.7b, we conclude that when x is 2010, y is approximately 6865.

Confirm Algebraically Evaluating Equation 1 for $x = 2010$ gives

$$y = 79.957(2010) - 153848.716$$
$$\approx 6865.$$

Interpret The linear regression equation suggests that the world population in the year 2010 will be about 6865 million, or approximately 53 million more than the Statistical Abstract prediction of 6812 million. ***Now try Exercise 45.***

Regression Analysis

Regression analysis has four steps:

1. Plot the data (scatter plot).
2. Find the regression equation. For a line, it has the form $y = mx + b$.
3. Superimpose the graph of the regression equation on the scatter plot to see the fit.
4. Use the regression equation to predict y-values for particular values of x.

Quick Review 1.1 *(For help, go to Section 1.1.)*

1. Find the value of y that corresponds to $x = 3$ in $y = -2 + 4(x - 3)$.

2. Find the value of x that corresponds to $y = 3$ in $y = 3 - 2(x + 1)$.

In Exercises 3 and 4, find the value of m that corresponds to the values of x and y.

3. $x = 5, \quad y = 2, \quad m = \dfrac{y - 3}{x - 4}$

4. $x = -1, \quad y = -3, \quad m = \dfrac{2 - y}{3 - x}$

In Exercises 5 and 6, determine whether the ordered pair is a solution to the equation.

5. $3x - 4y = 5$
(a) $(2, 1/4)$ **(b)** $(3, -1)$

6. $y = -2x + 5$
(a) $(-1, 7)$ **(b)** $(-2, 1)$

In Exercises 7 and 8, find the distance between the points.

7. $(1, 0), \quad (0, 1)$

8. $(2, 1), \quad (1, -1/3)$

In Exercises 9 and 10, solve for y in terms of x.

9. $4x - 3y = 7$

10. $-2x + 5y = -3$

Section 1.1 Exercises

In Exercises 1–4, find the coordinate increments from A to B.

1. $A(1, 2), \quad B(-1, -1)$

2. $A(-3, 2), \quad B(-1, -2)$

3. $A(-3, 1), \quad B(-8, 1)$

4. $A(0, 4), \quad B(0, -2)$

In Exercises 5–8, let L be the line determined by points A and B.
(a) Plot A and B. **(b)** Find the slope of L.
(c) Draw the graph of L.

5. $A(1, -2), \quad B(2, 1)$

6. $A(-2, -1), \quad B(1, -2)$

7. $A(2, 3), \quad B(-1, 3)$

8. $A(1, 2), \quad B(1, -3)$

In Exercise 9–12, write an equation for **(a)** the vertical line and **(b)** the horizontal line through the point P.

9. $P(3, 2)$

10. $P(-1, 4/3)$

11. $P(0, -\sqrt{2})$

12. $P(-\pi, 0)$

In Exercises 13–16, write the point-slope equation for the line through the point P with slope m.

13. $P(1, 1), \quad m = 1$

14. $P(-1, 1), \quad m = -1$

15. $P(0, 3), \quad m = 2$

16. $P(-4, 0), \quad m = -2$

In Exercises 17–20, write the slope-intercept equation for the line with slope m and y-intercept b.

17. $m = 3, \quad b = -2$

18. $m = -1, \quad b = 2$

19. $m = -1/2, \quad b = -3$

20. $m = 1/3, \quad b = -1$

In Exercises 21–24, write a general linear equation for the line through the two points.

21. $(0, 0), \quad (2, 3)$

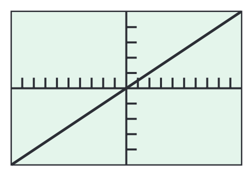
22. $(1, 1), \quad (2, 1)$

23. $(-2, 0), \quad (-2, -2)$

24. $(-2, 1), \quad (2, -2)$

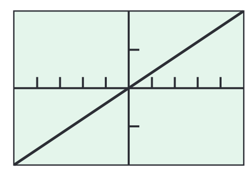
In Exercises 25 and 26, the line contains the origin and the point in the upper right corner of the grapher screen. Write an equation for the line.

25.



[–10, 10] by [–25, 25]

26.



[–5, 5] by [–2, 2]

In Exercises 27–30, find the **(a)** slope and **(b)** y-intercept, and **(c)** graph the line.

27. $3x + 4y = 12$

28. $x + y = 2$

29. $\dfrac{x}{3} + \dfrac{y}{4} = 1$

30. $y = 2x + 4$

In Exercises 31–34, write an equation for the line through P that is **(a)** parallel to L, and **(b)** perpendicular to L.

31. $P(0, 0), \quad L: y = -x + 2$

32. $P(-2, 2), \quad L: 2x + y = 4$

33. $P(-2, 4), \quad L: x = 5$

34. $P(-1, 1/2), \quad L: y = 3$

In Exercises 35 and 36, a table of values is given for the linear function $f(x) = mx + b$. Determine m and b.

35.

x	$f(x)$
1	2
3	9
5	16

36.

x	$f(x)$
2	−1
4	−4
6	−7

In Exercises 37 and 38, find the value of x or y for which the line through A and B has the given slope m.

37. $A(-2, 3), \quad B(4, y), \quad m = -2/3$

38. $A(-8, -2), \quad B(x, 2), \quad m = 2$

39. ***Revisiting Example 4*** Show that you get the same equation in Example 4 if you use the point (3, 4) to write the equation.

40. **Writing to Learn** ***x- and y-intercepts***

(a) Explain why c and d are the x-intercept and y-intercept, respectively, of the line

$$\frac{x}{c} + \frac{y}{d} = 1.$$

(b) How are the x-intercept and y-intercept related to c and d in the line

$$\frac{x}{c} + \frac{y}{d} = 2?$$

41. ***Parallel and Perpendicular Lines*** For what value of k are the two lines $2x + ky = 3$ and $x + y = 1$ **(a)** parallel? **(b)** perpendicular?

Group Activity In Exercises 42–44, work in groups of two or three to solve the problem.

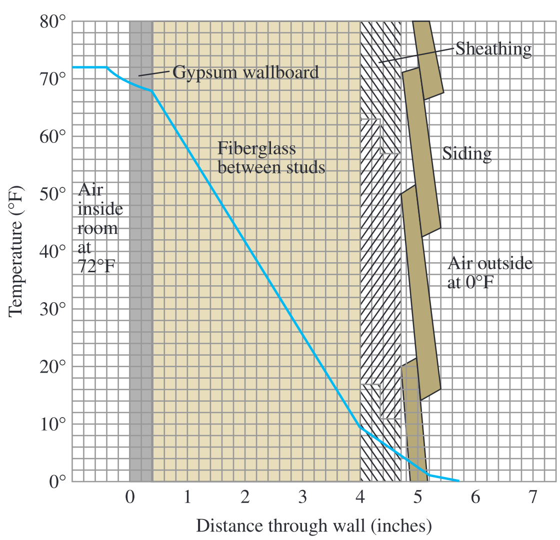
42. ***Insulation*** By measuring slopes in the figure below, find the temperature change in degrees per inch for the following materials.

(a) gypsum wallboard

(b) fiberglass insulation

(c) wood sheathing

(d) **Writing to Learn** Which of the materials in (a)–(c) is the best insulator? the poorest? Explain.



43. ***Pressure under Water*** The pressure p experienced by a diver under water is related to the diver's depth d by an equation of the form $p = kd + 1$(k a constant). When $d = 0$ meters, the pressure is 1 atmosphere. The pressure at 100 meters is 10.94 atmospheres. Find the pressure at 50 meters.

44. ***Modeling Distance Traveled*** A car starts from point P at time $t = 0$ and travels at 45 mph.

(a) Write an expression $d(t)$ for the distance the car travels from P.

(b) Graph $y = d(t)$.

(c) What is the slope of the graph in (b)? What does it have to do with the car?

(d) **Writing to Learn** Create a scenario in which t could have negative values.

(e) **Writing to Learn** Create a scenario in which the y-intercept of $y = d(t)$ could be 30.

In Exercises 45 and 46, use linear regression analysis.

45. Table 1.2 shows the mean annual compensation of construction workers.

Table 1.2 Construction Workers' Average Annual Compensation

Year	Annual Total Compensation (dollars)
1999	42,598
2000	44,764
2001	47,822
2002	48,966

Source: U.S. Bureau of the Census, *Statistical Abstract of the United States, 2004–2005.*

(a) Find the linear regression equation for the data.

(b) Find the slope of the regression line. What does the slope represent?

(c) Superimpose the graph of the linear regression equation on a scatter plot of the data.

(d) Use the regression equation to predict the construction workers' average annual compensation in the year 2008.

46. Table 1.3 lists the ages and weights of nine girls.

Table 1.3 Girls' Ages and Weights

Age (months)	Weight (pounds)
19	22
21	23
24	25
27	28
29	31
31	28
34	32
38	34
43	39

(a) Find the linear regression equation for the data.

(b) Find the slope of the regression line. What does the slope represent?

(c) Superimpose the graph of the linear regression equation on a scatter plot of the data.

(d) Use the regression equation to predict the approximate weight of a 30-month-old girl.

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

47. True or False The slope of a vertical line is zero. Justify your answer.

48. True or False The slope of a line perpendicular to the line $y = mx + b$ is $1/m$. Justify your answer.

49. Multiple Choice Which of the following is an equation of the line through $(-3, 4)$ with slope $1/2$?

(A) $y - 4 = \frac{1}{2}(x + 3)$ **(B)** $y + 3 = \frac{1}{2}(x - 4)$

(C) $y - 4 = -2(x + 3)$ **(D)** $y - 4 = 2(x + 3)$

(E) $y + 3 = 2(x - 4)$

50. Multiple Choice Which of the following is an equation of the vertical line through (–2, 4)?

(A) $y = 4$ **(B)** $x = 2$ **(C)** $y = -4$

(D) $x = 0$ **(E)** $x = -2$

51. Multiple Choice Which of the following is the x-intercept of the line $y = 2x - 5$?

(A) $x = -5$ **(B)** $x = 5$ **(C)** $x = 0$

(D) $x = 5/2$ **(E)** $x = -5/2$

52. Multiple Choice Which of the following is an equation of the line through $(-2, -1)$ parallel to the line $y = -3x + 1$?

(A) $y = -3x + 5$ **(B)** $y = -3x - 7$ **(C)** $y = \frac{1}{3}x - \frac{1}{3}$

(D) $y = -3x + 1$ **(E)** $y = -3x - 4$

Extending the Ideas

53. The median price of existing single-family homes has increased consistently during the past few years. However, the data in Table 1.4 show that there have been differences in various parts of the country.

Table 1.4 Median Price of Single-Family Homes

Year	South (dollars)	West (dollars)
1999	145,900	173,700
2000	148,000	196,400
2001	155,400	213,600
2002	163,400	238,500
2003	168,100	260,900

Source: U.S. Bureau of the Census, *Statistical Abstract of the United States, 2004–2005.*

(a) Find the linear regression equation for home cost in the South.

(b) What does the slope of the regression line represent?

(c) Find the linear regression equation for home cost in the West.

(d) Where is the median price increasing more rapidly, in the South or the West?

54. ***Fahrenheit versus Celsius*** We found a relationship between Fahrenheit temperature and Celsius temperature in Example 8.

(a) Is there a temperature at which a Fahrenheit thermometer and a Celsius thermometer give the same reading? If so, what is it?

(b) Writing to Learn Graph $y_1 = (9/5)x + 32$, $y_2 = (5/9)(x - 32)$, and $y_3 = x$ in the same viewing window. Explain how this figure is related to the question in part (a).

55. ***Parallelogram*** Three different parallelograms have vertices at $(-1, 1)$, $(2, 0)$, and $(2, 3)$. Draw the three and give the coordinates of the missing vertices.

56. ***Parallelogram*** Show that if the midpoints of consecutive sides of any quadrilateral are connected, the result is a parallelogram.

57. ***Tangent Line*** Consider the circle of radius 5 centered at $(0, 0)$. Find an equation of the line tangent to the circle at the point $(3, 4)$.

58. Group Activity *Distance From a Point to a Line* This activity investigates how to find the distance from a point $P(a, b)$ to a line $L: Ax + By = C$.

(a) Write an equation for the line M through P perpendicular to L.

(b) Find the coordinates of the point Q in which M and L intersect.

(c) Find the distance from P to Q.

1.2 Functions and Graphs

What you'll learn about

- Functions
- Domains and Ranges
- Viewing and Interpreting Graphs
- Even Functions and Odd Functions—Symmetry
- Functions Defined in Pieces
- Absolute Value Function
- Composite Functions

. . . and why

Functions and graphs form the basis for understanding mathematics and applications.

Functions

The values of one variable often depend on the values for another:

- The temperature at which water boils depends on elevation (the boiling point drops as you go up).
- The amount by which your savings will grow in a year depends on the interest rate offered by the bank.
- The area of a circle depends on the circle's radius.

In each of these examples, the value of one variable quantity depends on the value of another. For example, the boiling temperature of water, b, depends on the elevation, e; the amount of interest, I, depends on the interest rate, r. We call b and I **dependent variables** because they are determined by the values of the variables e and r on which they depend. The variables e and r are **independent variables.**

A rule that assigns to each element in one set a unique element in another set is called a *function.* The sets may be sets of any kind and do not have to be the same. A function is like a machine that assigns a unique output to every allowable input. The inputs make up the **domain** of the function; the outputs make up the **range** (Figure 1.8).

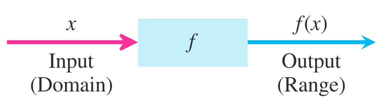


Figure 1.8 A "machine" diagram for a function.

DEFINITION Function

A **function** from a set D to a set R is a rule that assigns a unique element in R to each element in D.

In this definition, D is the domain of the function and R is a set *containing* the range (Figure 1.9).

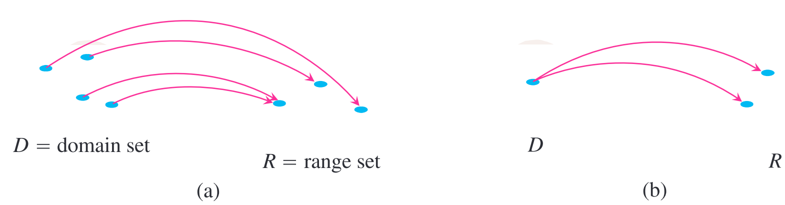


Figure 1.9 (a) A function from a set D to a set R. (b) *Not* a function. The assignment is not unique.

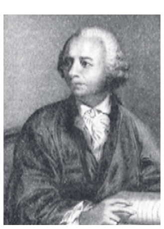
Euler invented a symbolic way to say "y is a function of x":

$$y = f(x),$$

which we read as "y equals f of x." This notation enables us to give different functions different names by changing the letters we use. To say that the boiling point of water is a function of elevation, we can write $b = f(e)$. To say that the area of a circle is a function of the circle's radius, we can write $A = A(r)$, giving the function the same name as the dependent variable.

Leonhard Euler

(1707–1783)



Leonhard Euler, the dominant mathematical figure of his century and the most prolific mathematician ever, was also an astronomer, physicist, botanist, and chemist, and an expert in oriental languages. His work was the first to give the function concept the prominence that it has in mathematics today. Euler's collected books and papers fill 72 volumes. This does not count his enormous correspondence to approximately 300 addresses. His introductory algebra text, written originally in German (Euler was Swiss), is still available in English translation.

The notation $y = f(x)$ gives a way to denote specific values of a function. The value of f at a can be written as $f(a)$, read "f of a."

EXAMPLE 1 The Circle-Area Function

Write a formula that expresses the area of a circle as a function of its radius. Use the formula to find the area of a circle of radius 2 in.

SOLUTION

If the radius of the circle is r, then the area $A(r)$ of the circle can be expressed as $A(r) = \pi r^2$. The area of a circle of radius 2 can be found by evaluating the function $A(r)$ at $r = 2$.

$$A(2) = \pi(2)^2 = 4\pi$$

The area of a circle of radius 2 is 4π in^2. ***Now try Exercise 3.***

Domains and Ranges

In Example 1, the domain of the function is restricted by context: the independent variable is a radius and must be positive. When we define a function $y = f(x)$ with a formula and the domain is not stated explicitly or restricted by context, the domain is assumed to be the largest set of x-values for which the formula gives real y-values—the so-called **natural domain.** If we want to restrict the domain, we must say so. The domain of $y = x^2$ is understood to be the entire set of real numbers. We must write "$y = x^2, x > 0$" if we want to restrict the function to positive values of x.

The domains and ranges of many real-valued functions of a real variable are intervals or combinations of intervals. The intervals may be open, closed, or half-open (Figures 1.10 and 1.11) and finite or infinite (Figure 1.12).

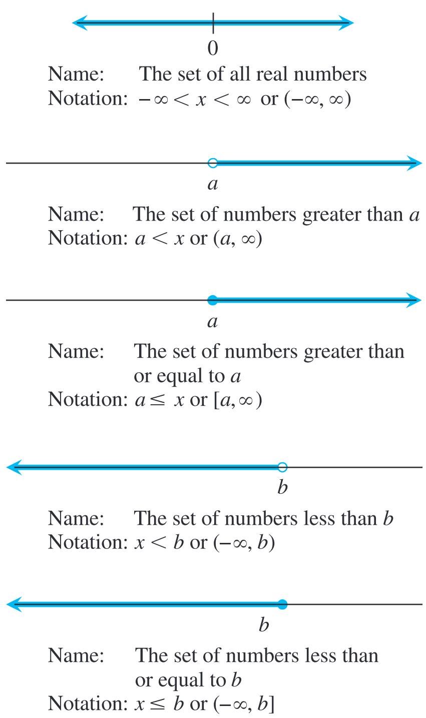


Figure 1.12 Infinite intervals—rays on the number line and the number line itself. The symbol ∞ (infinity) is used merely for convenience; it does not mean there is a number ∞.

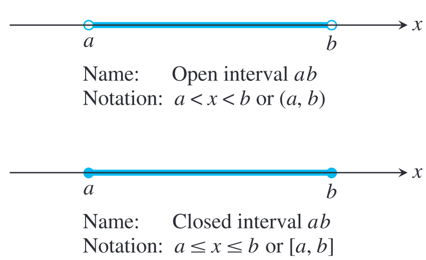


Figure 1.10 Open and closed finite intervals.

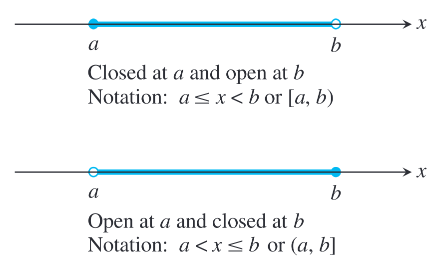


Figure 1.11 Half-open finite intervals.

The endpoints of an interval make up the interval's **boundary** and are called **boundary points.** The remaining points make up the interval's **interior** and are called **interior points. Closed intervals** contain their boundary points. **Open intervals** contain no boundary points. Every point of an open interval is an interior point of the interval.

Viewing and Interpreting Graphs

The points (x, y) in the plane whose coordinates are the input-output pairs of a function $y = f(x)$ make up the function's **graph.** The graph of the function $y = x + 2$, for example, is the set of points with coordinates (x, y) for which y equals $x + 2$.

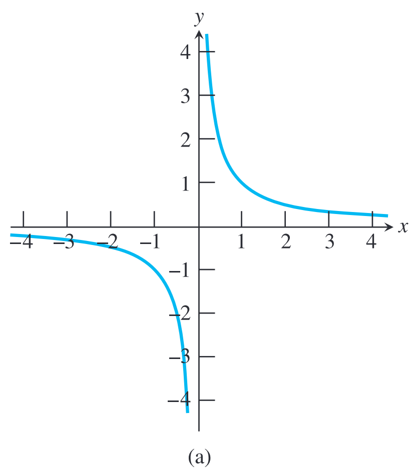
EXAMPLE 2 Identifying Domain and Range of a Function

Identify the domain and range, and then sketch a graph of the function.

(a) $y = \dfrac{1}{x}$ **(b)** $y = \sqrt{x}$

SOLUTION

(a) The formula gives a real y-value for every real x-value except $x = 0$. (*We cannot divide any number by* 0.) The domain is $(-\infty, 0) \cup (0, \infty)$. The value y takes on every real number except $y = 0$. ($y = c \neq 0$ if $x = 1/c$) The range is also $(-\infty, 0) \cup (0, \infty)$. A sketch is shown in Figure 1.13a.



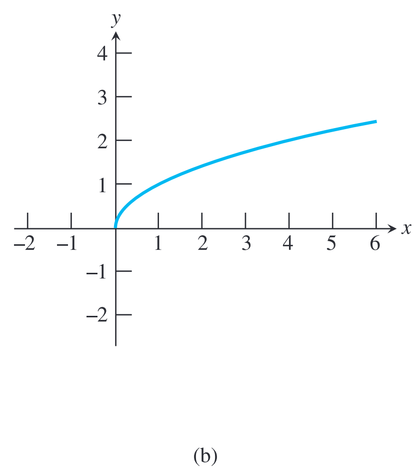


Figure 1.13 A sketch of the graph of (a) $y = 1/x$ and (b) $y = \sqrt{x}$. (Example 2)

(b) The formula gives a real number only when x is positive or zero. The domain is $[0, \infty)$. Because $\sqrt{x}$ denotes the principal square root of x, y is greater than or equal to zero. The range is also $[0, \infty)$. A sketch is shown in Figure 1.13b.

Now try Exercise 9.

Graphing with pencil and paper requires that you develop graph *drawing* skills. Graphing with a grapher (graphing calculator) requires that you develop graph *viewing* skills.

Graph Viewing Skills

1. Recognize that the graph is reasonable.
2. See all the important characteristics of the graph.
3. Interpret those characteristics.
4. Recognize grapher failure.

Being able to recognize that a graph is reasonable comes with experience. You need to know the basic functions, their graphs, and how changes in their equations affect the graphs.

Grapher failure occurs when the graph produced by a grapher is less than precise—or even incorrect—usually due to the limitations of the screen resolution of the grapher.

EXAMPLE 3 Identifying Domain and Range of a Function

Use a grapher to identify the domain and range, and then draw a graph of the function.

(a) $y = \sqrt{4 - x^2}$ **(b)** $y = x^{2/3}$

SOLUTION

(a) Figure 1.14a shows a graph of the function for $-4.7 \le x \le 4.7$ and $-3.1 \le y \le 3.1$, that is, the viewing window $[-4.7, 4.7]$ by $[-3.1, 3.1]$, with x-scale $= y$-scale $= 1$. The graph appears to be the upper half of a circle. The domain appears to be $[-2, 2]$. This observation is correct because we must have $4 - x^2 \ge 0$, or equivalently, $-2 \le x \le 2$. The range appears to be $[0, 2]$, which can also be verified algebraically.

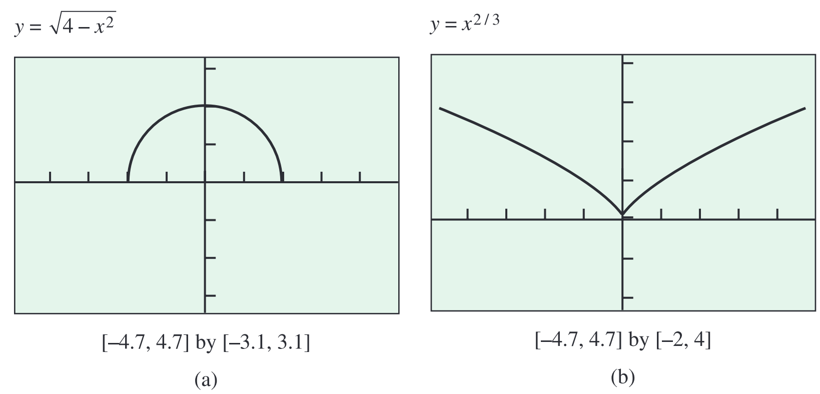


Figure 1.14 The graph of (a) $y = \sqrt{4 - x^2}$ and (b) $y = x^{2/3}$. (Example 3)

(b) Figure 1.14b shows a graph of the function in the viewing window $[-4.7, 4.7]$ by $[-2, 4]$, with x-scale $= y$-scale $= 1$. The domain appears to be $(-\infty, \infty)$, which we can verify by observing that $x^{2/3} = (\sqrt[3]{x})^2$. Also the range is $[0, \infty)$ by the same observation. ***Now try Exercise 15.***

Graphing $y = x^{2/3}$—Possible Grapher Failure

On some graphing calculators you need to enter this function as $y = (x^2)^{1/3}$ or $y = (x^{1/3})^2$ to obtain a correct graph. Try graphing this function on your grapher.

Even Functions and Odd Functions—Symmetry

The graphs of *even* and *odd* functions have important symmetry properties.

> **DEFINITIONS Even Function, Odd Function**
>
> A function $y = f(x)$ is an
>
> **even function of x** if $f(-x) = f(x)$,
>
> **odd function of x** if $f(-x) = -f(x)$,
>
> for every x in the function's domain.

The names even and odd come from powers of x. If y is an even power of x, as in $y = x^2$ or $y = x^4$, it is an even function of x (because $(-x)^2 = x^2$ and $(-x)^4 = x^4$). If y is an odd power of x, as in $y = x$ or $y = x^3$, it is an odd function of x (because $(-x)^1 = -x$ and $(-x)^3 = -x^3$).

The graph of an even function is **symmetric about the y-axis.** Since $f(-x) = f(x)$, a point (x, y) lies on the graph if and only if the point $(-x, y)$ lies on the graph (Figure 1.15a).

The graph of an odd function is **symmetric about the origin.** Since $f(-x) = -f(x)$, a point (x, y) lies on the graph if and only if the point $(-x, -y)$ lies on the graph (Figure 1.15b).

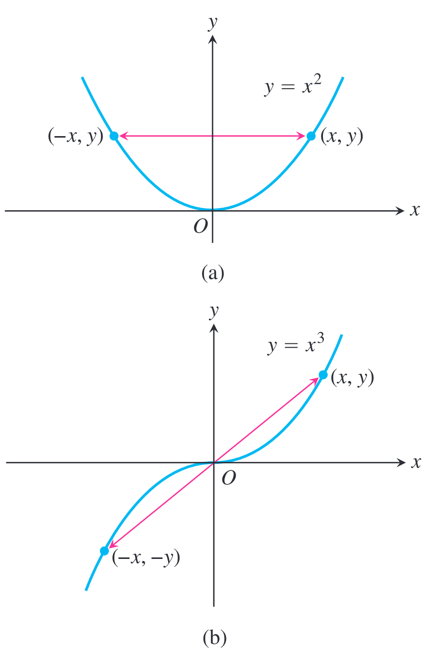
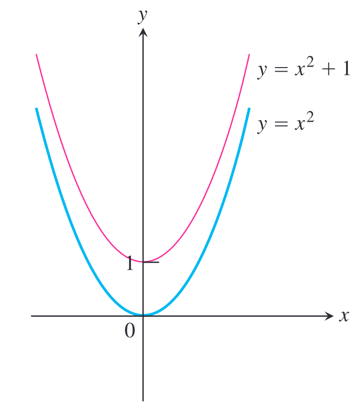
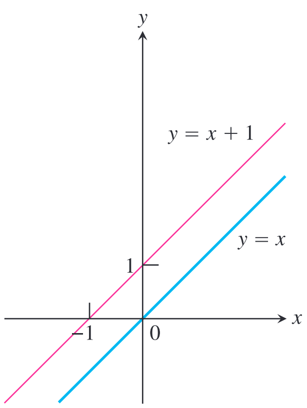


Figure 1.15 (a) The graph of $y = x^2$ (an even function) is symmetric about the y-axis. (b) The graph of $y = x^3$ (an odd function) is symmetric about the origin.



(a)



(b)

Figure 1.16 (a) When we add the constant term 1 to the function $y = x^2$, the resulting function $y = x^2 + 1$ is still even and its graph is still symmetric about the y-axis. (b) When we add the constant term 1 to the function $y = x$, the resulting function $y = x + 1$ is no longer odd. The symmetry about the origin is lost. (Example 4)

Equivalently, a graph is symmetric about the origin if a rotation of 180º about the origin leaves the graph unchanged.

EXAMPLE 4 Recognizing Even and Odd Functions

$f(x) = x^2$ Even function: $(-x)^2 = x^2$ for all x; symmetry about y-axis.

$f(x) = x^2 + 1$ Even function: $(-x)^2 + 1 = x^2 + 1$ for all x; symmetry about y-axis (Figure 1.16a).

$f(x) = x$ Odd function: $(-x) = -x$ for all x; symmetry about the origin.

$f(x) = x + 1$ Not odd: $f(-x) = -x + 1$, but $-f(x) = -x - 1$. The two are not equal.

Not even: $(-x) + 1 \neq x + 1$ for all $x \neq 0$ (Figure 1.16b).

Now try Exercises 21 and 23.

It is useful in graphing to recognize even and odd functions. Once we know the graph of either type of function on one side of the y-axis, we know its graph on both sides.

Functions Defined in Pieces

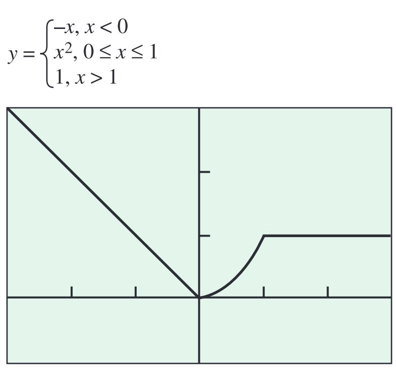
While some functions are defined by single formulas, others are defined by applying different formulas to different parts of their domains.

EXAMPLE 5 Graphing Piecewise-Defined Functions

Graph $y = f(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \le x \le 1 \\ 1, & x > 1. \end{cases}$

SOLUTION

The values of f are given by three separate formulas: $y = -x$ when $x < 0$, $y = x^2$ when $0 \le x \le 1$, and $y = 1$ when $x > 1$. However, the function is *just one function,* whose domain is the entire set of real numbers (Figure 1.17). ***Now try Exercise 33.***



[–3, 3] by [–1, 3]

Figure 1.17 The graph of a piecewise defined function. (Example 5).

EXAMPLE 6 Writing Formulas for Piecewise Functions

Write a formula for the function $y = f(x)$ whose graph consists of the two line segments in Figure 1.18.

SOLUTION

We find formulas for the segments from (0, 0) to (1, 1) and from (1, 0) to (2, 1) and piece them together in the manner of Example 5.

Segment from (0, 0) to (1, 1) The line through (0, 0) and (1, 1) has slope $m = (1 - 0)/(1 - 0) = 1$ and y-intercept $b = 0$. Its slope-intercept equation is $y = x$. The segment from (0, 0) to (1, 1) that includes the point (0, 0) but not the point (1, 1) is the graph of the function $y = x$ restricted to the half-open interval $0 \le x < 1$, namely,

$$y = x, \quad 0 \le x < 1.$$

continued

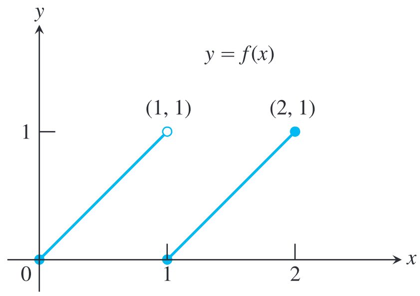


Figure 1.18 The segment on the left contains (0, 0) but not (1, 1). The segment on the right contains both of its endpoints. (Example 6)

Segment from (1, 0) to (2, 1) The line through (1, 0) and (2, 1) has slope $m = (1 - 0)/(2 - 1) = 1$ and passes through the point (1, 0). The corresponding point-slope equation for the line is

$$y = 1(x - 1) + 0, \quad \text{or} \quad y = x - 1.$$

The segment from (1, 0) to (2, 1) that includes both endpoints is the graph of $y = x - 1$ restricted to the closed interval $1 \le x \le 2$, namely,

$$y = x - 1, \quad 1 \le x \le 2.$$

Piecewise Formula Combining the formulas for the two pieces of the graph, we obtain

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ x - 1, & 1 \le x \le 2. \end{cases}$$

Now try Exercise 43.

Absolute Value Function

The **absolute value function** $y = |x|$ is defined piecewise by the formula

$$|x| = \begin{cases} -x, & x < 0 \\ x, & x \ge 0. \end{cases}$$

The function is even, and its graph (Figure 1.19) is symmetric about the y-axis.

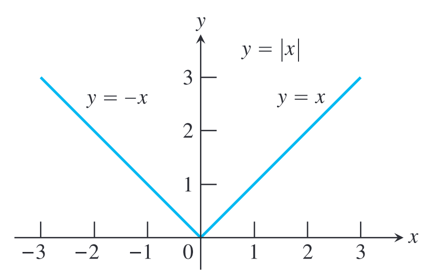


Figure 1.19 The absolute value function has domain $(-\infty, \infty)$ and range $[0, \infty)$.

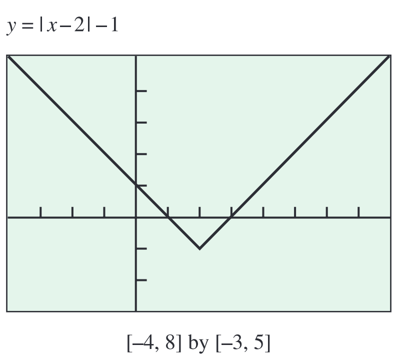


Figure 1.20 The lowest point of the graph of $f(x) = |x - 2| - 1$ is $(2, -1)$. (Example 7)

EXAMPLE 7 Using Transformations

Draw the graph of $f(x) = |x - 2| - 1$. Then find the domain and range.

SOLUTION

The graph of f is the graph of the absolute value function shifted 2 units horizontally to the right and 1 unit vertically downward (Figure 1.20). The domain of f is $(-\infty, \infty)$ and the range is $[-1, \infty)$.

Now try Exercise 49.

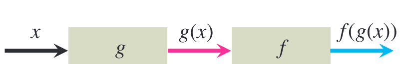


Figure 1.21 Two functions can be composed when a portion of the range of the first lies in the domain of the second.

Composite Functions

Suppose that some of the outputs of a function g can be used as inputs of a function f. We can then link g and f to form a new function whose inputs x are inputs of g and whose outputs are the numbers $f(g(x))$, as in Figure 1.21. We say that the function $f(g(x))$ (read

"f of g of x") is the **composite of g and f.** It is made by *composing* g and f in the order of first g, then f. The usual "stand-alone" notation for this composite is $f \circ g$, which is read as "f of g." Thus, the value of $f \circ g$ at x is $(f \circ g)(x) = f(g(x))$.

EXAMPLE 8 Composing Functions

Find a formula for $f(g(x))$ if $g(x) = x^2$ and $f(x) = x - 7$. Then find $f(g(2))$.

SOLUTION

To find $f(g(x))$, we replace x in the formula $f(x) = x - 7$ by the expression given for $g(x)$.

$$f(x) = x - 7$$
$$f(g(x)) = g(x) - 7 = x^2 - 7$$

We then find the value of $f(g(2))$ by substituting 2 for x.

$$f(g(2)) = (2)^2 - 7 = -3$$

Now try Exercise 51.

EXPLORATION 1 Composing Functions

Some graphers allow a function such as y_1 to be used as the independent variable of another function. With such a grapher, we can compose functions.

1. Enter the functions $y_1 = f(x) = 4 - x^2$, $y_2 = g(x) = \sqrt{x}$, $y_3 = y_2(y_1(x))$, and $y_4 = y_1(y_2(x))$. Which of y_3 and y_4 corresponds to $f \circ g$? to $g \circ f$?
2. Graph y_1, y_2, and y_3 and make conjectures about the domain and range of y_3.
3. Graph y_1, y_2, and y_4 and make conjectures about the domain and range of y_4.
4. Confirm your conjectures algebraically by finding formulas for y_3 and y_4.

Quick Review 1.2 *(For help, go to Appendix A1 and Section 1.2.)*

In Exercises 1–6, solve for x.

1. $3x - 1 \le 5x + 3$

2. $x(x - 2) > 0$

3. $|x - 3| \le 4$

4. $|x - 2| \ge 5$

5. $x^2 < 16$

6. $9 - x^2 \ge 0$

In Exercises 7 and 8, describe how the graph of f can be transformed to the graph of g.

7. $f(x) = x^2, \quad g(x) = (x + 2)^2 - 3$

8. $f(x) = |x|, \quad g(x) = |x - 5| + 2$

In Exercises 9–12, find all real solutions to the equations.

9. $f(x) = x^2 - 5$

(a) $f(x) = 4$ **(b)** $f(x) = -6$

10. $f(x) = 1/x$

(a) $f(x) = -5$ **(b)** $f(x) = 0$

11. $f(x) = \sqrt{x + 7}$

(a) $f(x) = 4$ **(b)** $f(x) = 1$

12. $f(x) = \sqrt[3]{x - 1}$

(a) $f(x) = -2$ **(b)** $f(x) = 3$

Section 1.2 Exercises

In Exercises 1–4, **(a)** write a formula for the function and **(b)** use the formula to find the indicated value of the function.

1. the area A of a circle as a function of its diameter d; the area of a circle of diameter 4 in.

2. the height h of an equilateral triangle as a function of its side length s; the height of an equilateral triangle of side length 3 m

3. the surface area S of a cube as a function of the length of the cube's edge e; the surface area of a cube of edge length 5 ft

4. the volume V of a sphere as a function of the sphere's radius r; the volume of a sphere of radius 3 cm

In Exercises 5–12, **(a)** identify the domain and range and **(b)** sketch the graph of the function.

5. $y = 4 - x^2$

6. $y = x^2 - 9$

7. $y = 2 + \sqrt{x - 1}$

8. $y = -\sqrt{-x}$

9. $y = \dfrac{1}{x - 2}$

10. $y = \sqrt[4]{-x}$

11. $y = 1 + \dfrac{1}{x}$

12. $y = 1 + \dfrac{1}{x^2}$

In Exercises 13–20, use a grapher to **(a)** identify the domain and range and **(b)** draw the graph of the function.

13. $y = \sqrt[3]{x}$

14. $y = 2\sqrt{3 - x}$

15. $y = \sqrt[3]{1 - x^2}$

16. $y = \sqrt{9 - x^2}$

17. $y = x^{2/5}$

18. $y = x^{3/2}$

19. $y = \sqrt[3]{x - 3}$

20. $y = \dfrac{1}{\sqrt{4 - x^2}}$

In Exercises 21–30, determine whether the function is even, odd, or neither. Try to answer without writing anything (except the answer).

21. $y = x^4$

22. $y = x + x^2$

23. $y = x + 2$

24. $y = x^2 - 3$

25. $y = \sqrt{x^2 + 2}$

26. $y = x + x^3$

27. $y = \dfrac{x^3}{x^2 - 1}$

28. $y = \sqrt[3]{2 - x}$

29. $y = \dfrac{1}{x - 1}$

30. $y = \dfrac{1}{x^2 - 1}$

In Exercises 31–34, graph the piecewise-defined functions.

31. $f(x) = \begin{cases} 3 - x, & x \le 1 \\ 2x, & 1 < x \end{cases}$

32. $f(x) = \begin{cases} 1, & x < 0 \\ \sqrt{x}, & x \ge 0 \end{cases}$

33. $f(x) = \begin{cases} 4 - x^2, & x < 1 \\ (3/2)x + 3/2, & 1 \le x \le 3 \\ x + 3, & x > 3 \end{cases}$

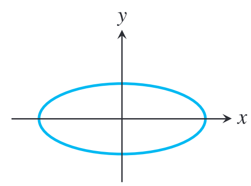
34. $f(x) = \begin{cases} x^2, & x < 0 \\ x^3, & 0 \le x \le 1 \\ 2x - 1, & x > 1 \end{cases}$

35. Writing to Learn The *vertical line test* to determine whether a curve is the graph of a function states: If every vertical line in the xy-plane intersects a given curve in at most one point, then the curve is the graph of a function. Explain why this is true.

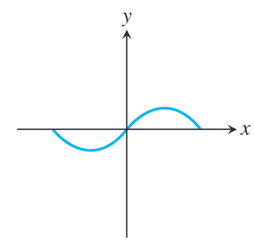
36. Writing to Learn For a curve to be *symmetric about the x-axis*, the point (x, y) must lie on the curve if and only if the point $(x, -y)$ lies on the curve. Explain why a curve that is symmetric about the x-axis is not the graph of a function, unless the function is $y = 0$.

In Exercises 37–40, use the vertical line test (see Exercise 35) to determine whether the curve is the graph of a function.

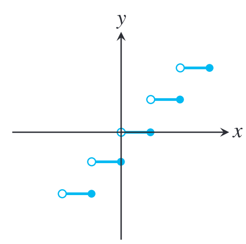
37.



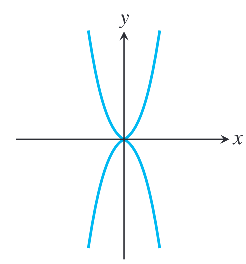
38.



39.

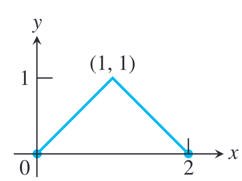


40.

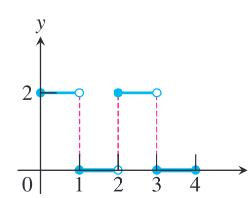


In Exercises 41–48, write a piecewise formula for the function.

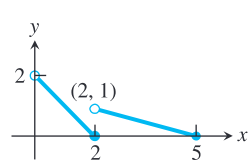
41.



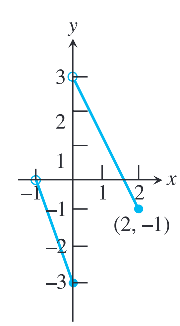
42.



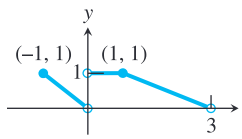
43.



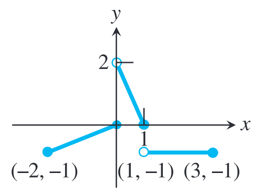
44.



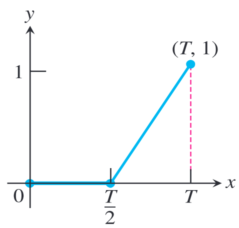
45.



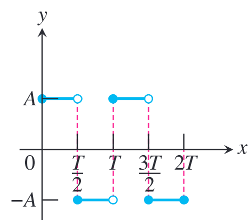
46.



47.



48.



In Exercises 49 and 50, **(a)** draw the graph of the function. Then find its **(b)** domain and **(c)** range.

49. $f(x) = -|3 - x| + 2$ **50.** $f(x) = 2|x + 4| - 3$

In Exercises 51 and 52, find

(a) $f(g(x))$ **(b)** $g(f(x))$ **(c)** $f(g(0))$

(d) $g(f(0))$ **(e)** $g(g(-2))$ **(f)** $f(f(x))$

51. $f(x) = x + 5, \quad g(x) = x^2 - 3$

52. $f(x) = x + 1, \quad g(x) = x - 1$

53. Copy and complete the following table.

	$g(x)$	$f(x)$	$(f \circ g)(x)$
(a)	?	$\sqrt{x - 5}$	$\sqrt{x^2 - 5}$
(b)	?	$1 + 1/x$	x
(c)	$1/x$	?	x
(d)	$\sqrt{x}$	?	$\lvert x \rvert, x \geq 0$

54. ***Broadway Season Statistics*** Table 1.5 shows the gross revenue for the Broadway season in millions of dollars for several years.

Table 1.5 Broadway Season Revenue

Year	Amount ($ millions)
1997	558
1998	588
1999	603
2000	666
2001	643
2002	721
2003	771

Source: The League of American Theatres and Producers, Inc., New York, NY, as reported in *The World Almanac and Book of Facts, 2005.*

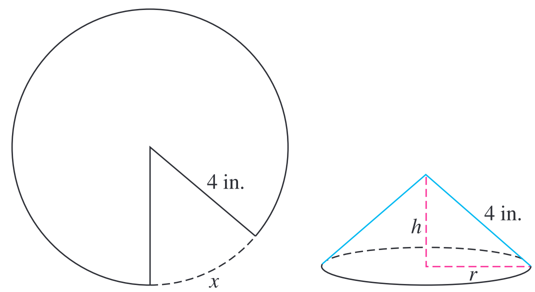
(a) Find the quadratic regression for the data in Table 1.5. Let $x = 0$ represent 1990, $x = 1$ represent 1991, and so forth.

(b) Superimpose the graph of the quadratic regression equation on a scatter plot of the data.

(c) Use the quadratic regression to predict the amount of revenue in 2008.

(d) Now find the linear regression for the data and use it to predict the amount of revenue in 2008.

55. ***The Cone Problem*** Begin with a circular piece of paper with a 4-in. radius as shown in (a). Cut out a sector with an arc length of x. Join the two edges of the remaining portion to form a cone with radius r and height h, as shown in (b).



(a) (b)

(a) Explain why the circumference of the base of the cone is $8\pi - x$.

(b) Express the radius r as a function of x.

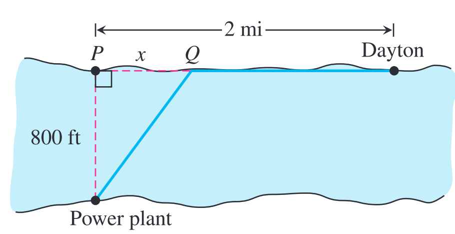
(c) Express the height h as a function of x.

(d) Express the volume V of the cone as a function of x.

56. ***Industrial Costs*** Dayton Power and Light, Inc., has a power plant on the Miami River where the river is 800 ft wide. To lay a new cable from the plant to a location in the city 2 mi downstream on the opposite side costs $180 per foot across the river and $100 per foot along the land.

(a) Suppose that the cable goes from the plant to a point Q on the opposite side that is x ft from the point P directly opposite the plant. Write a function $C(x)$ that gives the cost of laying the cable in terms of the distance x.

(b) Generate a table of values to determine if the least expensive location for point Q is less than 2000 ft or greater than 2000 ft from point P.



(Not to scale)

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

57. True or False The function $f(x) = x^4 + x^2 + x$ is an even function. Justify your answer.

58. True or False The function $f(x) = x^{-3}$ is an odd function. Justify your answer.

59. Multiple Choice Which of the following gives the domain of $f(x) = \dfrac{x}{\sqrt{9 - x^2}}$?

(A) $x \neq \pm 3$ **(B)** $(-3, 3)$ **(C)** $[-3, 3]$
(D) $(-\infty, -3) \cup (3, \infty)$ **(E)** $(3, \infty)$

60. Multiple Choice Which of the following gives the range of $f(x) = 1 + \dfrac{1}{x - 1}$?

(A) $(-\infty, 1) \cup (1, \infty)$ **(B)** $x \neq 1$ **(C)** all real numbers
(D) $(-\infty, 0) \cup (0, \infty)$ **(E)** $x \neq 0$

61. Multiple Choice If $f(x) = 2x - 1$ and $g(x) = x + 3$, which of the following gives $(f \circ g)(2)$?

(A) 2 **(B)** 6 **(C)** 7 **(D)** 9 **(E)** 10

62. Multiple Choice The length L of a rectangle is twice as long as its width W. Which of the following gives the area A of the rectangle as a function of its width?

(A) $A(W) = 3W$ **(B)** $A(W) = \frac{1}{2}W^2$ **(C)** $A(W) = 2W^2$
(D) $A(W) = W^2 + 2W$ **(E)** $A(W) = W^2 - 2W$

Explorations

In Exercises 63–66, **(a)** graph $f \circ g$ and $g \circ f$ and make a conjecture about the domain and range of each function. **(b)** Then confirm your conjectures by finding formulas for $f \circ g$ and $g \circ f$.

63. $f(x) = x - 7, \quad g(x) = \sqrt{x}$

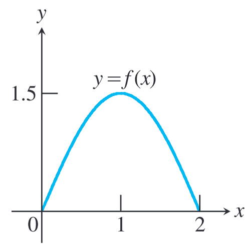
64. $f(x) = 1 - x^2, \quad g(x) = \sqrt{x}$

65. $f(x) = x^2 - 3, \quad g(x) = \sqrt{x + 2}$

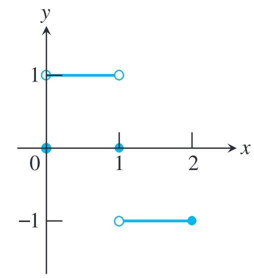
66. $f(x) = \dfrac{2x - 1}{x + 3}, \quad g(x) = \dfrac{3x + 1}{2 - x}$

Group Activity In Exercises 67–70, a portion of the graph of a function defined on $[-2, 2]$ is shown. Complete each graph assuming that the graph is **(a)** even, **(b)** odd.

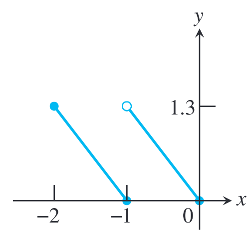
67.



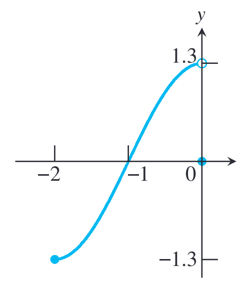
68.



69.



70.



Extending the Ideas

71. Enter $y_1 = \sqrt{x}$, $y_2 = \sqrt{1 - x}$ and $y_3 = y_1 + y_2$ on your grapher.

(a) Graph y_3 in $[-3, 3]$ by $[-1, 3]$.

(b) Compare the domain of the graph of y_3 with the domains of the graphs of y_1 and y_2.

(c) Replace y_3 by

$$y_1 - y_2, \quad y_2 - y_1, \quad y_1 \cdot y_2, \quad y_1/y_2, \quad \text{and} \quad y_2/y_1,$$

in turn, and repeat the comparison of part (b).

(d) Based on your observations in (b) and (c), what would you conjecture about the domains of sums, differences, products, and quotients of functions?

72. ***Even and Odd Functions***

(a) Must the product of two even functions always be even? Give reasons for your answer.

(b) Can anything be said about the product of two odd functions? Give reasons for your answer.

1.3 Exponential Functions

What you'll learn about

- Exponential Growth
- Exponential Decay
- Applications
- The Number e

. . . and why

Exponential functions model many growth patterns.

Exponential Growth

Table 1.6 shows the growth of \$100 invested in 1996 at an interest rate of 5.5%, compounded annually.

Table 1.6 Savings Account Growth

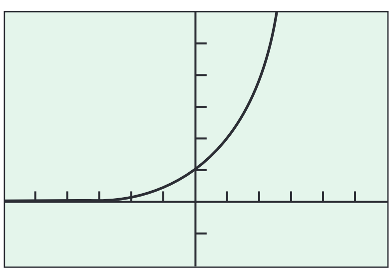
Year	Amount (dollars)	Increase (dollars)
1996	100	
1997	$100(1.055) = 105.50$	5.50
1998	$100(1.055)^2 = 111.30$	5.80
1999	$100(1.055)^3 = 117.42$	6.12
2000	$100(1.055)^4 = 123.88$	6.46

After the first year, the value of the account is always 1.055 times its value in the previous year. After n years, the value is $y = 100 \cdot (1.055)^n$.

Compound interest provides an example of *exponential growth* and is modeled by a function of the form $y = P \cdot a^x$, where P is the initial investment and a is equal to 1 plus the interest rate expressed as a decimal.

The equation $y = P \cdot a^x$, $a > 0$, $a \neq 1$, identifies a family of functions called *exponential functions.* Notice that the ratio of consecutive amounts in Table 1.6 is always the same: $111.30/105.30 = 117.42/111.30 = 123.88/117.42 \approx 1.055$. This fact is an important feature of exponential curves that has widespread application, as we will see.

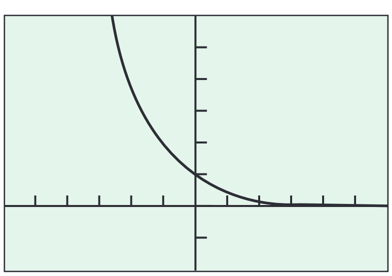
$y = 2^x$



[–6, 6] by [–2, 6]

(a)

$y = 2^{-x}$



[–6, 6] by [–2, 6]

(b)

Figure 1.22 A graph of (a) $y = 2^x$ and (b) $y = 2^{-x}$.

EXPLORATION 1 Exponential Functions

1. Graph the function $y = a^x$ for $a = 2, 3, 5$, in a $[-5, 5]$ by $[-2, 5]$ viewing window.
2. For what values of x is it true that $2^x < 3^x < 5^x$?
3. For what values of x is it true that $2^x > 3^x > 5^x$?
4. For what values of x is it true that $2^x = 3^x = 5^x$?
5. Graph the function $y = (1/a)^x = a^{-x}$ for $a = 2, 3, 5$.
6. Repeat parts 2–4 for the functions in part 5.

DEFINITION Exponential Function

Let a be a positive real number other than 1. The function

$$f(x) = a^x$$

is the **exponential function with base *a*.**

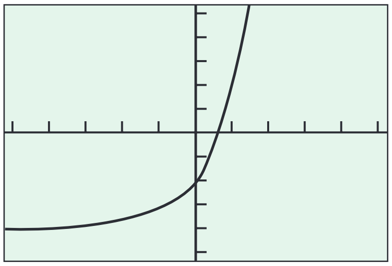
The domain of $f(x) = a^x$ is $(-\infty, \infty)$ and the range is $(0, \infty)$. If $a > 1$, the graph of f looks like the graph of $y = 2^x$ in Figure 1.22a. If $0 < a < 1$, the graph of f looks like the graph of $y = 2^{-x}$ in Figure 1.22b.

EXAMPLE 1 Graphing an Exponential Function

Graph the function $y = 2(3^x) - 4$. State its domain and range.

continued

$y = 2(3^x) - 4$



[–5, 5] by [–5, 5]

Figure 1.23 The graph of $y = 2(3^x) - 4$. (Example 1)

SOLUTION

Figure 1.23 shows the graph of the function y. It appears that the domain is $(-\infty, \infty)$. The range is $(-4, \infty)$ because $2(3^x) > 0$ for all x. ***Now try Exercise 1.***

EXAMPLE 2 Finding Zeros

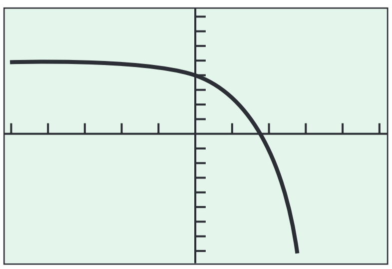
Find the zeros of $f(x) = 5 - 2.5^x$ graphically.

SOLUTION

Figure 1.24a suggests that f has a zero between $x = 1$ and $x = 2$, closer to 2. We can use our grapher to find that the zero is approximately 1.756 (Figure 1.24b).

Now try Exercise 9.

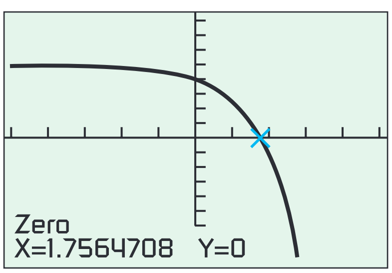
$y = 5 - 2.5^x$



[–5, 5] by [–8, 8]

(a)

$y = 5 - 2.5^x$



[–5, 5] by [–8, 8]

(b)

Figure 1.24 (a) A graph of $f(x) = 5 - 2.5^x$. (b) Showing the use of the ZERO feature to approximate the zero of f. (Example 2)

Exponential functions obey the rules for exponents.

Rules for Exponents

If $a > 0$ and $b > 0$, the following hold for all real numbers x and y.

1. $a^x \cdot a^y = a^{x+y}$

2. $\dfrac{a^x}{a^y} = a^{x-y}$

3. $(a^x)^y = (a^y)^x = a^{xy}$

4. $a^x \cdot b^x = (ab)^x$

5. $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

In Table 1.6 we observed that the ratios of the amounts in consecutive years were always the same, namely the interest rate. Population growth can sometimes be modeled with an exponential function, as we see in Table 1.7 and Example 3.

Table 1.7 gives the United States population for several recent years. In this table we have divided the population in one year by the population in the previous year to get an idea of how the population is growing. These ratios are given in the third column.

Table 1.7 United States Population

Year	Population (millions)	Ratio
1998	276.1	
		$279.3/276.1 \approx 1.0116$
1999	279.3	
		$282.4/279.3 \approx 1.0111$
2000	282.4	
		$285.3/282.4 \approx 1.0102$
2001	285.3	
		$288.2/285.3 \approx 1.0102$
2002	288.2	
		$291.0/288.2 \approx 1.0097$
2003	291.0	

Source: Statistical Abstract of the United States, 2004–2005.

EXAMPLE 3 Predicting United States Population

Use the data in Table 1.7 and an exponential model to predict the population of the United States in the year 2010.

continued

SOLUTION

Based on the third column of Table 1.7, we might be willing to conjecture that the population of the United States in any year is about 1.01 times the population in the previous year.

If we start with the population in 1998, then according to the model the population (in millions) in 2010 would be about

$$276.1(1.01)^{12} \approx 311.1,$$

or about 311.1 million people. ***Now try Exercise 19.***

Exponential Decay

Exponential functions can also model phenomena that produce a decrease over time, such as happens with radioactive decay. The **half-life** of a radioactive substance is the amount of time it takes for half of the substance to change from its original radioactive state to a nonradioactive state by emitting energy in the form of radiation.

EXAMPLE 4 Modeling Radioactive Decay

Suppose the half-life of a certain radioactive substance is 20 days and that there are 5 grams present initially. When will there be only 1 gram of the substance remaining?

SOLUTION

Model The number of grams remaining after 20 days is

$$5\left(\frac{1}{2}\right) = \frac{5}{2}.$$

The number of grams remaining after 40 days is

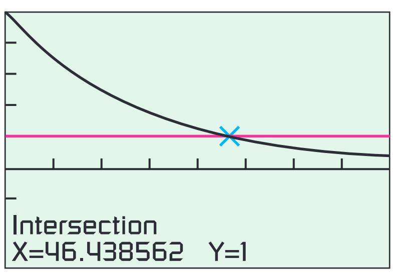
$$5\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = 5\left(\frac{1}{2}\right)^2 = \frac{5}{4}.$$

The function $y = 5(1/2)^{t/20}$ models the mass in grams of the radioactive substance after t days.

Solve Graphically Figure 1.25 shows that the graphs of $y_1 = 5(1/2)^{t/20}$ and $y_2 = 1$ (for 1 gram) intersect when t is approximately 46.44.

Interpret There will be 1 gram of the radioactive substance left after approximately 46.44 days, or about 46 days 10.5 hours. ***Now try Exercise 23.***

$y = 5\left(\frac{1}{2}\right)^{t/20}$, $y = 1$



[0, 80] by [–3, 5]

Figure 1.25 (Example 4)

Compound interest investments, population growth, and radioactive decay are all examples of *exponential growth and decay.*

> **DEFINITIONS Exponential Growth, Exponential Decay**
>
> The function $y = k \cdot a^x$, $k > 0$ is a model for **exponential growth** if $a > 1$, and a model for **exponential decay** if $0 < a < 1$.

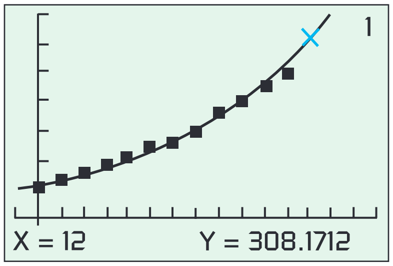
Applications

Most graphers have the exponential growth and decay model $y = k \cdot a^x$ built in as an exponential regression equation. We use this feature in Example 5 to analyze the U.S. population from the data in Table 1.8.

Table 1.8 U.S. Population

Year	Population (millions)
1880	50.2
1890	63.0
1900	76.2
1910	92.2
1920	106.0
1930	123.2
1940	132.1
1950	151.3
1960	179.3
1970	203.3
1980	226.5
1990	248.7

Source: The Statistical Abstract of the United States, 2004–2005.



[–1, 15] by [–50, 350]

Figure 1.26 (Example 5)

EXAMPLE 5 Predicting the U.S. Population

Use the population data in Table 1.8 to estimate the population for the year 2000. Compare the result with the actual 2000 population of approximately 281.4 million.

SOLUTION

Model Let $x = 0$ represent 1880, $x = 1$ represent 1890, and so on. We enter the data into the grapher and find the exponential regression equation to be

$$f(x) = (56.4696)(1.1519)^x.$$

Figure 1.26 shows the graph of f superimposed on the scatter plot of the data.

Solve Graphically The year 2000 is represented by $x = 12$. Reading from the curve, we find

$$f(12) \approx 308.2.$$

The exponential model estimates the 2000 population to be 308.2 million, an overestimate of approximately 26.8 million, or about 9.5%.

Now try Exercise 39(a, b).

EXAMPLE 6 Interpreting Exponential Regression

What *annual* rate of growth can we infer from the exponential regression equation in Example 5?

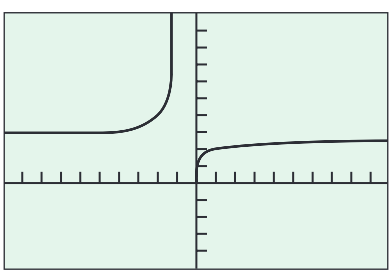
SOLUTION

Let r be the annual rate of growth of the U.S. population, expressed as a decimal. Because the time increments we used were 10-year intervals, we have

$$(1 + r)^{10} \approx 1.1519$$
$$r \approx \sqrt[10]{1.1519} - 1$$
$$r \approx 0.014$$

The annual rate of growth is about 1.4%. ***Now try Exercise 39(c).***

$y = (1 + 1/x)^x$



[–10, 10] by [–5, 10]

X	Y1
1000	2.7169
2000	2.7176
3000	2.7178
4000	2.7179
5000	2.718
6000	2.7181
7000	2.7181

Y1 = (1+1/X)^X

Figure 1.27 A graph and table of values for $f(x) = (1 + 1/x)^x$ both suggest that as $x \to \infty$, $f(x) \to e \approx 2.718$.

The Number e

Many natural, physical, and economic phenomena are best modeled by an exponential function whose base is the famous number e, which is 2.718281828 to nine decimal places. We can define e to be the number that the function $f(x) = (1 + 1/x)^x$ approaches as x approaches infinity. The graph and table in Figure 1.27 strongly suggest that such a number exists.

The exponential functions $y = e^x$ and $y = e^{-x}$ are frequently used as models of exponential growth or decay. For example, interest **compounded continuously** uses the model $y = P \cdot e^{rt}$, where P is the initial investment, r is the interest rate as a decimal, and t is time in years.

Quick Review 1.3 *(For help, go to Section 1.3.)*

In Exercises 1–3, evaluate the expression. Round your answers to 3 decimal places.

1. $5^{2/3}$ **2.** $3^{\sqrt{2}}$

3. $3^{-1.5}$

In Exercises 4–6, solve the equation. Round your answers to 4 decimal places.

4. $x^3 = 17$ **5.** $x^5 = 24$

6. $x^{10} = 1.4567$

In Exercises 7 and 8, find the value of investing P dollars for n years with the interest rate r compounded annually.

7. $P = \$500, \quad r = 4.75\%, \quad n = 5$ years

8. $P = \$1000, \quad r = 6.3\%, \quad n = 3$ years

In Exercises 9 and 10, simplify the exponential expression.

9. $\dfrac{(x^{-3}y^2)^2}{(x^4y^3)^3}$ **10.** $\left(\dfrac{a^3b^{-2}}{c^4}\right)^2\left(\dfrac{a^4c^{-2}}{b^3}\right)^{-1}$

Section 1.3 Exercises

In Exercises 1–4, graph the function. State its domain and range.

1. $y = -2^x + 3$ **2.** $y = e^x + 3$

3. $y = 3 \cdot e^{-x} - 2$ **4.** $y = -2^{-x} - 1$

In Exercises 5–8, rewrite the exponential expression to have the indicated base.

5. 9^{2x}, base 3 **6.** 16^{3x}, base 2

7. $(1/8)^{2x}$, base 2 **8.** $(1/27)^x$, base 3

In Exercises 9–12, use a graph to find the zeros of the function.

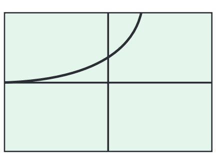
9. $f(x) = 2^x - 5$ **10.** $f(x) = e^x - 4$

11. $f(x) = 3^x - 0.5$ **12.** $f(x) = 3 - 2^x$

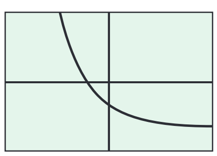
In Exercises 13–18, match the function with its graph. Try to do it without using your grapher.

13. $y = 2^x$ **14.** $y = 3^{-x}$ **15.** $y = -3^{-x}$

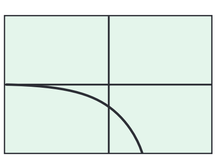
16. $y = -0.5^{-x}$ **17.** $y = 2^{-x} - 2$ **18.** $y = 1.5^x - 2$



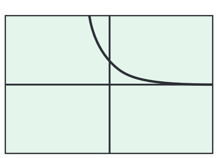
(a)



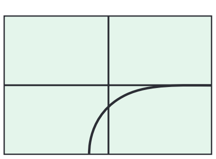
(b)



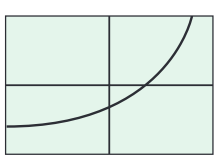
(c)



(d)



(e)



(f)

19. ***Population of Nevada*** Table 1.9 gives the population of Nevada for several years.

Table 1.9 Population of Nevada

Year	Population (thousands)
1998	1,853
1999	1,935
2000	1,998
2001	2,095
2002	2,167
2003	2,241

Source: Statistical Abstract of the United States, 2004–2005.

(a) Compute the ratios of the population in one year by the population in the previous year.

(b) Based on part (a), create an exponential model for the population of Nevada.

(c) Use your model in part (b) to predict the population of Nevada in 2010.

20. ***Population of Virginia*** Table 1.10 gives the population of Virginia for several years.

Table 1.10 Population of Virginia

Year	Population (thousands)
1998	6,901
1999	7,000
2000	7,078
2001	7,193
2002	7,288
2003	7,386

Source: Statistical Abstract of the United States, 2004–2005.

(a) Compute the ratios of the population in one year by the population in the previous year.

(b) Based on part (a), create an exponential model for the population of Virginia.

(c) Use your model in part (b) to predict the population of Virginia in 2008.

In Exercises 21–32, use an exponential model to solve the problem.

21. ***Population Growth*** The population of Knoxville is 500,000 and is increasing at the rate of 3.75% each year. Approximately when will the population reach 1 million?

22. ***Population Growth*** The population of Silver Run in the year 1890 was 6250. Assume the population increased at a rate of 2.75% per year.

(a) Estimate the population in 1915 and 1940.

(b) Approximately when did the population reach 50,000?

23. ***Radioactive Decay*** The half-life of phosphorus-32 is about 14 days. There are 6.6 grams present initially.

(a) Express the amount of phosphorus-32 remaining as a function of time t.

(b) When will there be 1 gram remaining?

24. ***Finding Time*** If John invests \$2300 in a savings account with a 6% interest rate compounded annually, how long will it take until John's account has a balance of \$4150?

25. ***Doubling Your Money*** Determine how much time is required for an investment to double in value if interest is earned at the rate of 6.25% compounded annually.

26. ***Doubling Your Money*** Determine how much time is required for an investment to double in value if interest is earned at the rate of 6.25% compounded monthly.

27. ***Doubling Your Money*** Determine how much time is required for an investment to double in value if interest is earned at the rate of 6.25% compounded continuously.

28. ***Tripling Your Money*** Determine how much time is required for an investment to triple in value if interest is earned at the rate of 5.75% compounded annually.

29. ***Tripling Your Money*** Determine how much time is required for an investment to triple in value if interest is earned at the rate of 5.75% compounded daily.

30. ***Tripling Your Money*** Determine how much time is required for an investment to triple in value if interest is earned at the rate of 5.75% compounded continuously.

31. ***Cholera Bacteria*** Suppose that a colony of bacteria starts with 1 bacterium and doubles in number every half hour. How many bacteria will the colony contain at the end of 24 h?

32. ***Eliminating a Disease*** Suppose that in any given year, the number of cases of a disease is reduced by 20%. If there are 10,000 cases today, how many years will it take

(a) to reduce the number of cases to 1000?

(b) to eliminate the disease; that is, to reduce the number of cases to less than 1?

Group Activity In Exercises 33–36, copy and complete the table for the function.

33. $y = 2x - 3$

x	y	Change (Δy)
1	?	
		?
2	?	
		?
3	?	
		?
4	?	

34. $y = -3x + 4$

x	y	Change (Δy)
1	?	
		?
2	?	
		?
3	?	
		?
4	?	

35. $y = x^2$

x	y	Change (Δy)
1	?	
		?
2	?	
		?
3	?	
		?
4	?	

36. $y = 3e^x$

x	y	Ratio (y_i/y_{i-1})
1	?	
		?
2	?	
		?
3	?	
		?
4	?	

37. **Writing to Learn** Explain how the change Δy is related to the slopes of the lines in Exercises 33 and 34. If the changes in x are constant for a linear function, what would you conclude about the corresponding changes in y?

38. ***Bacteria Growth*** The number of bacteria in a petri dish culture after t hours is

$$B = 100e^{0.693t}.$$

(a) What was the initial number of bacteria present?

(b) How many bacteria are present after 6 hours?

(c) Approximately when will the number of bacteria be 200? Estimate the doubling time of the bacteria.

39. ***Population of Texas*** Table 1.11 gives the population of Texas for several years.

Table 1.11 Population of Texas

Year	Population (thousands)
1980	14,229
1990	16,986
1995	18,959
1998	20,158
1999	20,558
2000	20,852

Source: Statistical Abstract of the United States, 2004–2005.

(a) Let $x = 0$ represent 1980, $x = 1$ represent 1981, and so forth. Find an exponential regression for the data, and superimpose its graph on a scatter plot of the data.

(b) Use the exponential regression equation to estimate the population of Texas in 2003. How close is the estimate to the actual population of 22,119,000 in 2003?

(c) Use the exponential regression equation to estimate the annual rate of growth of the population of Texas.

40. ***Population of California*** Table 1.12 gives the population of California for several years.

Table 1.12 Population of California

Year	Population (thousands)
1980	23,668
1990	29,811
1995	31,697
1998	32,988
1999	33,499
2000	33,872

Source: Statistical Abstract of the United States, 2004–2005.

(a) Let $x = 0$ represent 1980, $x = 1$ represent 1981, and so forth. Find an exponential regression for the data, and superimpose its graph on a scatter plot of the data.

(b) Use the exponential regression equation to estimate the population of California in 2003. How close is the estimate to the actual population of 35,484,000 in 2003?

(c) Use the exponential regression equation to estimate the annual rate of growth of the population of California.

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

41. **True or False** The number 3^{-2} is negative. Justify your answer.

42. **True or False** If $4^3 = 2^a$, then $a = 6$. Justify your answer.

43. **Multiple Choice** John invests \$200 at 4.5% compounded annually. About how long will it take for John's investment to double in value?

(A) 6 yrs **(B)** 9 yrs **(C)** 12 yrs **(D)** 16 yrs **(E)** 20 yrs

44. **Multiple Choice** Which of the following gives the domain of $y = 2e^{-x} - 3$?

(A) $(-\infty, \infty)$ **(B)** $[-3, \infty)$ **(C)** $[-1, \infty)$ **(D)** $(-\infty, 3]$

(E) $x \neq 0$

45. **Multiple Choice** Which of the following gives the range of $y = 4 - 2^{-x}$?

(A) $(-\infty, \infty)$ **(B)** $(-\infty, 4)$ **(C)** $[-4, \infty)$

(D) $(-\infty, 4]$ **(E)** all reals

46. **Multiple Choice** Which of the following gives the best approximation for the zero of $f(x) = 4 - e^x$?

(A) $x = -1.386$ **(B)** $x = 0.386$ **(C)** $x = 1.386$

(D) $x = 3$ **(E)** there are no zeros

Exploration

47. Let $y_1 = x^2$ and $y_2 = 2^x$.

(a) Graph y_1 and y_2 in $[-5, 5]$ by $[-2, 10]$. How many times do you think the two graphs cross?

(b) Compare the corresponding changes in y_1 and y_2 as x changes from 1 to 2, 2 to 3, and so on. How large must x be for the changes in y_2 to overtake the changes in y_1?

(c) Solve for x: $x^2 = 2^x$. **(d)** Solve for x: $x^2 < 2^x$.

Extending the Ideas

In Exercises 48 and 49, assume that the graph of the exponential function $f(x) = k \cdot a^x$ passes through the two points. Find the values of a and k.

48. $(1, 4.5), (-1, 0.5)$ **49.** $(1, 1.5), (-1, 6)$

Quick Quiz for AP* Preparation: Sections 1.1–1.3

You may use graphing calculator to solve the following problems.

1. **Multiple Choice** Which of the following gives an equation for the line through (3, −1) and parallel to the line $y = -2x + 1$?

   (A) $= \frac{1}{2}x + \frac{7}{2}$ **(B)** $y = \frac{1}{2}x - \frac{5}{2}$ **(C)** $y = -2x + 5$
   (D) $y = -2x - 7$ **(E)** $y = -2x + 1$

2. **Multiple Choice** If $f(x) = x^2 + 1$ and $g(x) = 2x - 1$, which of the following gives $f \circ g(2)$?

   (A) 2 **(B)** 5 **(C)** 9 **(D)** 10 **(E)** 15

3. **Multiple Choice** The half-life of a certain radioactive substance is 8 hrs. There are 5 grams present initially. Which of the following gives the best approximation when there will be 1 gram remaining?

   (A) 2 **(B)**10 **(C)** 15 **(D)** 16 **(E)** 19

4. **Free Response** Let $f(x) = e^{-x} - 2$.

   (a) Find the domain of f. **(b)** Find the range of f.
   (c) Find the zeros of f.

1.4 Parametric Equations

What you'll learn about

- Relations
- Circles
- Ellipses
- Lines and Other Curves

. . . and why

Parametric equations can be used to obtain graphs of relations and functions.

Relations

A **relation** is a set of ordered pairs (x, y) of real numbers. The **graph of a relation** is the set of points in the plane that correspond to the ordered pairs of the relation. If x and y are *functions* of a third variable t, called a *parameter*, then we can use the *parametric mode* of a grapher to obtain a graph of the relation.

EXAMPLE 1 Graphing Half a Parabola

Describe the graph of the relation determined by

$$x = \sqrt{t}, \quad y = t, \quad t \geq 0.$$

Indicate the direction in which the curve is traced. Find a Cartesian equation for a curve that contains the parametrized curve.

SOLUTION

Set $x_1 = \sqrt{t}$, $y_1 = t$, and use the parametric mode of the grapher to draw the graph in Figure 1.28. The graph appears to be the right half of the parabola $y = x^2$. Notice that there is no information about t on the graph itself. The curve appears to be traced to the upper right with starting point (0, 0).

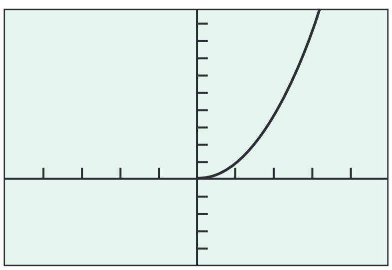
Confirm Algebraically Both x and y will be greater than or equal to zero because $t \geq 0$. Eliminating t we find that for every value of t,

$$y = t = (\sqrt{t})^2 = x^2.$$

Thus, the relation is the function $y = x^2$, $x \geq 0$.

Now try Exercise 5.

$x = \sqrt{t}$, $y = t$



[–5, 5] by [–5, 10]

Figure 1.28 You must choose a *smallest* and *largest* value for t in parametric mode. Here we used 0 and 10, respectively. (Example 1)

DEFINITIONS Parametric Curve, Parametric Equations

If x and y are given as functions

$$x = f(t), \quad y = g(t)$$

over an interval of t-values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a **parametric curve.** The equations are **parametric equations** for the curve.

The variable t is a **parameter** for the curve and its domain I is the **parameter interval.** If I is a closed interval, $a \leq t \leq b$, the point $(f(a), g(a))$ is the **initial point of the curve** and the point $(f(b), g(b))$ is the **terminal point of the curve.** When we give parametric equations and a parameter interval for a curve, we say that we have **parametrized** the curve. The equations and interval constitute a **parametrization of the curve.**

In Example 1, the parameter interval is $[0, \infty)$, so (0, 0) is the initial point and there is no terminal point.

A grapher can draw a parametrized curve only over a closed interval, so the portion it draws has endpoints even when the curve being graphed does not. Keep this in mind when you graph.

Circles

In applications, t often denotes time, an angle, or the distance a particle has traveled along its path from its starting point. In fact, parametric graphing can be used to simulate the motion of the particle.

EXPLORATION 1 **Parametrizing Circles**

Let $x = a \cos t$ and $y = a \sin t$.

1. Let $a = 1$, 2, or 3 and graph the parametric equations in a *square viewing window* using the parameter interval $[0, 2\pi]$. How does changing a affect this graph?
2. Let $a = 2$ and graph the parametric equations using the following parameter intervals: $[0, \pi/2]$, $[0, \pi]$, $[0, 3\pi/2]$, $[2\pi, 4\pi]$, and $[0, 4\pi]$. Describe the role of the length of the parameter interval.
3. Let $a = 3$ and graph the parametric equations using the intervals $[\pi/2, 3\pi/2]$, $[\pi, 2\pi]$, $[3\pi/2, 3\pi]$, and $[\pi, 5\pi]$. What are the initial point and terminal point in each case?
4. Graph $x = 2 \cos(-t)$ and $y = 2 \sin(-t)$ using the parameter intervals $[0, 2\pi]$, $[\pi, 3\pi]$, and $[\pi/2, 3\pi/2]$. In each case, describe how the graph is traced.

For $x = a \cos t$ and $y = a \sin t$, we have

$$x^2 + y^2 = a^2 \cos^2 t + a^2 \sin^2 t = a^2(\cos^2 t + \sin^2 t) = a^2(1) = a^2,$$

using the identity $\cos^2 t + \sin^2 t = 1$. Thus, the curves in Exploration 1 were either circles or portions of circles, each with center at the origin.

EXAMPLE 2 **Graphing a Circle**

Describe the graph of the relation determined by

$$x = 2 \cos t, \quad y = 2 \sin t, \quad 0 \le t \le 2\pi.$$

Find the initial and terminal points, if any, and indicate the direction in which the curve is traced. Find a Cartesian equation for a curve that contains the parametrized curve.

SOLUTION

Figure 1.29 shows that the graph appears to be a circle with radius 2. By watching the graph develop we can see that the curve is traced exactly once counterclockwise. The initial point at $t = 0$ is $(2, 0)$, and the terminal point at $t = 2\pi$ is also $(2, 0)$.

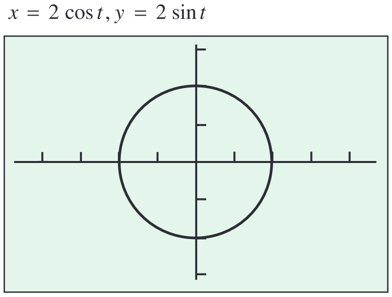
Next we eliminate the variable t.

$$\begin{aligned} x^2 + y^2 &= 4 \cos^2 t + 4 \sin^2 t \\ &= 4(\cos^2 t + \sin^2 t) \\ &= 4 \end{aligned}$$

The parametrized curve is a circle centered at the origin of radius 2.

Now try Exercise 9.

$x = 2 \cos t, y = 2 \sin t$



[–4.7, 4.7] by [–3.1, 3.1]

Figure 1.29 A graph of the parametric curve $x = 2 \cos t$, $y = 2 \sin t$, with Tmin = 0, Tmax = 2π, and Tstep = $\pi/24 \approx 0.131$. (Example 2)

Ellipses

Parametrizations of ellipses are similar to parametrizations of circles. Recall that the standard form of an ellipse centered at (0, 0) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

EXAMPLE 3 Graphing an Ellipse

Graph the parametric curve $x = 3 \cos t$, $y = 4 \sin t$, $0 \le t \le 2\pi$.

Find a Cartesian equation for a curve that contains the parametric curve. What portion of the graph of the Cartesian equation is traced by the parametric curve? Indicate the direction in which the curve is traced and the initial and terminal points, if any.

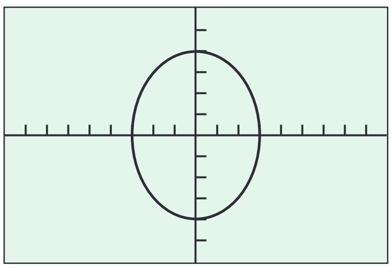
SOLUTION

Figure 1.30 suggests that the curve is an ellipse. The Cartesian equation is

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = \cos^2 t + \sin^2 t = 1,$$

so the parametrized curve lies along an ellipse with major axis endpoints $(0, \pm 4)$ and minor axis endpoints $(\pm 3, 0)$. As t increases from 0 to 2π, the point $(x, y) = (3 \cos t, 4 \sin t)$ starts at (3, 0) and traces the entire ellipse once counterclockwise. Thus, (3, 0) is both the initial point and the terminal point. ***Now try Exercise 13.***

$x = 3 \cos t$, $y = 4 \sin t$



[–9, 9] by [–6, 6]

Figure 1.30 A graph of the parametric equations $x = 3 \cos t$, $y = 4 \sin t$ for $0 \le t \le 2\pi$. (Example 3)

EXPLORATION 2 Parametrizing Ellipses

Let $x = a \cos t$ and $y = b \sin t$.

1. Let $a = 2$ and $b = 3$. Then graph using the parameter interval $[0, 2\pi]$. Repeat, changing b to 4, 5, and 6.
2. Let $a = 3$ and $b = 4$. Then graph using the parameter interval $[0, 2\pi]$. Repeat, changing a to 5, 6, and 7.
3. Based on parts 1 and 2, how do you identify the axis that contains the major axis of the ellipse? the minor axis?
4. Let $a = 4$ and $b = 3$. Then graph using the parameter intervals $[0, \pi/2]$, $[0, \pi]$, $[0, 3\pi/2]$, and $[0, 4\pi]$. Describe the role of the length of the parameter interval.
5. Graph $x = 5 \cos(-t)$ and $y = 2 \sin(-t)$ using the parameter intervals $(0, 2\pi]$, $[\pi, 3\pi]$, and $[\pi/2, 3\pi/2]$. Describe how the graph is traced. What are the initial point and terminal point in each case?

For $x = a \cos t$ and $y = b \sin t$, we have $(x/a)^2 + (y/b)^2 = \cos^2 t + \sin^2 t = 1$. Thus, the curves in Exploration 2 were either ellipses or portions of ellipses, each with center at the origin.

In the exercises you will see how to graph hyperbolas parametrically.

Lines and Other Curves

Lines, line segments, and many other curves can be defined parametrically.

EXAMPLE 4 Graphing a Line Segment

Draw and identify the graph of the parametric curve determined by

$$x = 3t, \quad y = 2 - 2t, \quad 0 \le t \le 1.$$

SOLUTION

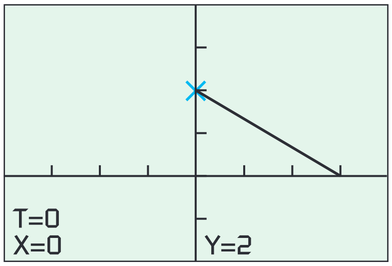
The graph (Figure 1.31) appears to be a line segment with endpoints (0, 2) and (3, 0).

Confirm Algebraically When $t = 0$, the equations give $x = 0$ and $y = 2$. When $t = 1$, they give $x = 3$ and $y = 0$. When we substitute $t = x/3$ into the y equation, we obtain

$$y = 2 - 2\left(\frac{x}{3}\right) = -\frac{2}{3}x + 2.$$

Thus, the parametric curve traces the segment of the line $y = -(2/3)x + 2$ from the point (0, 2) to (3, 0). ***Now try Exercise 17.***

$x = 3t,\ y = 2 - 2t$



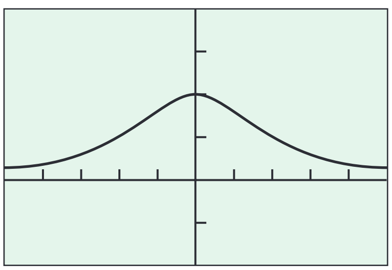
[–4, 4] by [–2, 4]

Figure 1.31 The graph of the line segment $x = 3t$, $y = 2 - 2t$, $0 \le t \le 1$, with trace on the initial point (0, 2). (Example 4)

If we change the parameter interval [0, 1] in Example 4 to $(-\infty, \infty)$, the parametrization will trace the entire line $y = -(2/3)x + 2$.

The bell-shaped curve in Exploration 3 is the famous witch of Agnesi. You will find more information about this curve in Exercise 47.

$x = 2 \cot t,\ y = 2 \sin^2 t$



[–5, 5] by [–2, 4]

Figure 1.32 The witch of Agnesi (Exploration 3)

EXPLORATION 3 Graphing the Witch of Agnesi

The witch of Agnesi is the curve

$$x = 2 \cot t, \quad y = 2 \sin^2 t, \quad 0 < t < \pi.$$

1. Draw the curve using the window in Figure 1.32. What did you choose as a closed parameter interval for your grapher? In what direction is the curve traced? How far to the left and right of the origin do you think the curve extends?
2. Graph the same parametric equations using the parameter intervals $(-\pi/2, \pi/2)$, $(0, \pi/2)$, and $(\pi/2, \pi)$. In each case, describe the curve you see and the direction in which it is traced by your grapher.
3. What happens if you replace $x = 2 \cot t$ by $x = -2 \cot t$ in the original parametrization? What happens if you use $x = 2 \cot(\pi - t)$?

Maria Agnesi (1718–1799)

The first text to include differential and integral calculus along with analytic geometry, infinite series, and differential equations was written in the 1740s by the Italian mathematician Maria Gaetana Agnesi. Agnesi, a gifted scholar and linguist whose Latin essay defending higher education for women was published when she was only nine years old, was a well-published scientist by age 20, and an honorary faculty member of the University of Bologna by age 30.

Today, Agnesi is remembered chiefly for a bell-shaped curve called *the witch of Agnesi*. This name, found only in English texts, is the result of a mistranslation. Agnesi's own name for the curve was *versiera* or "turning curve." John Colson, a noted Cambridge mathematician, probably confused versiera with *avversiera*, which means "wife of the devil" and translated it into "witch."

EXAMPLE 5 Parametrizing a Line Segment

Find a parametrization for the line segment with endpoints $(-2, 1)$ and $(3, 5)$.

SOLUTION

Using $(-2, 1)$ we create the parametric equations

$$x = -2 + at, \qquad y = 1 + bt.$$

These represent a line, as we can see by solving each equation for t and equating to obtain

$$\frac{x + 2}{a} = \frac{y - 1}{b}.$$

continued

This line goes through the point $(-2, 1)$ when $t = 0$. We determine a and b so that the line goes through $(3, 5)$ when $t = 1$.

$$3 = -2 + a \quad \Rightarrow \quad a = 5$$
$$5 = 1 + b \quad \Rightarrow \quad b = 4$$

Therefore,

$$x = -2 + 5t, \quad y = 1 + 4t, \quad 0 \le t \le 1$$

is a parametrization of the line segment with initial point $(-2, 1)$ and terminal point $(3, 5)$.

Now try Exercise 23.

Quick Review 1.4 *(For help, go to Section 1.1 and Appendix A1.)*

In Exercises 1–3, write an equation for the line.

1. the line through the points $(1, 8)$ and $(4, 3)$

2. the horizontal line through the point $(3, -4)$

3. the vertical line through the point $(2, -3)$

In Exercises 4–6, find the x- and y-intercepts of the graph of the relation.

4. $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$ **5.** $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$

6. $2y^2 = x + 1$

In Exercises 7 and 8, determine whether the given points lie on the graph of the relation.

7. $2x^2y + y^2 = 3$

(a) $(1, 1)$ **(b)** $(-1, -1)$ **(c)** $(1/2, -2)$

8. $9x^2 - 18x + 4y^2 = 27$

(a) $(1, 3)$ **(b)** $(1, -3)$ **(c)** $(-1, 3)$

9. Solve for t.

(a) $2x + 3t = -5$ **(b)** $3y - 2t = -1$

10. For what values of a is each equation true?

(a) $\sqrt{a^2} = a$ **(b)** $\sqrt{a^2} = \pm a$ **(c)** $\sqrt{4a^2} = 2|a|$

Section 1.4 Exercises

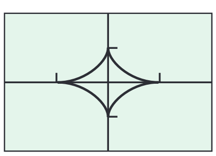
In Exercises 1–4, match the parametric equations with their graph. State the approximate dimensions of the viewing window. Give a parameter interval that traces the curve exactly once.

1. $x = 3\sin(2t), \quad y = 1.5\cos t$

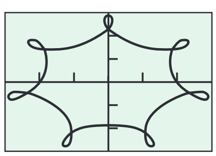
2. $x = \sin^3 t, \quad y = \cos^3 t$

3. $x = 7\sin t - \sin(7t), \quad y = 7\cos t - \cos(7t)$

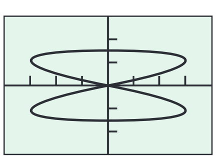
4. $x = 12\sin t - 3\sin(6t), \quad y = 12\cos t + 3\cos(6t)$



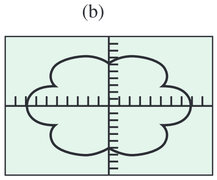
(a)



(b)



(c)



(d)

In Exercises 5–22, a parametrization is given for a curve.

(a) Graph the curve. What are the initial and terminal points, if any? Indicate the direction in which the curve is traced.

(b) Find a Cartesian equation for a curve that contains the parametrized curve. What portion of the graph of the Cartesian equation is traced by the parametrized curve?

5. $x = 3t, \quad y = 9t^2, \quad -\infty < t < \infty$

6. $x = -\sqrt{t}, \quad y = t, \quad t \ge 0$

7. $x = t, \quad y = \sqrt{t}, \quad t \ge 0$

8. $x = (\sec^2 t) - 1, \quad y = \tan t, \quad -\pi/2 < t < \pi/2$

9. $x = \cos t, \quad y = \sin t, \quad 0 \le t \le \pi$

10. $x = \sin(2\pi t), \quad y = \cos(2\pi t), \quad 0 \le t \le 1$

11. $x = \cos(\pi - t), \quad y = \sin(\pi - t), \quad 0 \le t \le \pi$

12. $x = 4\cos t, \quad y = 2\sin t, \quad 0 \le t \le 2\pi$

13. $x = 4\sin t, \quad y = 2\cos t, \quad 0 \le t \le \pi$

14. $x = 4\sin t, \quad y = 5\cos t, \quad 0 \le t \le 2\pi$

15. $x = 2t - 5, \quad y = 4t - 7, \quad -\infty < t < \infty$

16. $x = 1 - t, \quad y = 1 + t, \quad -\infty < t < \infty$

17. $x = t, \quad y = 1 - t, \quad 0 \le t \le 1$

18. $x = 3 - 3t, \quad y = 2t, \quad 0 \le t \le 1$

19. $x = 4 - \sqrt{t}, \quad y = \sqrt{t}, \quad 0 \le t$

20. $x = t^2, \quad y = \sqrt{4 - t^2}, \quad 0 \le t \le 2$

21. $x = \sin t, \quad y = \cos 2t, \quad -\infty < t < \infty$

22. $x = t^2 - 3, \quad y = t, \quad t \le 0$

In Exercises 23–28, find a parametrization for the curve.

23. the line segment with endpoints $(-1, -3)$ and $(4, 1)$

24. the line segment with endpoints $(-1, 3)$ and $(3, -2)$

25. the lower half of the parabola $x - 1 = y^2$

26. the left half of the parabola $y = x^2 + 2x$

27. the ray (half line) with initial point $(2, 3)$ that passes through the point $(-1, -1)$

28. the ray (half line) with initial point $(-1, 2)$ that passes through the point $(0, 0)$

Group Activity In Exercises 29–32, refer to the graph of

$$x = 3 - |t|, \quad y = t - 1, \quad -5 \le t \le 5,$$

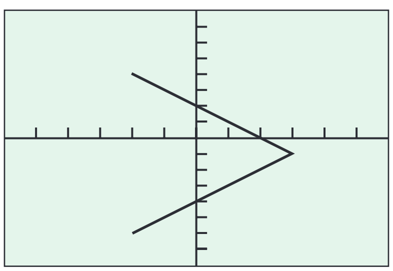
shown in the figure. Find the values of t that produce the graph in the given quadrant.

29. Quadrant I

30. Quadrant II

31. Quadrant III

32. Quadrant IV



[–6, 6] by [–8, 8]

In Exercises 33 and 34, find a parametrization for the part of the graph that lies in Quadrant I.

33. $y = x^2 + 2x + 2$

34. $y = \sqrt{x + 3}$

35. ***Circles*** Find parametrizations to model the motion of a particle that starts at $(a, 0)$ and traces the circle $x^2 + y^2 = a^2$, $a > 0$, as indicated.

(a) once clockwise **(b)** once counterclockwise

(c) twice clockwise **(d)** twice counterclockwise

36. ***Ellipses*** Find parametrizations to model the motion of a particle that starts at $(-a, 0)$ and traces the ellipse

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1, \quad a > 0, b > 0,$$

as indicated.

(a) once clockwise **(b)** once counterclockwise

(c) twice clockwise **(d)** twice counterclockwise

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

37. True or False The graph of the parametric curve $x = 3 \cos t$, $y = 4 \sin t$ is a circle. Justify your answer.

38. True or False The parametric curve $x = 2 \cos(-t)$, $y = 2 \sin(-t)$, $0 \le t \le 2\pi$ is traced clockwise. Justify your answer.

In Exercises 39 and 40, use the parametric curve $x = 5t$, $y = 3 - 3t$, $0 \le t \le 1$.

39. Multiple Choice Which of the following describes its graph?

(A) circle **(B)** parabola **(C)** ellipse

(D) line segment **(E)** line

40. Multiple Choice Which of the following is the initial point of the curve?

(A) $(-5, 6)$ **(B)** $(0, -3)$ **(C)** $(0, 3)$ **(D)** $(5, 0)$

(E) $(10, -3)$

41. Multiple Choice Which of the following describes the graph of the parametric curve $x = -3 \sin t$, $y = -3 \cos t$?

(A) circle **(B)** parabola **(C)** ellipse

(D) hyperbola **(E)** line

42. Multiple Choice Which of the following describes the graph of the parametric curve $x = 3t$, $y = 2t$, $t \ge 1$?

(A) circle **(B)** parabola **(C)** line segment

(D) line **(E)** ray

Explorations

43. ***Hyperbolas*** Let $x = a \sec t$ and $y = b \tan t$.

(a) Writing to Learn Let $a = 1, 2$, or 3, $b = 1, 2$, or 3, and graph using the parameter interval $(-\pi/2, \pi/2)$. Explain what you see, and describe the role of a and b in these parametric equations. (Caution: If you get what appear to be asymptotes, try using the approximation $[-1.57, 1.57]$ for the parameter interval.)

(b) Let $a = 2$, $b = 3$, and graph in the parameter interval $(\pi/2, 3\pi/2)$. Explain what you see.

(c) Writing to Learn Let $a = 2$, $b = 3$, and graph using the parameter interval $(-\pi/2, 3\pi/2)$. Explain why you must be careful about graphing in this interval or any interval that contains $\pm\pi/2$.

(d) Use algebra to explain why

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1.$$

(e) Let $x = a \tan t$ and $y = b \sec t$. Repeat (a), (b), and (d) using an appropriate version of (d).

44. ***Transformations*** Let $x = (2 \cos t) + h$ and $y = (2 \sin t) + k$.

(a) Writing to Learn Let $k = 0$ and $h = -2, -1, 1$, and 2, in turn. Graph using the parameter interval $[0, 2\pi]$. Describe the role of h.

(b) **Writing to Learn** Let $h = 0$ and $k = -2, -1, 1,$ and 2, in turn. Graph using the parameter interval $[0, 2\pi]$. Describe the role of k.

(c) Find a parametrization for the circle with radius 5 and center at $(2, -3)$.

(d) Find a parametrization for the ellipse centered at $(-3, 4)$ with semimajor axis of length 5 parallel to the x-axis and semiminor axis of length 2 parallel to the y-axis.

In Exercises 45 and 46, a parametrization is given for a curve.

(a) Graph the curve. What are the initial and terminal points, if any? Indicate the direction in which the curve is traced.

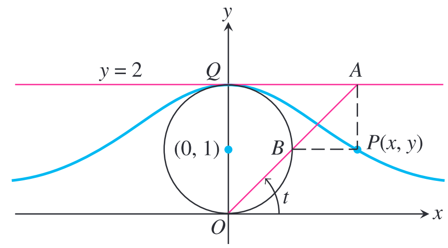
(b) Find a Cartesian equation for a curve that contains the parametrized curve. What portion of the graph of the Cartesian equation is traced by the parametrized curve?

45. $x = -\sec t, \quad y = \tan t, \quad -\pi/2 < t < \pi/2$

46. $x = \tan t, \quad y = -2\sec t, \quad -\pi/2 < t < \pi/2$

Extending the Ideas

47. ***The Witch of Agnesi*** The bell-shaped witch of Agnesi can be constructed as follows. Start with the circle of radius 1, centered at the point (0, 1) as shown in the figure.



Choose a point A on the line $y = 2$, and connect it to the origin with a line segment. Call the point where the segment crosses the circle B. Let P be the point where the vertical line through A crosses the horizontal line through B. The witch is the curve traced by P as A moves along the line $y = 2$.

Find a parametrization for the witch by expressing the coordinates of P in terms of t, the radian measure of the angle that segment OA makes with the positive x-axis. The following equalities (which you may assume) will help:

(i) $x = AQ$ **(ii)** $y = 2 - AB\sin t$ **(iii)** $AB \cdot AO = (AQ)^2$

48. ***Parametrizing Lines and Segments***

(a) Show that $x = x_1 + (x_2 - x_1)t, \quad y = y_1 + (y_2 - y_1)t,$ $-\infty < t < \infty$ is a parametrization for the line through the points (x_1, y_1) and (x_2, y_2).

(b) Find a parametrization for the line segment with endpoints (x_1, y_1) and (x_2, y_2).

1.5 Functions and Logarithms

What you'll learn about

- One-to-One Functions
- Inverses
- Finding Inverses
- Logarithmic Functions
- Properties of Logarithms
- Applications

. . . and why

Logarithmic functions are used in many applications, including finding time in investment problems.

One-to-One Functions

As you know, a function is a rule that assigns a single value in its range to each point in its domain. Some functions assign the same output to more than one input. For example, $f(x) = x^2$ assigns the output 4 to both 2 and -2. Other functions never output a given value more than once. For example, the cubes of different numbers are always different.

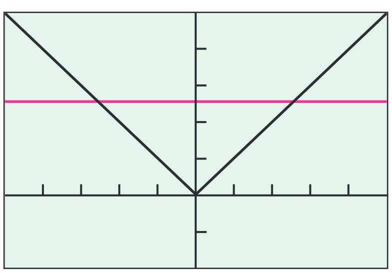
If each output value of a function is associated with exactly one input value, the function is *one-to-one.*

DEFINITION One-to-One Function

A function $f(x)$ is **one-to-one** on a domain D if $f(a) \neq f(b)$ whenever $a \neq b$.

The graph of a one-to-one function $y = f(x)$ can intersect any horizontal line at most once (the *horizontal line test*). If it intersects such a line more than once it assumes the same y-value more than once, and is therefore not one-to-one (Figure 1.33).

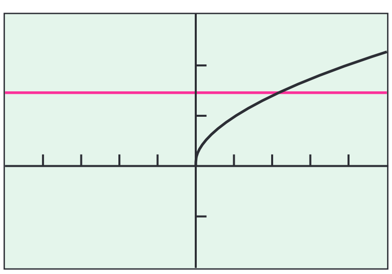
$y = |x|$



[–5, 5] by [–2, 5]

(a)

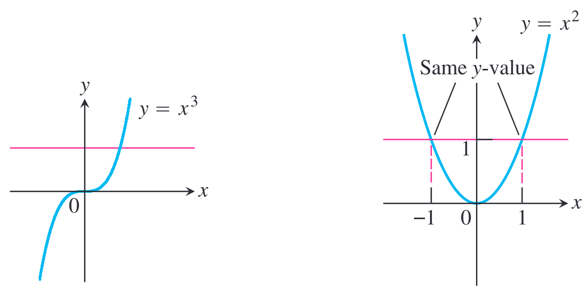
$y = \sqrt{x}$



[–5, 5] by [–2, 3]

(b)

Figure 1.34 (a) The graph of $f(x) = |x|$ and a horizontal line. (b) The graph of $g(x) = \sqrt{x}$ and a horizontal line. (Example 1)



One-to-one: Graph meets each horizontal line once.

Not one-to-one: Graph meets some horizontal lines more than once.

Figure 1.33 Using the horizontal line test, we see that $y = x^3$ is one-to-one and $y = x^2$ is not.

EXAMPLE 1 Using the Horizontal Line Test

Determine whether the functions are one-to-one.

(a) $f(x) = |x|$ **(b)** $g(x) = \sqrt{x}$

SOLUTION

(a) As Figure 1.34a suggests, each horizontal line $y = c$, $c > 0$, intersects the graph of $f(x) = |x|$ twice. So f is not one-to-one.

(b) As Figure 1.34b suggests, each horizontal line intersects the graph of $g(x) = \sqrt{x}$ either once or not at all. The function g is one-to-one.

Now try Exercise 1.

Inverses

Since each output of a one-to-one function comes from just one input, a one-to-one function can be reversed to send outputs back to the inputs from which they came. The function defined by reversing a one-to-one function f is the **inverse of f.** The functions in Tables 1.13 and 1.14 are inverses of one another. The symbol for the inverse of f is f^{-1}, read "f inverse." The -1 in f^{-1} is not an exponent; $f^{-1}(x)$ does not mean $1/f(x)$.

Table 1.13 Rental Charge versus Time

Time x (hours)	Charge y (dollars)
1	5.00
2	7.50
3	10.00
4	12.50
5	15.00
6	17.50

Table 1.14 Time versus Rental Charge

Charge x (dollars)	Time y (hours)
5.00	1
7.50	2
10.00	3
12.50	4
15.00	5
17.50	6

As Tables 1.13 and 1.14 suggest, composing a function with its inverse in either order sends each output back to the input from which it came. In other words, the result of composing a function and its inverse in either order is the **identity function,** the function that assigns each number to itself. This gives a way to test whether two functions f and g are inverses of one another. Compute $f \circ g$ and $g \circ f$. If $(f \circ g)(x) = (g \circ f)(x) = x$, then f and g are inverses of one another; otherwise they are not. The functions $f(x) = x^3$ and $g(x) = x^{1/3}$ are inverses of one another because $(x^3)^{1/3} = x$ and $(x^{1/3})^3 = x$ for every number x.

EXPLORATION 1 Testing for Inverses Graphically

For each of the function pairs below,

(a) Graph f and g together in a square window.

(b) Graph $f \circ g$. **(c)** Graph $g \circ f$.

What can you conclude from the graphs?

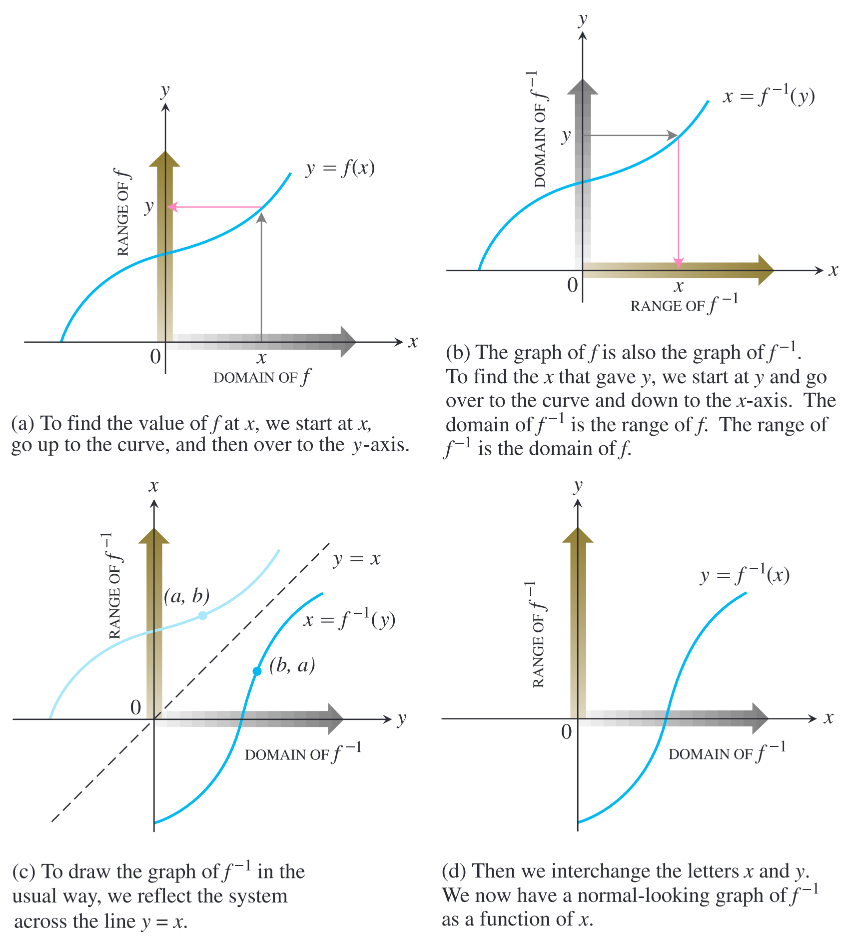
1. $f(x) = x^3, \quad g(x) = x^{1/3}$ **2.** $f(x) = x, \quad g(x) = 1/x$

3. $f(x) = 3x, \quad g(x) = x/3$ **4.** $f(x) = e^x, \quad g(x) = \ln x$

Finding Inverses

How do we find the graph of the inverse of a function? Suppose, for example, that the function is the one pictured in Figure 1.35a. To read the graph, we start at the point x on the x-axis, go up to the graph, and then move over to the y-axis to read the value of y. If we start with y and want to find the x from which it came, we reverse the process (Figure 1.35b).

The graph of f is already the graph of f^{-1}, although the latter graph is not drawn in the usual way with the domain axis horizontal and the range axis vertical. For f^{-1}, the input-output pairs are reversed. To display the graph of f^{-1} in the usual way, we have to reverse the pairs by reflecting the graph across the 45° line $y = x$ (Figure 1.35c) and interchanging the letters x and y (Figure 1.35d). This puts the independent variable, now called x, on the horizontal axis and the dependent variable, now called y, on the vertical axis.



(a) To find the value of f at x, we start at x, go up to the curve, and then over to the y-axis.

(b) The graph of f is also the graph of f^{-1}. To find the x that gave y, we start at y and go over to the curve and down to the x-axis. The domain of f^{-1} is the range of f. The range of f^{-1} is the domain of f.

(c) To draw the graph of f^{-1} in the usual way, we reflect the system across the line $y = x$.

(d) Then we interchange the letters x and y. We now have a normal-looking graph of f^{-1} as a function of x.

Figure 1.35 The graph of $y = f^{-1}(x)$.

The fact that the graphs of f and f^{-1} are reflections of each other across the line $y = x$ is to be expected because the input-output pairs (a, b) of f have been reversed to produce the input-output pairs (b, a) of f^{-1}.

The pictures in Figure 1.35 tell us how to express f^{-1} as a function of x algebraically.

Writing f^{-1} as a Function of x

1. Solve the equation $y = f(x)$ for x in terms of y.
2. Interchange x and y. The resulting formula will be $y = f^{-1}(x)$.

EXAMPLE 2 Finding the Inverse Function

Show that the function $y = f(x) = -2x + 4$ is one-to-one and find its inverse function.

SOLUTION

Every horizontal line intersects the graph of f exactly once, so f is one-to-one and has an inverse.

Step 1:

Solve for x in terms of y:
$$y = -2x + 4$$
$$x = -\frac{1}{2}y + 2$$

continued

Step 2:

Interchange x and y: $y = -\frac{1}{2}x + 2$

The inverse of the function $f(x) = -2x + 4$ is the function $f^{-1}(x) = -(1/2)x + 2$. We can verify that both composites are the identity function.

$$f^{-1}(f(x)) = -\frac{1}{2}(-2x + 4) + 2 = x - 2 + 2 = x$$

$$f(f^{-1}(x)) = -2\left(-\frac{1}{2}x + 2\right) + 4 = x - 4 + 4 = x$$

Now try Exercise 13.

Graphing $y = f(x)$ and $y = f^{-1}(x)$ Parametrically

We can graph any function $y = f(x)$ as

$$x_1 = t, \quad y_1 = f(t).$$

Interchanging t and $f(t)$ produces parametric equations for the inverse:

$$x_2 = f(t), \quad y_2 = t.$$

We can use parametric graphing to graph the inverse of a function without finding an explicit rule for the inverse, as illustrated in Example 3.

EXAMPLE 3 Graphing the Inverse Parametrically

(a) Graph the one-to-one function $f(x) = x^2$, $x \geq 0$, together with its inverse and the line $y = x$, $x \geq 0$.

(b) Express the inverse of f as a function of x.

SOLUTION

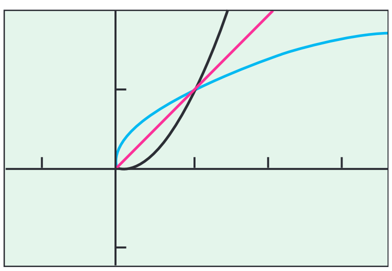
(a) We can graph the three functions parametrically as follows:

Graph of f: $x_1 = t, \quad y_1 = t^2, \quad t \geq 0$

Graph of f^{-1}: $x_2 = t^2, \quad y_2 = t$

Graph of $y = x$: $x_3 = t, \quad y_3 = t$

Figure 1.36 shows the three graphs.



[–1.5, 3.5] by [–1, 2]

Figure 1.36 The graphs of f and are reflections of each other across the line $y = x$. (Example 3)

(b) Next we find a formula for $f^{-1}(x)$.

Step 1:

Solve for x in terms of y.

$$y = x^2$$
$$\sqrt{y} = \sqrt{x^2}$$
$$\sqrt{y} = x$$

Step 2:

Interchange x and y.

$$\sqrt{x} = y$$

Thus, $f^{-1}(x) = \sqrt{x}$.

Now try Exercise 27.

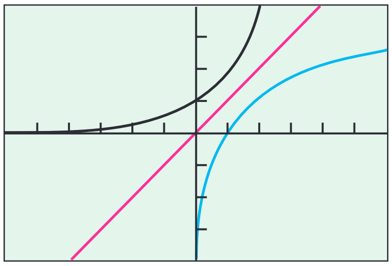
Logarithmic Functions

If a is any positive real number other than 1, the base a exponential function $f(x) = a^x$ is one-to-one. It therefore has an inverse. Its inverse is called the *base a logarithm function.*

DEFINITION Base *a* Logarithm Function

The **base *a* logarithm function** $y = \log_a x$ is the inverse of the base a exponential function $y = a^x$ $(a > 0, a \neq 1)$.

The domain of $\log_a x$ is $(0, \infty)$, the range of a^x. The range of $\log_a x$ is $(-\infty, \infty)$, the domain of a^x.



[–6, 6] by [–4, 4]

Figure 1.37 The graphs of $y = 2^x$ ($x_1 = t$, $y_1 = 2^t$), its inverse $y_2 = t$), and $y = x$ ($x_3 = t$, $y_3 = t$).

Because we have no technique for solving for x in terms of y in the equation $y = a^x$, we do not have an explicit formula for the logarithm function as a function of x. However, the graph of $y = \log_a x$ can be obtained by reflecting the graph of $y = a^x$ across the line $y = x$, or by using parametric graphing (Figure 1.37).

Logarithms with base e and base 10 are so important in applications that calculators have special keys for them. They also have their own special notation and names:

$$\log_e x = \ln x,$$

$$\log_{10} x = \log x.$$

The function $y = \ln x$ is called the **natural logarithm function** and $y = \log x$ is often called the **common logarithm function.**

Properties of Logarithms

Because a^x and $\log_a x$ are inverses of each other, composing them in either order gives the identity function. This gives two useful properties.

Inverse Properties for a^x and $\log_a x$

1. Base a: $a^{\log_a x} = x, \quad \log_a a^x = x, \quad a > 1, x > 0$
2. Base e: $e^{\ln x} = x, \quad \ln e^x = x, \quad x > 0$

These properties help us with the solution of equations that contain logarithms and exponential functions.

EXAMPLE 4 Using the Inverse Properties

Solve for x: **(a)** $\ln x = 3t + 5$ **(b)** $e^{2x} = 10$

SOLUTION

(a) $\ln x = 3t + 5$

$$e^{\ln x} = e^{3t+5}$$

$$x = e^{3t+5}$$

(b) $e^{2x} = 10$

$$\ln e^{2x} = \ln 10$$

$$2x = \ln 10$$

$$x = \frac{1}{2}\ln 10 \approx 1.15$$

Now try Exercises 33 and 37.

The logarithm function has the following useful arithmetic properties.

Properties of Logarithms

For any real numbers $x > 0$ and $y > 0$,

1. *Product Rule*: $\log_a xy = \log_a x + \log_a y$
2. *Quotient Rule*: $\log_a \frac{x}{y} = \log_a x - \log_a y$
3. *Power Rule*: $\log_a x^y = y \log_a x$

EXPLORATION 2 **Supporting the Product Rule**

Let $y_1 = \ln(ax)$, $y_2 = \ln x$, and $y_3 = y_1 - y_2$.

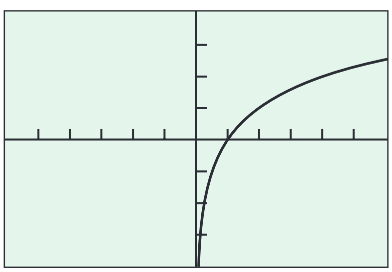
1. Graph y_1 and y_2 for $a = 2, 3, 4$, and 5. How do the graphs of y_1 and y_2 appear to be related?
2. Support your finding by graphing y_3.
3. Confirm your finding algebraically.

The following formula allows us to evaluate $\log_a x$ for any base $a > 0$, $a \neq 1$, and to obtain its graph using the natural logarithm function on our grapher.

Change of Base Formula

$$\log_a x = \frac{\ln x}{\ln a}$$

$y = \dfrac{\ln x}{\ln 2}$



[–6, 6] by [–4, 4]

Figure 1.38 The graph of $f(x) = \log_2 x$ using $f(x) = (\ln x)/(\ln 2)$. (Example 5)

EXAMPLE 5 **Graphing a Base *a* Logarithm Function**

Graph $f(x) = \log_2 x$.

SOLUTION

We use the change of base formula to rewrite $f(x)$.

$$f(x) = \log_2 x = \frac{\ln x}{\ln 2}.$$

Figure 1.38 gives the graph of f. ***Now try Exercise 41.***

Applications

In Section 1.3 we used graphical methods to solve exponential growth and decay problems. Now we can use the properties of logarithms to solve the same problems algebraically.

EXAMPLE 6 **Finding Time**

Sarah invests \$1000 in an account that earns 5.25% interest compounded annually. How long will it take the account to reach \$2500?

SOLUTION

Model The amount in the account at any time t in years is $1000(1.0525)^t$, so we need to solve the equation

$$1000(1.0525)^t = 2500.$$

continued

Table 1.15 Saudi Arabia's Natural Gas Production

Year	Cubic Feet (trillions)
1997	1.60
1998	1.65
1999	1.63
2000	1.76
2001	1.90

Source: Statistical Abstract of the United States, 2004–2005.

Solve Algebraically

$$(1.0525)^t = 2.5$$
$$\ln (1.0525)^t = \ln 2.5$$
$$t \ln 1.0525 = \ln 2.5$$
$$t = \frac{\ln 2.5}{\ln 1.0525} \approx 17.9$$

Interpret The amount in Sarah's account will be \$2500 in about 17.9 years, or about 17 years and 11 months. *Now try Exercise 47.*

EXAMPLE 7 Estimating Natural Gas Production

Table 1.15 shows the annual number of cubic feet in trillions of natural gas produced by Saudi Arabia for several years.

Find the natural logarithm regression equation for the data in Table 1.15 and use it to estimate the number of cubic feet of natural gas produced by Saudi Arabia in 2002. Compare with the actual amount of 2.00 trillion cubic feet in 2002.

SOLUTION

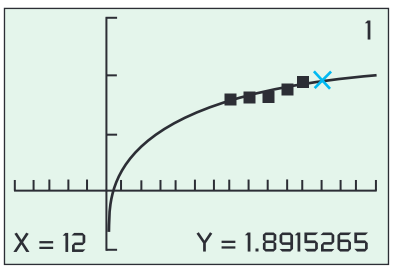
Model We let $x = 0$ represent 1990, $x = 1$ represent 1991, and so forth. We compute the natural logarithm regression equation to be

$$f(x) = 0.3730 + (0.611)\ln(x).$$

Solve Graphically Figure 1.39 shows the graph of f superimposed on the scatter plot of the data. The year 2002 is represented by $x = 12$. Reading from the graph we find $f(12) = 1.89$ trillion cubic feet.

Interpret The natural logarithmic model gives an underestimate of 0.11 trillion cubic feet of the 2002 natural gas production. *Now try Exercise 49.*

$f(x) = 0.3730 + (0.611) \ln x$



[–5, 15] by [–1, 3]

Figure 1.39 The value of f at $x = 12$ is about 1.89. (Example 7)

Quick Review 1.5 *(For help, go to Sections 1.2, 1.3, and 1.4.)*

In Exercises 1–4, let $f(x) = \sqrt[3]{x - 1}$, $g(x) = x^2 + 1$, and evaluate the expression.

1. $(f \circ g)(1)$
2. $(g \circ f)(-7)$
3. $(f \circ g)(x)$
4. $(g \circ f)(x)$

In Exercises 5 and 6, choose parametric equations and a parameter interval to represent the function on the interval specified.

5. $y = \dfrac{1}{x - 1}, \quad x \geq 2$
6. $y = x, \quad x < -3$

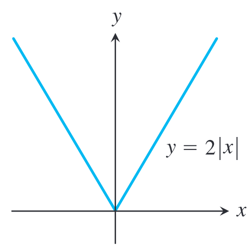
In Exercises 7–10, find the points of intersection of the two curves. Round your answers to 2 decimal places.

7. $y = 2x - 3, \quad y = 5$
8. $y = -3x + 5, \quad y = -3$
9. **(a)** $y = 2^x, \quad y = 3$
   (b) $y = 2^x, \quad y = -1$
10. **(a)** $y = e^{-x}, \quad y = 4$
    (b) $y = e^{-x}, \quad y = -1$

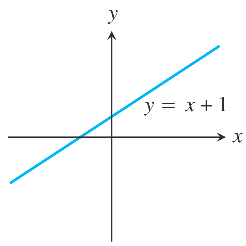
Section 1.5 Exercises

In Exercises 1–6, determine whether the function is one-to-one.

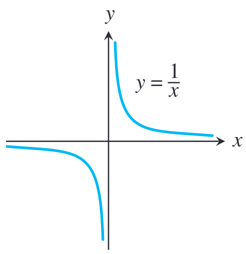
1.



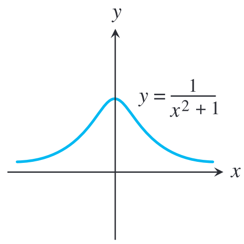
2.



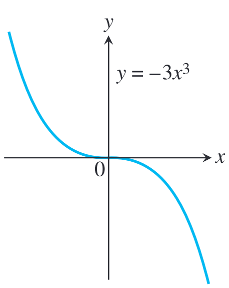
3.



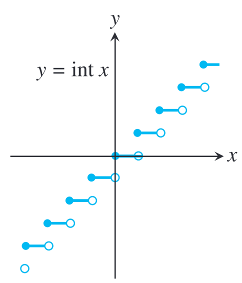
4.



5.



6.



In Exercises 7–12, determine whether the function has an inverse function.

7. $y = \dfrac{3}{x-2} - 1$ **8.** $y = x^2 + 5x$ **9.** $y = x^3 - 4x + 6$

10. $y = x^3 + x$ **11.** $y = \ln x^2$ **12.** $y = 2^{3-x}$

In Exercises 13–24, find f^{-1} and verify that

$$(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x.$$

13. $f(x) = 2x + 3$

14. $f(x) = 5 - 4x$

15. $f(x) = x^3 - 1$

16. $f(x) = x^2 + 1, \quad x \geq 0$

17. $f(x) = x^2, \quad x \leq 0$

18. $f(x) = x^{2/3}, \quad x \geq 0$

19. $f(x) = -(x-2)^2, \quad x \leq 2$

20. $f(x) = x^2 + 2x + 1, \quad x \geq -1$

21. $f(x) = \dfrac{1}{x^2}, \quad x > 0$

22. $f(x) = \dfrac{1}{x^3}$

23. $f(x) = \dfrac{2x+1}{x+3}$

24. $f(x) = \dfrac{x+3}{x-2}$

In Exercises 25–32, use parametric graphing to graph f, f^{-1}, and $y = x$.

25. $f(x) = e^x$ **26.** $f(x) = 3^x$ **27.** $f(x) = 2^{-x}$

28. $f(x) = 3^{-x}$ **29.** $f(x) = \ln x$ **30.** $f(x) = \log x$

31. $f(x) = \sin^{-1} x$ **32.** $f(x) = \tan^{-1} x$

In Exercises 33–36, solve the equation algebraically. Support your solution graphically.

33. $(1.045)^t = 2$

34. $e^{0.05t} = 3$

35. $e^x + e^{-x} = 3$

36. $2^x + 2^{-x} = 5$

In Exercises 37 and 38, solve for y.

37. $\ln y = 2t + 4$

38. $\ln(y-1) - \ln 2 = x + \ln x$

In Exercises 39–42, draw the graph and determine the domain and range of the function.

39. $y = 2\ln(3-x) - 4$

40. $y = -3\log(x+2) + 1$

41. $y = \log_2(x+1)$

42. $y = \log_3(x-4)$

In Exercises 43 and 44, find a formula for f^{-1} and verify that $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$.

43. $f(x) = \dfrac{100}{1 + 2^{-x}}$

44. $f(x) = \dfrac{50}{1 + 1.1^{-x}}$

45. ***Self-inverse*** Prove that the function f is its own inverse.

(a) $f(x) = \sqrt{1 - x^2}, \quad x \geq 0$ **(b)** $f(x) = 1/x$

46. ***Radioactive Decay*** The half-life of a certain radioactive substance is 12 hours. There are 8 grams present initially.

(a) Express the amount of substance remaining as a function of time t.

(b) When will there be 1 gram remaining?

47. ***Doubling Your Money*** Determine how much time is required for a \$500 investment to double in value if interest is earned at the rate of 4.75% compounded annually.

48. ***Population Growth*** The population of Glenbrook is 375,000 and is increasing at the rate of 2.25% per year. Predict when the population will be 1 million.

In Exercises 49 and 50, let $x = 0$ represent 1990, $x = 1$ represent 1991, and so forth.

49. ***Natural Gas Production***

(a) Find a natural logarithm regression equation for the data in Table 1.16 and superimpose its graph on a scatter plot of the data.

Table 1.16 Canada's Natural Gas Production

Year	Cubic Feet (trillions)
1997	5.76
1998	5.98
1999	6.26
2000	6.47
2001	6.60

Source: Statistical Abstract of the United States, 2004–2005.

(b) Estimate the number of cubic feet of natural gas produced by Canada in 2002. Compare with the actual amount of 6.63 trillion cubic feet in 2002.

(c) Predict when Canadian natural gas production will reach 7 trillion cubic feet.

50. ***Natural Gas Production***

(a) Find a natural logarithm regression equation for the data in Table 1.17 and superimpose its graph on a scatter plot of the data.

Table 1.17 China's Natural Gas Production

Year	Cubic Feet (trillions)
1997	0.75
1998	0.78
1999	0.85
2000	0.96
2001	1.07

Source: Statistical Abstract of the United States, 2004–2005.

(b) Estimate the number of cubic feet of natural gas produced by China in 2002. Compare with the actual amount of 1.15 trillion cubic feet in 2002.

(c) Predict when China's natural gas production will reach 1.5 trillion cubic feet.

51. **Group Activity** ***Inverse Functions*** Let $y = f(x) = mx + b$, $m \neq 0$.

(a) **Writing to Learn** Give a convincing argument that f is a one-to-one function.

(b) Find a formula for the inverse of f. How are the slopes of f and f^{-1} related?

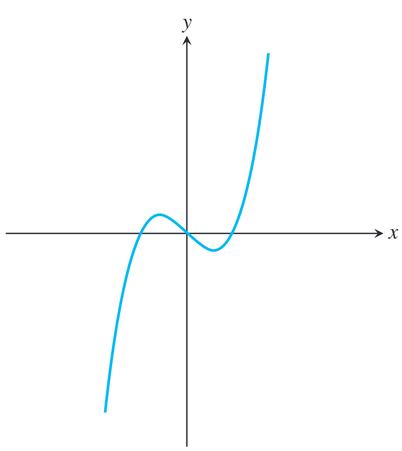
(c) If the graphs of two functions are parallel lines with a nonzero slope, what can you say about the graphs of the inverses of the functions?

(d) If the graphs of two functions are perpendicular lines with a nonzero slope, what can you say about the graphs of the inverses of the functions?

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

52. **True or False** The function displayed in the graph below is one-to-one. Justify your answer.



53. **True or False** If $(f \circ g)(x) = x$, then g is the inverse function of f. Justify your answer.

In Exercises 54 and 55, use the function $f(x) = 3 - \ln(x + 2)$.

54. **Multiple Choice** Which of the following is the domain of f?

(A) $x \neq -2$ **(B)** $(-\infty, \infty)$ **(C)** $(-2, \infty)$

(D) $[-1.9, \infty)$ **(E)** $(0, \infty)$

55. **Multiple Choice** Which of the following is the range of f?

(A) $(-\infty, \infty)$ **(B)** $(-\infty, 0)$ **(C)** $(-2, \infty)$

(D) $(0, \infty)$ **(E)** $(0, 5.3)$

56. **Multiple Choice** Which of the following is the inverse of $f(x) = 3x - 2$?

(A) $g(x) = \dfrac{1}{3x - 2}$ **(B)** $g(x) = x$ **(C)** $g(x) = 3x - 2$

(D) $g(x) = \dfrac{x - 2}{3}$ **(E)** $g(x) = \dfrac{x + 2}{3}$

57. **Multiple Choice** Which of the following is a solution of the equation $2 - 3^{-x} = -1$?

(A) $x = -2$ **(B)** $x = -1$ **(C)** $x = 0$

(D) $x = 1$ **(E)** There are no solutions.

Exploration

58. ***Supporting the Quotient Rule*** Let $y_1 = \ln(x/a)$, $y_2 = \ln x$, $y_3 = y_2 - y_1$, and $y_4 = e^{y_3}$.

(a) Graph y_1 and y_2 for $a = 2, 3, 4$, and 5. How are the graphs of y_1 and y_2 related?

(b) Graph y_3 for $a = 2, 3, 4$, and 5. Describe the graphs.

(c) Graph y_4 for $a = 2, 3, 4$, and 5. Compare the graphs to the graph of $y = a$.

(d) Use $e^{y_3} = e^{y_2 - y_1} = a$ to solve for y_1.

Extending the Ideas

59. ***One-to-One Functions*** If f is a one-to-one function, prove that $g(x) = -f(x)$ is also one-to-one.

60. ***One-to-One Functions*** If f is a one-to-one function and $f(x)$ is never zero, prove that $g(x) = 1/f(x)$ is also one-to-one.

61. ***Domain and Range*** Suppose that $a \neq 0$, $b \neq 1$, and $b > 0$. Determine the domain and range of the function.

(a) $y = a(b^{c-x}) + d$ **(b)** $y = a \log_b(x - c) + d$

62. **Group Activity** ***Inverse Functions***

Let $f(x) = \dfrac{ax + b}{cx + d}$, $c \neq 0$, $ad - bc \neq 0$.

(a) **Writing to Learn** Give a convincing argument that f is one-to-one.

(b) Find a formula for the inverse of f.

(c) Find the horizontal and vertical asymptotes of f.

(d) Find the horizontal and vertical asymptotes of f^{-1}. How are they related to those of f?

1.6 Trigonometric Functions

What you'll learn about

- Radian Measure
- Graphs of Trigonometric Functions
- Periodicity
- Even and Odd Trigonometric Functions
- Transformations of Trigonometric Graphs
- Inverse Trigonometric Functions

. . . and why

Trigonometric functions can be used to model periodic behavior and applications such as musical notes.

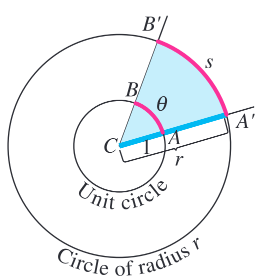


Figure 1.40 The radian measure of angle ACB is the length θ of arc AB on the unit circle centered at C. The value of θ can be found from any other circle, however, as the ratio s/r.

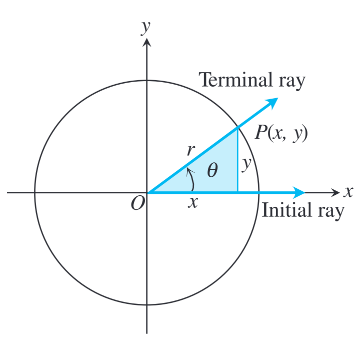


Figure 1.41 An angle θ in standard position.

Radian Measure

The **radian measure** of the angle ACB at the center of the unit circle (Figure 1.40) equals the length of the arc that ACB cuts from the unit circle.

EXAMPLE 1 Finding Arc Length

Find the length of an arc subtended on a circle of radius 3 by a central angle of measure $2\pi/3$.

SOLUTION

According to Figure 1.40, if s is the length of the arc, then

$$s = r\theta = 3(2\pi/3) = 2\pi.$$

Now try Exercise 1.

When an angle of measure θ is placed in *standard position* at the center of a circle of radius r (Figure 1.41), the six basic trigonometric functions of θ are defined as follows:

$$\textbf{sine: } \sin\theta = \frac{y}{r} \qquad \textbf{cosecant: } \csc\theta = \frac{r}{y}$$

$$\textbf{cosine: } \cos\theta = \frac{x}{r} \qquad \textbf{secant: } \sec\theta = \frac{r}{x}$$

$$\textbf{tangent: } \tan\theta = \frac{y}{x} \qquad \textbf{cotangent: } \cot\theta = \frac{x}{y}$$

Graphs of Trigonometric Functions

When we graph trigonometric functions in the coordinate plane, we usually denote the independent variable (radians) by x instead of θ. Figure 1.42 on the next page shows sketches of the six trigonometric functions. It is a good exercise for you to compare these with what you see in a grapher viewing window. (Some graphers have a "trig viewing window.")

EXPLORATION 1 Unwrapping Trigonometric Functions

Set your grapher in *radian mode,* parametric mode, and *simultaneous mode* (all three). Enter the parametric equations

$$x_1 = \cos t, \quad y_1 = \sin t \quad \text{and} \quad x_2 = t, \quad y_2 = \sin t.$$

1. Graph for $0 \le t \le 2\pi$ in the window $[-1.5, 2\pi]$ by $[-2.5, 2.5]$. Describe the two curves. (You may wish to make the viewing window square.)
2. Use trace to compare the y-values of the two curves.
3. Repeat part 2 in the window $[-1.5, 4\pi]$ by $[-5, 5]$, using the parameter interval $0 \le t \le 4\pi$.
4. Let $y_2 = \cos t$. Use trace to compare the x-values of curve 1 (the unit circle) with the y-values of curve 2 using the parameter intervals $[0, 2\pi]$ and $[0, 4\pi]$.
5. Set $y_2 = \tan t$, $\csc t$, $\sec t$, and $\cot t$. Graph each in the window $[-1.5, 2\pi]$ by $[-2.5, 2.5]$ using the interval $0 \le t \le 2\pi$. How is a y-value of curve 2 related to the corresponding point on curve 1? (Use trace to explore the curves.)

Angle Convention: Use Radians

From now on in this book it is assumed that all angles are measured in radians unless degrees or some other unit is stated explicitly. When we talk about the angle $\pi/3$, we mean $\pi/3$ radians (which is 60°), not $\pi/3$ degrees. When you do calculus, keep your calculator in radian mode.

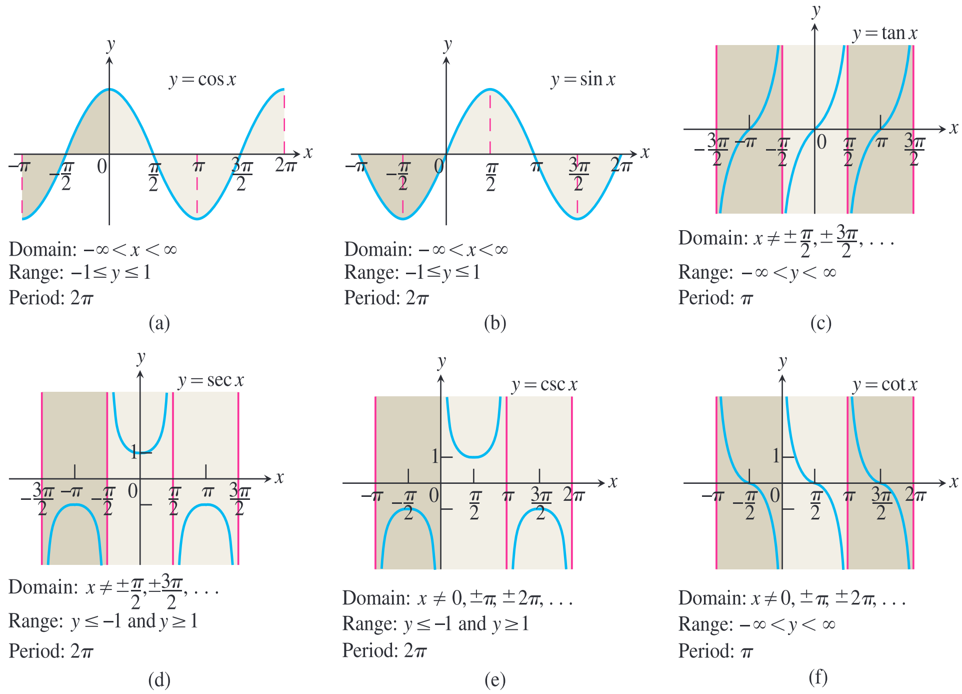


Figure 1.42 Graphs of the (a) cosine, (b) sine, (c) tangent, (d) secant, (e) cosecant, and (f) cotangent functions using radian measure.

Periods of Trigonometric Functions

Period π: $\tan(x + \pi) = \tan x$
$\cot(x + \pi) = \cot x$

Period 2π: $\sin(x + 2\pi) = \sin x$
$\cos(x + 2\pi) = \cos x$
$\sec(x + 2\pi) = \sec x$
$\csc(x + 2\pi) = \csc x$

Periodicity

When an angle of measure θ and an angle of measure $\theta + 2\pi$ are in standard position, their terminal rays coincide. The two angles therefore have the same trigonometric function values:

$$\begin{aligned} \cos(\theta + 2\pi) &= \cos\theta \quad & \sin(\theta + 2\pi) &= \sin\theta \quad & \tan(\theta + 2\pi) &= \tan\theta \\ \sec(\theta + 2\pi) &= \sec\theta \quad & \csc(\theta + 2\pi) &= \csc\theta \quad & \cot(\theta + 2\pi) &= \cot\theta \end{aligned} \tag{1}$$

Similarly, $\cos(\theta - 2\pi) = \cos\theta$, $\sin(\theta - 2\pi) = \sin\theta$, and so on.

We see the values of the trigonometric functions repeat at regular intervals. We describe this behavior by saying that the six basic trigonometric functions are *periodic*.

DEFINITION Periodic Function, Period

A function $f(x)$ is **periodic** if there is a positive number p such that $f(x + p) = f(x)$ for every value of x. The smallest such value of p is the **period** of f.

As we can see in Figure 1.42, the functions $\cos x$, $\sin x$, $\sec x$, and $\csc x$ are periodic with period 2π. The functions $\tan x$ and $\cot x$ are periodic with period π.

Even and Odd Trigonometric Functions

The graphs in Figure 1.42 suggest that $\cos x$ and $\sec x$ are even functions because their graphs are symmetric about the y-axis. The other four basic trigonometric functions are odd.

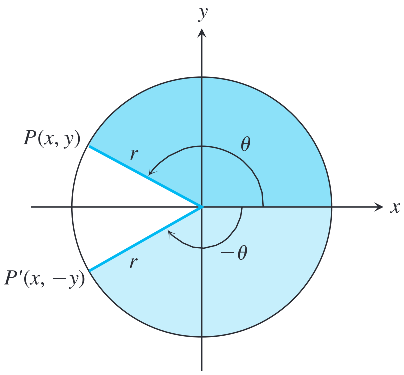


Figure 1.43 Angles of opposite sign. (Example 2)

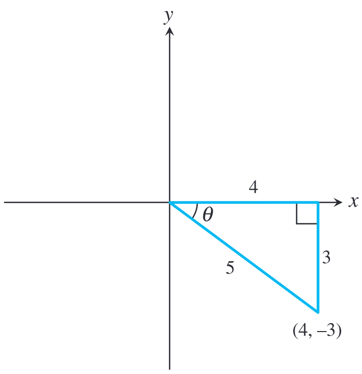


Figure 1.44 The angle θ in standard position. (Example 3)

EXAMPLE 2 Confirming Even and Odd

Show that cosine is an even function and sine is odd.

SOLUTION

From Figure 1.43 it follows that

$$\cos(-\theta) = \frac{x}{r} = \cos\theta, \quad \sin(-\theta) = \frac{-y}{r} = -\sin\theta,$$

so cosine is an even function and sine is odd. ***Now try Exercise 5.***

EXAMPLE 3 Finding Trigonometric Values

Find all the trigonometric values of θ if $\sin\theta = -3/5$ and $\tan\theta < 0$.

SOLUTION

The angle θ is in the fourth quadrant, as shown in Figure 1.44, because its sine and tangent are negative. From this figure we can read that $\cos\theta = 4/5$, $\tan\theta = -3/4$, $\csc\theta = -5/3$, $\sec\theta = 5/4$, and $\cot\theta = -4/3$. ***Now try Exercise 9.***

Transformations of Trigonometric Graphs

The rules for shifting, stretching, shrinking, and reflecting the graph of a function apply to the trigonometric functions. The following diagram will remind you of the controlling parameters.

Vertical stretch or shrink; reflection about x-axis — Vertical shift

$$y = af(b(x + c)) + d$$

Horizontal stretch or shrink; reflection about y-axis — Horizontal shift

The general sine function or **sinusoid** can be written in the form

$$f(x) = A\sin\left[\frac{2\pi}{B}(x - C)\right] + D,$$

where $|A|$ is the *amplitude,* $|B|$ is the *period,* C is the *horizontal shift,* and D is the *vertical shift*.

EXAMPLE 4 Graphing a Trigonometric Function

Determine the **(a)** period, **(b)** domain, **(c)** range, and **(d)** draw the graph of the function $y = 3\cos(2x - \pi) + 1$.

SOLUTION

We can rewrite the function in the form

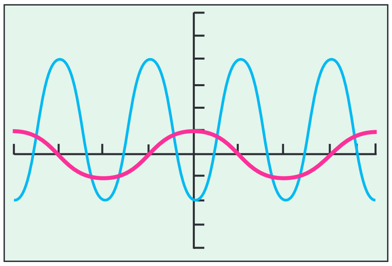
$$y = 3\cos\left[2\left(x - \frac{\pi}{2}\right)\right] + 1.$$

(a) The period is given by $2\pi/B$, where $2\pi/B = 2$. The period is π.

(b) The domain is $(-\infty, \infty)$.

(c) The graph is a basic cosine curve with amplitude 3 that has been shifted up 1 unit. Thus, the range is $[-2, 4]$.

continued



$[-2\pi, 2\pi]$ by $[-4, 6]$

Figure 1.45 The graph of $y = 3\cos(2x - \pi) + 1$ (blue) and the graph of $y = \cos x$ (red). (Example 4)

(d) The graph has been shifted to the right $\pi/2$ units. The graph is shown in Figure 1.45 together with the graph of $y = \cos x$. Notice that four periods of $y = 3\cos(2x - \pi) + 1$ are drawn in this window. ***Now try Exercise 13.***

Musical notes are pressure waves in the air. The wave behavior can be modeled with great accuracy by general sine curves. Devices called Calculator Based Laboratory™ (CBL) systems can record these waves with the aid of a microphone. The data in Table 1.18 give pressure displacement versus time in seconds of a musical note produced by a tuning fork and recorded with a CBL system.

Table 1.18 Tuning Fork Data

Time	Pressure	Time	Pressure	Time	Pressure
0.00091	−0.080	0.00271	−0.141	0.00453	0.749
0.00108	0.200	0.00289	−0.309	0.00471	0.581
0.00125	0.480	0.00307	−0.348	0.00489	0.346
0.00144	0.693	0.00325	−0.248	0.00507	0.077
0.00162	0.816	0.00344	−0.041	0.00525	−0.164
0.00180	0.844	0.00362	0.217	0.00543	−0.320
0.00198	0.771	0.00379	0.480	0.00562	−0.354
0.00216	0.603	0.00398	0.681	0.00579	−0.248
0.00234	0.368	0.00416	0.810	0.00598	−0.035
0.00253	0.099	0.00435	0.827		

EXAMPLE 5 Finding the Frequency of a Musical Note

Consider the tuning fork data in Table 1.18.

(a) Find a sinusoidal regression equation (general sine curve) for the data and superimpose its graph on a scatter plot of the data.

(b) The *frequency* of a musical note, or wave, is measured in cycles per second, or hertz (1 Hz = 1 cycle per second). The frequency is the reciprocal of the *period* of the wave, which is measured in seconds per cycle. Estimate the frequency of the note produced by the tuning fork.

SOLUTION

(a) The sinusoidal regression equation produced by our calculator is approximately

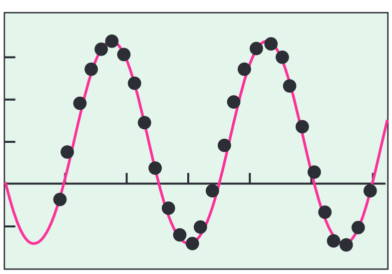
$$y = 0.6\sin(2488.6x - 2.832) + 0.266.$$

Figure 1.46 shows its graph together with a scatter plot of the tuning fork data.

(b) The period is $\dfrac{2\pi}{2488.6}$ sec, so the frequency is $\dfrac{2488.6}{2\pi} \approx 396$ Hz.

Interpretation The tuning fork is vibrating at a frequency of about 396 Hz. On the pure tone scale, this is the note G above middle C. It is a few cycles per second different from the frequency of the G we hear on a piano's tempered scale, 392 Hz.

Now try Exercise 23.



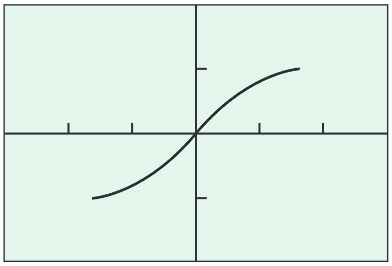
$[0, 0.0062]$ by $[-0.5, 1]$

Figure 1.46 A sinusoidal regression model for the tuning fork data in Table 1.18. (Example 5)

Inverse Trigonometric Functions

None of the six basic trigonometric functions graphed in Figure 1.42 is one-to-one. These functions do not have inverses. However, in each case the domain can be restricted to produce a new function that does have an inverse, as illustrated in Example 6.

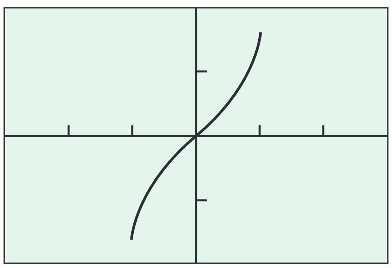
$x = t,\ y = \sin t,\ -\frac{\pi}{2} \le t \le \frac{\pi}{2}$



[–3, 3] by [–2, 2]

(a)

$x = \sin t,\ y = t,\ -\frac{\pi}{2} \le t \le \frac{\pi}{2}$



[–3, 3] by [–2, 2]

(b)

Figure 1.47 (a) A restricted sine function and (b) its inverse. (Example 6)

EXAMPLE 6 Restricting the Domain of the Sine

Show that the function $y = \sin x,\ -\pi/2 \le x \le \pi/2$, is one-to-one, and graph its inverse.

SOLUTION

Figure 1.47a shows the graph of this restricted sine function using the parametric equations

$$x_1 = t, \quad y_1 = \sin t, \quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}.$$

This restricted sine function is one-to-one because it does not repeat any output values. It therefore has an inverse, which we graph in Figure 1.47b by interchanging the ordered pairs using the parametric equations

$$x_2 = \sin t, \quad y_2 = t, \quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}.$$

Now try Exercise 25.

The inverse of the restricted sine function of Example 6 is called the *inverse sine function.* The inverse sine of x is the angle whose sine is x. It is denoted by $\sin^{-1} x$ or $\arcsin x$. Either notation is read "arcsine of x" or "the inverse sine of x."

The domains of the other basic trigonometric functions can also be restricted to produce a function with an inverse. The domains and ranges of the resulting inverse functions become parts of their definitions.

DEFINITIONS Inverse Trigonometric Functions

Function	Domain	Range
$y = \cos^{-1} x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \sin^{-1} x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \tan^{-1} x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$y = \sec^{-1} x$	$\lvert x \rvert \ge 1$	$0 \le y \le \pi,\ y \ne \frac{\pi}{2}$
$y = \csc^{-1} x$	$\lvert x \rvert \ge 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2},\ y \ne 0$
$y = \cot^{-1} x$	$-\infty < x < \infty$	$0 < y < \pi$

The graphs of the six inverse trigonometric functions are shown in Figure 1.48.

EXAMPLE 7 Finding Angles in Degrees and Radians

Find the measure of $\cos^{-1}(-0.5)$ in degrees and radians.

SOLUTION

Put the calculator in degree mode and enter $\cos^{-1}(-0.5)$. The calculator returns 120, which means 120 degrees. Now put the calculator in radian mode and enter $\cos^{-1}(-0.5)$. The calculator returns 2.094395102, which is the measure of the angle in radians. You can check that $2\pi/3 \approx 2.094395102$.

Now try Exercise 27.

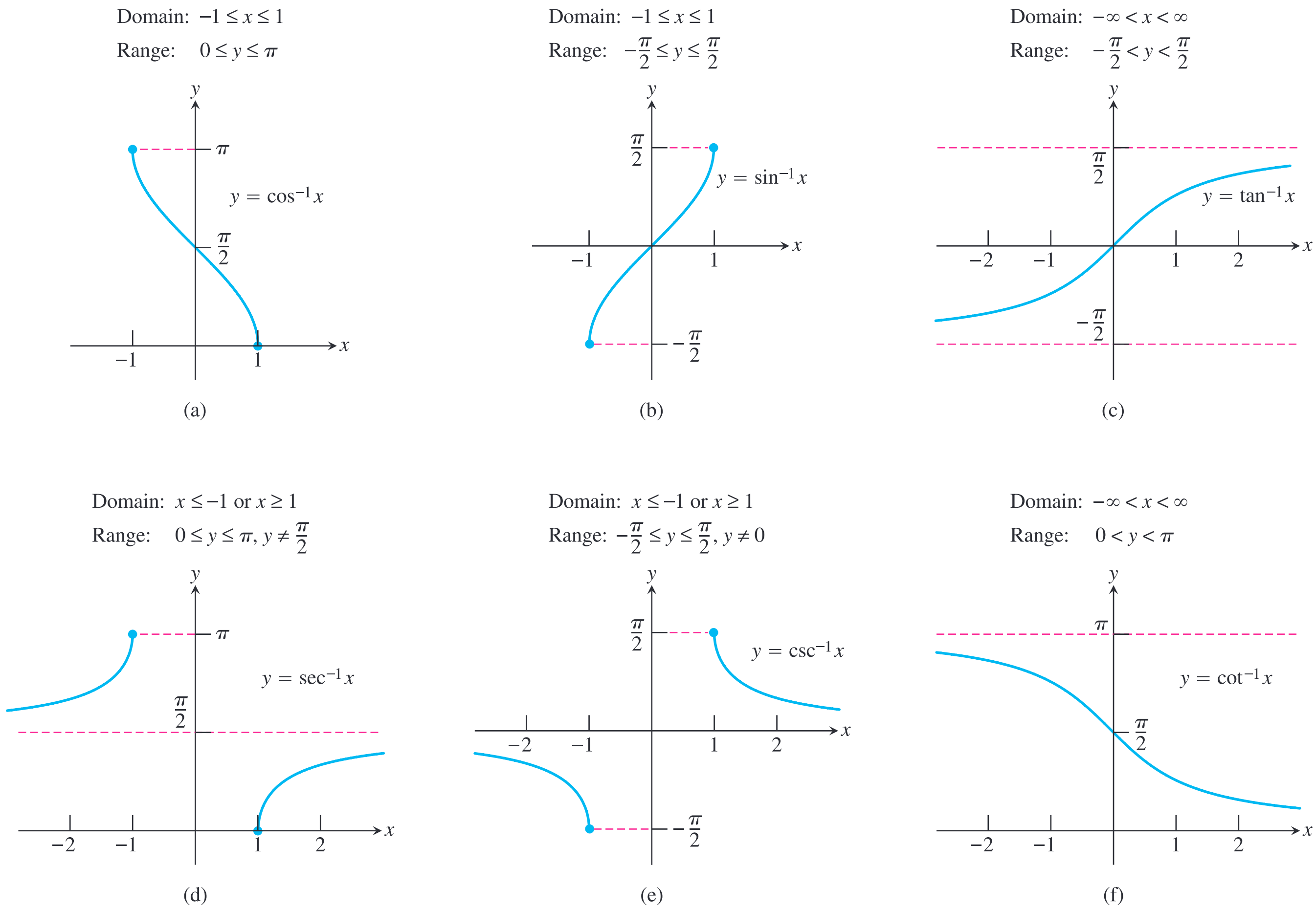


Figure 1.48 Graphs of (a) $y = \cos^{-1} x$, (b) $y = \sin^{-1} x$, (c) $y = \tan^{-1} x$, (d) $y = \sec^{-1} x$, (e) $y = \csc^{-1} x$, and (f) $y = \cot^{-1} x$.

EXAMPLE 8 Using the Inverse Trigonometric Functions

Solve for x.

(a) $\sin x = 0.7$ in $0 \le x < 2\pi$

(b) $\tan x = -2$ in $-\infty < x < \infty$

SOLUTION

(a) Notice that $x = \sin^{-1}(0.7) \approx 0.775$ is in the first quadrant, so 0.775 is one solution of this equation. The angle $\pi - x$ is in the second quadrant and has sine equal to 0.7. Thus two solutions in this interval are

$$\sin^{-1}(0.7) \approx 0.775 \quad \text{and} \quad \pi - \sin^{-1}(0.7) \approx 2.366.$$

(b) The angle $x = \tan^{-1}(-2) \approx -1.107$ is in the fourth quadrant and is the only solution to this equation in the interval $-\pi/2 < x < \pi/2$ where $\tan x$ is one-to-one. Since $\tan x$ is periodic with period π, the solutions to this equation are of the form

$$\tan^{-1}(-2) + k\pi \approx -1.107 + k\pi$$

where k is any integer. *Now try Exercise 31.*

Quick Review 1.6 *(For help, go to Sections 1.2 and 1.6.)*

In Exercises 1–4, convert from radians to degrees or degrees to radians.

1. $\pi/3$ **2.** -2.5 **3.** $-40°$ **4.** $45°$

In Exercises 5–7, solve the equation graphically in the given interval.

5. $\sin x = 0.6, \quad 0 \le x \le 2\pi$ **6.** $\cos x = -0.4, \quad 0 \le x \le 2\pi$

7. $\tan x = 1, \quad -\frac{\pi}{2} \le x < \frac{3\pi}{2}$

8. Show that $f(x) = 2x^2 - 3$ is an even function. Explain why its graph is symmetric about the y-axis.

9. Show that $f(x) = x^3 - 3x$ is an odd function. Explain why its graph is symmetric about the origin.

10. Give one way to restrict the domain of the function $f(x) = x^4 - 2$ to make the resulting function one-to-one.

Section 1.6 Exercises

In Exercises 1–4, the angle lies at the center of a circle and subtends an arc of the circle. Find the missing angle measure, circle radius, or arc length.

Angle	Radius	Arc Length
1. $5\pi/8$	2	?
2. 175°	?	10
3. ?	14	7
4. ?	6	$3\pi/2$

In Exercises 5–8, determine if the function is even or odd.

5. secant **6.** tangent

7. cosecant **8.** cotangent

In Exercises 9 and 10, find all the trigonometric values of θ with the given conditions.

9. $\cos\theta = -\frac{15}{17}, \quad \sin\theta > 0$

10. $\tan\theta = -1, \quad \sin\theta < 0$

In Exercises 11–14, determine **(a)** the period, **(b)** the domain, **(c)** the range, and **(d)** draw the graph of the function.

11. $y = 3\csc(3x + \pi) - 2$ **12.** $y = 2\sin(4x + \pi) + 3$

13. $y = -3\tan(3x + \pi) + 2$

14. $y = 2\sin\left(2x + \frac{\pi}{3}\right)$

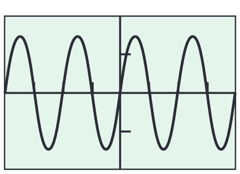
In Exercises 15 and 16, choose an appropriate viewing window to display two complete periods of each trigonometric function in radian mode.

15. **(a)** $y = \sec x$ **(b)** $y = \csc x$ **(c)** $y = \cot x$

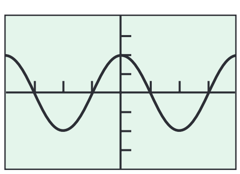
16. **(a)** $y = \sin x$ **(b)** $y = \cos x$ **(c)** $y = \tan x$

In Exercises 17–22, specify **(a)** the period, **(b)** the amplitude, and **(c)** identify the viewing window that is shown.

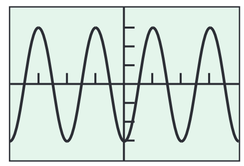
17. $y = 1.5\sin 2x$



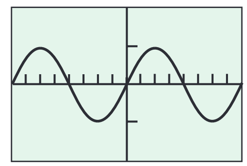
18. $y = 2\cos 3x$



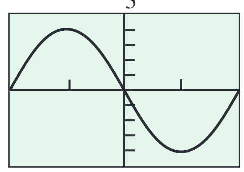
19. $y = -3\cos 2x$



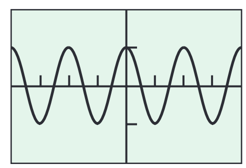
20. $y = 5\sin\frac{x}{2}$



21. $y = -4\sin\frac{\pi}{3}x$



22. $y = \cos\pi x$



23. Group Activity A musical note like that produced with a tuning fork or pitch meter is a pressure wave. Table 1.19 gives frequencies (in Hz) of musical notes on the tempered scale. The pressure versus time tuning fork data in Table 1.20 were collected using a CBL™ and a microphone.

Table 1.19 Frequencies of Notes

Note	Frequency (Hz)
C	262
C♯ or D♭	277
D	294
D♯ or E♭	311
E	330
F	349
F♯ or G♭	370
G	392
G♯ or A♭	415
A	440
A♯ or B♭	466
B	494
C (next octave)	524

Source: CBL™ System Experimental Workbook, Texas Instruments, Inc., 1994.

Table 1.20 Tuning Fork Data

Time (s)	Pressure	Time (s)	Pressure
0.0002368	1.29021	0.0049024	−1.06632
0.0005664	1.50851	0.0051520	0.09235
0.0008256	1.51971	0.0054112	1.44694
0.0010752	1.51411	0.0056608	1.51411
0.0013344	1.47493	0.0059200	1.51971
0.0015840	0.45619	0.0061696	1.51411
0.0018432	−0.89280	0.0064288	1.43015
0.0020928	−1.51412	0.0066784	0.19871
0.0023520	−1.15588	0.0069408	−1.06072
0.0026016	−0.04758	0.0071904	−1.51412
0.0028640	1.36858	0.0074496	−0.97116
0.0031136	1.50851	0.0076992	0.23229
0.0033728	1.51971	0.0079584	1.46933
0.0036224	1.51411	0.0082080	1.51411
0.0038816	1.45813	0.0084672	1.51971
0.0041312	0.32185	0.0087168	1.50851
0.0043904	−0.97676	0.0089792	1.36298
0.0046400	−1.51971		

(a) Find a sinusoidal regression equation for the data in Table 1.20 and superimpose its graph on a scatter plot of the data.

(b) Determine the frequency of and identify the musical note produced by the tuning fork.

24. ***Temperature Data*** Table 1.21 gives the average monthly temperatures for St. Louis for a 12-month period starting with January. Model the monthly temperature with an equation of the form

$$y = a \sin [b(t - h)] + k,$$

y in degrees Fahrenheit, t in months, as follows:

Table 1.21 Temperature Data for St. Louis

Time (months)	Temperature (ºF)
1	34
2	30
3	39
4	44
5	58
6	67
7	78
8	80
9	72
10	63
11	51
12	40

(a) Find the value of b assuming that the period is 12 months.

(b) How is the amplitude a related to the difference 80º − 30º?

(c) Use the information in (b) to find k.

(d) Find h, and write an equation for y.

(e) Superimpose a graph of y on a scatter plot of the data.

In Exercises 25–26, show that the function is one-to-one, and graph its inverse.

25. $y = \cos x, \quad 0 \le x \le \pi$ **26.** $y = \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

In Exercises 27–30, give the measure of the angle in radians and degrees. Give exact answers whenever possible.

27. $\sin^{-1}(0.5)$ **28.** $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

29. $\tan^{-1}(-5)$ **30.** $\cos^{-1}(0.7)$

In Exercises 31–36, solve the equation in the specified interval.

31. $\tan x = 2.5, \quad 0 \le x \le 2\pi$

32. $\cos x = -0.7, \quad 2\pi \le x < 4\pi$

33. $\csc x = 2, \quad 0 < x < 2\pi$ **34.** $\sec x = -3, \quad -\pi \le x < \pi$

35. $\sin x = -0.5, \quad -\infty < x < \infty$ **36.** $\cot x = -1, \quad -\infty < x < \infty$

In Exercises 37–40, use the given information to find the values of the six trigonometric functions at the angle θ. Give exact answers.

37. $\theta = \sin^{-1}\left(\frac{8}{17}\right)$ **38.** $\theta = \tan^{-1}\left(-\frac{5}{12}\right)$

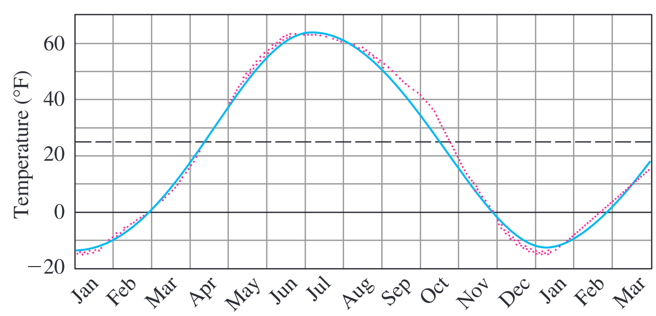
39. The point $P(-3, 4)$ is on the terminal side of θ.

40. The point $P(-2, 2)$ is on the terminal side of θ.

In Exercises 41 and 42, evaluate the expression.

41. $\sin\left(\cos^{-1}\left(\frac{7}{11}\right)\right)$ **42.** $\tan\left(\sin^{-1}\left(\frac{9}{13}\right)\right)$

43. ***Temperatures in Fairbanks, Alaska*** Find the **(a)** amplitude, **(b)** period, **(c)** horizontal shift, and **(d)** vertical shift of the model used in the figure below. **(e)** Then write the equation for the model.



Normal mean air temperature for Fairbanks, Alaska, plotted as data points (red). The approximating sine function $f(x)$ is drawn in blue. *Source: "Is the Curve of Temperature Variation a Sine Curve?"* by B. M. Lando and C. A. Lando, *The Mathematics Teacher, 7.6,* Fig. 2, p. 535 (Sept . 1977).

44. ***Temperatures in Fairbanks, Alaska*** Use the equation of Exercise 43 to approximate the answers to the following questions about the temperatures in Fairbanks, Alaska, shown in the figure in Exercise 43. Assume that the year has 365 days.

(a) What are the highest and lowest mean daily temperatures?

(b) What is the average of the highest and lowest mean daily temperatures? Why is this average the vertical shift of the function?

45. ***Even-Odd***

(a) Show that $\cot x$ is an odd function of x.

(b) Show that the quotient of an even function and an odd function is an odd function.

46. ***Even-Odd***

(a) Show that $\csc x$ is an odd function of x.

(b) Show that the reciprocal of an odd function is odd.

47. ***Even-Odd*** Show that the product of an even function and an odd function is an odd function.

48. ***Finding the Period*** Give a convincing argument that the period of $\tan x$ is π.

49. ***Sinusoidal Regression*** Table 1.22 gives the values of the function

$$f(x) = a \sin(bx + c) + d$$

accurate to two decimals.

Table 1.22 Values of a Function

x	$f(x)$
1	3.42
2	0.73
3	0.12
4	2.16
5	4.97
6	5.97

(a) Find a sinusoidal regression equation for the data.

(b) Rewrite the equation with a, b, c, and d rounded to the nearest integer.

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

50. **True or False** The period of $y = \sin(x/2)$ is π. Justify your answer.

51. **True or False** The amplitude of $y = \frac{1}{2}\cos x$ is 1. Justify your answer.

In Exercises 52–54, $f(x) = 2\cos(4x + \pi) - 1$.

52. **Multiple Choice** Which of the following is the domain of f?

(A) $[-\pi, \pi]$ **(B)** $[-3, 1]$ **(C)** $[-1, 4]$
(D) $(-\infty, \infty)$ **(E)** $x \neq 0$

53. **Multiple Choice** Which of the following is the range of f?

(A) $(-3, 1)$ **(B)** $[-3, 1]$ **(C)** $(-1, 4)$
(D) $[-1, 4]$ **(E)** $(-\infty, \infty)$

54. **Multiple Choice** Which of the following is the period of f?

(A) 4π **(B)** 3π **(C)** 2π **(D)** π **(E)** $\pi/2$

55. **Multiple Choice** Which of the following is the measure of $\tan^{-1}(-\sqrt{3})$ in degrees?

(A) $-60°$ **(B)** $-30°$ **(C)** $30°$ **(D)** $60°$ **(E)** $120°$

Exploration

56. ***Trigonometric Identities*** Let $f(x) = \sin x + \cos x$.

(a) Graph $y = f(x)$. Describe the graph.

(b) Use the graph to identify the amplitude, period, horizontal shift, and vertical shift.

(c) Use the formula

$$\sin\alpha\cos\beta + \cos\alpha\sin\beta = \sin(\alpha + \beta)$$

for the sine of the sum of two angles to confirm your answers.

Extending the Ideas

57. **Exploration** Let $y = \sin(ax) + \cos(ax)$.

Use the symbolic manipulator of a computer algebra system (CAS) to help you with the following:

(a) Express y as a sinusoid for $a = 2, 3, 4$, and 5.

(b) Conjecture another formula for y for a equal to any positive integer n.

(c) Check your conjecture with a CAS.

(d) Use the formula for the sine of the sum of two angles (see Exercise 56c) to confirm your conjecture.

58. **Exploration** Let $y = a\sin x + b\cos x$.

Use the symbolic manipulator of a computer algebra system (CAS) to help you with the following:

(a) Express y as a sinusoid for the following pairs of values:
$a = 2, b = 1$; $a = 1, b = 2$; $a = 5, b = 2$; $a = 2, b = 5$; $a = 3, b = 4$.

(b) Conjecture another formula for y for any pair of positive integers. Try other values if necessary.

(c) Check your conjecture with a CAS.

(d) Use the following formulas for the sine or cosine of a sum or difference of two angles to confirm your conjecture.

$$\sin\alpha\cos\beta \pm \cos\alpha\sin\beta = \sin(\alpha \pm \beta)$$

$$\cos\alpha\cos\beta \pm \sin\alpha\sin\beta = \cos(\alpha \mp \beta)$$

In Exercises 59 and 60, show that the function is periodic and find its period.

59. $y = \sin^3 x$

60. $y = |\tan x|$

In Exercises 61 and 62, graph one period of the function.

61. $f(x) = \sin(60x)$

62. $f(x) = \cos(60\pi x)$

Quick Quiz for AP* Preparation: Sections 1.4–1.6

You should solve the following problems without using a graphing calculator.

1. **Multiple Choice** Which of the following is the domain of $f(x) = -\log_2(x + 3)$?

   (A) $(-\infty, \infty)$ **(B)** $(-\infty, 3)$ **(C)** $(-3, \infty)$

   (D) $[-3, \infty)$ **(E)** $(-\infty, 3]$

2. **Multiple Choice** Which of the following is the range of $f(x) = 5\cos(x + \pi) + 3$?

   (A) $(-\infty, \infty)$ **(B)** $[2, 4]$ **(C)** $[-8, 2]$

   (D) $[-2, 8]$ **(E)** $\left[-\frac{2}{5}, \frac{8}{5}\right]$

3. **Multiple Choice** Which of the following gives the solution of $\tan x = -1$ in $\pi < x < \frac{3\pi}{2}$?

   (A) $-\frac{\pi}{4}$ **(B)** $\frac{\pi}{4}$ **(C)** $\frac{\pi}{3}$ **(D)** $\frac{3\pi}{4}$ **(E)** $\frac{5\pi}{4}$

4. **Free Response** Let $f(x) = 5x - 3$.

   (a) Find the inverse g of f.

   (b) Compute $f \circ g(x)$. Show your work.

   (c) Compute $g \circ f(x)$. Show your work.

Chapter 1 Key Terms

absolute value function (p. 17)
base a logarithm function (p. 40)
boundary of an interval (p. 13)
boundary points (p. 13)
change of base formula (p. 42)
closed interval (p. 13)
common logarithm function (p. 41)
composing (p. 18)
composite function (p. 17)
compounded continuously (p. 25)
cosecant function (p. 46)
cosine function (p. 46)
cotangent function (p. 46)
dependent variable (p. 12)
domain (p. 12)
even function (p. 15)
exponential decay (p. 24)
exponential function base a (p. 22)
exponential growth (p. 24)
function (p. 12)
general linear equation (p. 5)
graph of a function (p. 13)
graph of a relation (p. 30)
grapher failure (p. 15)
half-life (p. 24)
half-open interval (p. 13)
identity function (p. 38)
increments (p. 3)
independent variable (p. 12)
initial point of parametrized curve (p. 30)
interior of an interval (p. 13)
interior points of an interval (p. 13)
inverse cosecant function (p. 50)
inverse cosine function (p. 50)
inverse cotangent function (p. 50)
inverse function (p. 38)
inverse properties for a^x and $\log_a x$ (p. 41)
inverse secant function (p. 50)
inverse sine function (p. 50)
inverse tangent function (p. 50)
linear regression (p. 7)
natural domain (p. 13)
natural logarithm function (p. 41)
odd function (p. 15)
one-to-one function (p. 37)
open interval (p. 13)
parallel lines (p. 4)
parameter (p. 30)
parameter interval (p. 30)
parametric curve (p. 30)
parametric equations (p. 30)
parametrization of a curve (p. 30)
parametrize (p. 30)
period of a function (p. 47)
periodic function (p. 47)
perpendicular lines (p. 4)
piecewise defined function (p. 16)
point-slope equation (p. 4)
power rule for logarithms (p. 41)
product rule for logarithms (p. 41)
quotient rule for logarithms (p. 41)
radian measure (p. 46)
range (p. 12)
regression analysis (p. 7)
regression curve (p. 7)
relation (p. 30)
rise (p. 3)
rules for exponents (p. 23)
run (p. 3)
scatter plot (p. 7)
secant function (p. 46)
sine function (p. 46)
sinusoid (p. 48)
sinusoidal regression (p. 49)
slope (p. 4)
slope-intercept equation (p. 5)
symmetry about the origin (p. 15)
symmetry about the y-axis (p. 15)
tangent function (p. 46)
terminal point of parametrized curve (p. 30)
witch of Agnesi (p. 33)
x-intercept (p. 5)
y-intercept (p. 5)

Chapter 1 Review Exercises

The collection of exercises marked in red could be used as a chapter test.

In Exercises 1–14, write an equation for the specified line.

1. through $(1, -6)$ with slope 3

2. through $(-1, 2)$ with slope $-1/2$

3. the vertical line through $(0, -3)$

4. through $(-3, 6)$ and $(1, -2)$

5. the horizontal line through $(0, 2)$

6. through $(3, 3)$ and $(-2, 5)$

7. with slope -3 and y-intercept 3

8. through $(3, 1)$ and parallel to $2x - y = -2$

9. through $(4, -12)$ and parallel to $4x + 3y = 12$

10. through $(-2, -3)$ and perpendicular to $3x - 5y = 1$

11. through $(-1, 2)$ and perpendicular to $\frac{1}{2}x + \frac{1}{3}y = 1$

12. with x-intercept 3 and y-intercept -5

13. the line $y = f(x)$, where f has the following values:

x	-2	2	4
$f(x)$	4	2	1

14. through $(4, -2)$ with x-intercept -3

In Exercises 15–18, determine whether the graph of the function is symmetric about the y-axis, the origin, or neither.

15. $y = x^{1/5}$

16. $y = x^{2/5}$

17. $y = x^2 - 2x - 1$

18. $y = e^{-x^2}$

In Exercises 19–26, determine whether the function is even, odd, or neither.

19. $y = x^2 + 1$

20. $y = x^5 - x^3 - x$

21. $y = 1 - \cos x$

22. $y = \sec x \tan x$

23. $y = \dfrac{x^4 + 1}{x^3 - 2x}$

24. $y = 1 - \sin x$

25. $y = x + \cos x$

26. $y = \sqrt{x^4 - 1}$

In Exercises 27–38, find the **(a)** domain and **(b)** range, and **(c)** graph the function.

27. $y = |x| - 2$

28. $y = -2 + \sqrt{1 - x}$

29. $y = \sqrt{16 - x^2}$

30. $y = 3^{2-x} + 1$

31. $y = 2e^{-x} - 3$

32. $y = \tan(2x - \pi)$

33. $y = 2\sin(3x + \pi) - 1$

34. $y = x^{2/5}$

35. $y = \ln(x - 3) + 1$

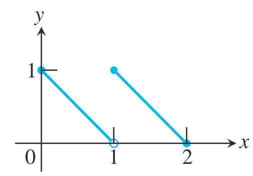
36. $y = -1 + \sqrt[3]{2 - x}$

37. $y = \begin{cases} \sqrt{-x}, & -4 \le x \le 0 \\ \sqrt{x}, & 0 < x \le 4 \end{cases}$

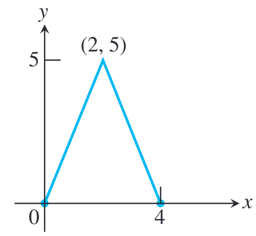
38. $y = \begin{cases} -x - 2, & -2 \le x \le -1 \\ x, & -1 < x \le 1 \\ -x + 2, & 1 < x \le 2 \end{cases}$

In Exercises 39 and 40, write a piecewise formula for the function.

39.



40.



In Exercises 41 and 42, find

(a) $(f \circ g)(-1)$ **(b)** $(g \circ f)(2)$ **(c)** $(f \circ f)(x)$ **(d)** $(g \circ g)(x)$

41. $f(x) = \dfrac{1}{x}$, $g(x) = \dfrac{1}{\sqrt{x + 2}}$

42. $f(x) = 2 - x$, $g(x) = \sqrt[3]{x + 1}$

In Exercises 43 and 44, **(a)** write a formula for $f \circ g$ and $g \circ f$ and find the **(b)** domain and **(c)** range of each.

43. $f(x) = 2 - x^2$, $g(x) = \sqrt{x + 2}$

44. $f(x) = \sqrt{x}$, $g(x) = \sqrt{1 - x}$

In Exercises 45–48, a parametrization is given for a curve.

(a) Graph the curve. Identify the initial and terminal points, if any. Indicate the direction in which the curve is traced.

(b) Find a Cartesian equation for a curve that contains the parametrized curve. What portion of the graph of the Cartesian equation is traced by the parametrized curve?

45. $x = 5\cos t$, $y = 2\sin t$, $0 \le t \le 2\pi$

46. $x = 4\cos t$, $y = 4\sin t$, $\pi/2 \le t < 3\pi/2$

47. $x = 2 - t$, $y = 11 - 2t$, $-2 \le t \le 4$

48. $x = 1 + t$, $y = \sqrt{4 - 2t}$, $t \le 2$

In Exercises 49–52, give a parametrization for the curve.

49. the line segment with endpoints $(-2, 5)$ and $(4, 3)$

50. the line through $(-3, -2)$ and $(4, -1)$

51. the ray with initial point $(2, 5)$ that passes through $(-1, 0)$

52. $y = x(x - 4)$, $x \le 2$

Group Activity In Exercises 53 and 54, do the following.

(a) Find f^{-1} and show that $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$.
(b) Graph f and f^{-1} in the same viewing window.

53. $f(x) = 2 - 3x$

54. $f(x) = (x + 2)^2$, $x \ge -2$

In Exercises 55 and 56, find the measure of the angle in radians and degrees.

55. $\sin^{-1}(0.6)$

56. $\tan^{-1}(-2.3)$

57. Find the six trigonometric values of $\theta = \cos^{-1}(3/7)$. Give exact answers.

58. Solve the equation $\sin x = -0.2$ in the following intervals.

(a) $0 \le x < 2\pi$ **(b)** $-\infty < x < \infty$

59. Solve for x: $e^{-0.2x} = 4$

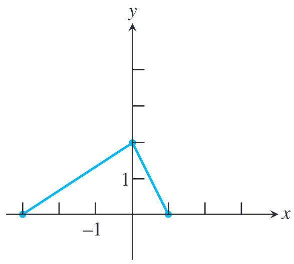
60. The graph of f is shown. Draw the graph of each function.

(a) $y = f(-x)$

(b) $y = -f(x)$

(c) $y = -2f(x + 1) + 1$

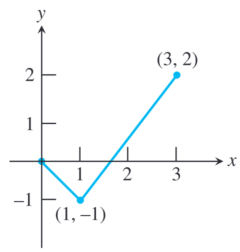
(d) $y = 3f(x - 2) - 2$



61. A portion of the graph of a function defined on $[-3, 3]$ is shown. Complete the graph assuming that the function is

(a) even.

(b) odd.



62. *Depreciation* Smith Hauling purchased an 18-wheel truck for \$100,000. The truck depreciates at the constant rate of \$10,000 per year for 10 years.

(a) Write an expression that gives the value y after x years.

(b) When is the value of the truck \$55,000?

63. *Drug Absorption* A drug is administered intravenously for pain. The function

$$f(t) = 90 - 52 \ln(1 + t), \quad 0 \le t \le 4$$

gives the number of units of the drug in the body after t hours.

(a) What was the initial number of units of the drug administered?

(b) How much is present after 2 hours? **(c)** Draw the graph of f.

64. *Finding Time* If Joenita invests \$1500 in a retirement account that earns 8% compounded annually, how long will it take this single payment to grow to \$5000?

65. *Guppy Population* The number of guppies in Susan's aquarium doubles every day. There are four guppies initially.

(a) Write the number of guppies as a function of time t.

(b) How many guppies were present after 4 days? after 1 week?

(c) When will there be 2000 guppies?

(d) Writing to Learn Give reasons why this might not be a good model for the growth of Susan's guppy population.

66. *Doctoral Degrees* Table 1.23 shows the number of doctoral degrees earned by Hispanic students for several years. Let $x = 0$ represent 1980, $x = 1$ represent 1981, and so forth.

Table 1.23 Doctorates Earned by Hispanic Americans

Year	Number of Degrees
1981	456
1985	677
1990	780
1995	984
2000	1305

Source: Statistical Abstract of the United States, 2004–2005.

(a) Find a linear regression equation for the data and superimpose its graph on a scatter plot of the data.

(b) Use the regression equation to predict the number of doctoral degrees that will be earned by Hispanic Americans in 2002. How close is the estimate to the actual number in 2002 of 1432?

(c) Writing to Learn Find the slope of the regression line. What does the slope represent?

67. *Population of New York* Table 1.24 shows the population of New York State for several years. Let $x = 0$ represent 1980, $x = 1$ represent 1981, and so forth.

Table 1.24 Population of New York State

Year	Population (thousands)
1980	17,558
1990	17,991
1995	18,524
1998	18,756
1999	18,883
2000	18,977

Source: Statistical Abstract of the United States, 2004–2005.

(a) Find the exponential regression equation for the data and superimpose its graph on a scatter plot of the data.

(b) Use the regression equation to predict the population in 2003. How close is the estimate to the actual number in 2003 of 19,190 thousand?

(c) Use the exponential regression equation to estimate the annual rate of growth of the population of New York State.

AP* Examination Preparation

You may use a graphing calculator to solve the following problems.

68. Consider the point $P(-2, 1)$ and the line L: $x + y = 2$.

(a) Find the slope of L.

(b) Write an equation for the line through P and parallel to L.

(c) Write an equation for the line through P and perpendicular to L.

(d) What is the x-intercept of L?

69. Let $f(x) = 1 - \ln(x - 2)$.

(a) What is the domain of f? **(b)** What is the range of f?

(c) What are the x-intercepts of the graph of f?

(d) Find f^{-1}. **(e)** Confirm your answer algebraically in part (d).

70. Let $f(x) = 1 - 3\cos(2x)$.

(a) What is the domain of f? **(b)** What is the range of f?

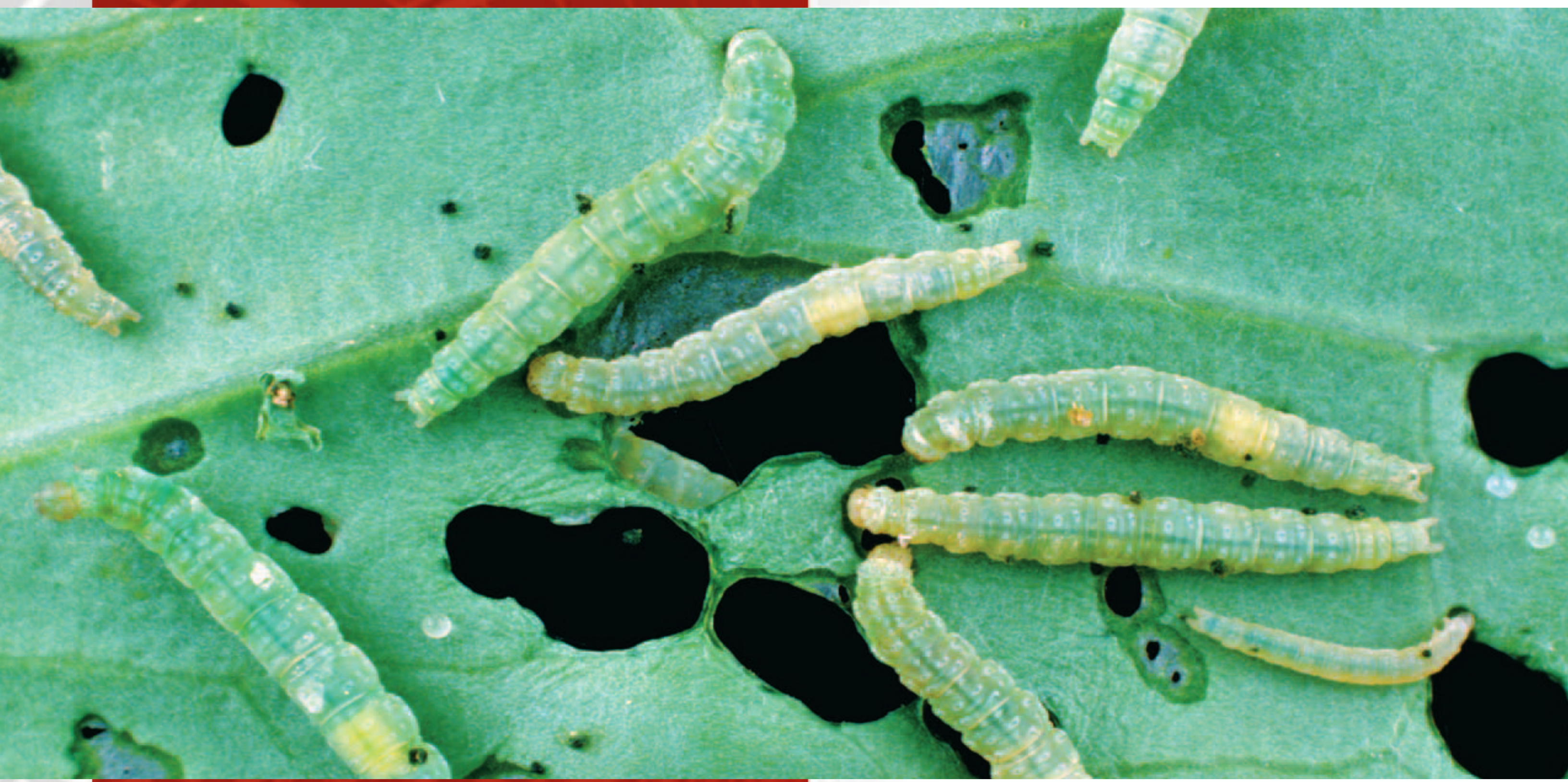
(c) What is the period of f?

(d) Is f an even function, odd function, or neither?

(e) Find all the zeros of f in $\pi/2 \le x \le \pi$.

Chapter 2

Limits and Continuity



An Economic Injury Level (EIL) is a measurement of the fewest number of insect pests that will cause economic damage to a crop or forest. It has been estimated that monitoring pest populations and establishing EILs can reduce pesticide use by 30%–50%.

Accurate population estimates are crucial for determining EILs. A population density of one insect pest can be approximated by

$$D(t) = \frac{t^2}{90} + \frac{t}{3}$$

pests per plant, where t is the number of days since initial infestation. What is the rate of change of this population density when the population density is equal to the EIL of 20 pests per plant? Section 2.4 can help answer this question.

Chapter 2 Overview

The concept of limit is one of the ideas that distinguish calculus from algebra and trigonometry.

In this chapter, we show how to define and calculate limits of function values. The calculation rules are straightforward and most of the limits we need can be found by substitution, graphical investigation, numerical approximation, algebra, or some combination of these.

One of the uses of limits is to test functions for continuity. Continuous functions arise frequently in scientific work because they model such an enormous range of natural behavior. They also have special mathematical properties, not otherwise guaranteed.

2.1 Rates of Change and Limits

What you'll learn about

- Average and Instantaneous Speed
- Definition of Limit
- Properties of Limits
- One-sided and Two-sided Limits
- Sandwich Theorem

. . . and why

Limits can be used to describe continuity, the derivative, and the integral: the ideas giving the foundation of calculus.

Free Fall

Near the surface of the earth, all bodies fall with the same constant acceleration. The distance a body falls after it is released from rest is a constant multiple of the square of the time fallen. At least, that is what happens when a body falls in a vacuum, where there is no air to slow it down. The square-of-time rule also holds for dense, heavy objects like rocks, ball bearings, and steel tools during the first few seconds of fall through air, before the velocity builds up to where air resistance begins to matter. When air resistance is absent or insignificant and the only force acting on a falling body is the force of gravity, we call the way the body falls *free fall.*

Average and Instantaneous Speed

A moving body's **average speed** during an interval of time is found by dividing the distance covered by the elapsed time. The unit of measure is length per unit time—kilometers per hour, feet per second, or whatever is appropriate to the problem at hand.

EXAMPLE 1 Finding an Average Speed

A rock breaks loose from the top of a tall cliff. What is its average speed during the first 2 seconds of fall?

SOLUTION

Experiments show that a dense solid object dropped from rest to fall freely near the surface of the earth will fall

$$y = 16t^2$$

feet in the first t seconds. The average speed of the rock over any given time interval is the distance traveled, Δy, divided by the length of the interval Δt. For the first 2 seconds of fall, from $t = 0$ to $t = 2$, we have

$$\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(0)^2}{2 - 0} = 32\frac{\text{ft}}{\text{sec}}.$$

Now try Exercise 1.

EXAMPLE 2 Finding an Instantaneous Speed

Find the speed of the rock in Example 1 at the instant $t = 2$.

SOLUTION

Solve Numerically We can calculate the average speed of the rock over the interval from time $t = 2$ to any slightly later time $t = 2 + h$ as

$$\frac{\Delta y}{\Delta t} = \frac{16(2 + h)^2 - 16(2)^2}{h}. \quad (1)$$

We cannot use this formula to calculate the speed at the exact instant $t = 2$ because that would require taking $h = 0$, and $0/0$ is undefined. However, we can get a good idea of what is happening at $t = 2$ by evaluating the formula at values of h *close* to 0. When we do, we see a clear pattern (Table 2.1 on the next page). As h approaches 0, the average speed approaches the limiting value 64 ft/sec.

continued

Table 2.1 Average Speeds over Short Time Intervals Starting at $t = 2$

$$\frac{\Delta y}{\Delta t} = \frac{16(2+h)^2 - 16(2)^2}{h}$$

Length of Time Interval, h (sec)	Average Speed for Interval $\Delta y/\Delta t$ (ft/sec)
1	80
0.1	65.6
0.01	64.16
0.001	64.016
0.0001	64.0016
0.00001	64.00016

Confirm Algebraically If we expand the numerator of Equation 1 and simplify, we find that

$$\frac{\Delta y}{\Delta t} = \frac{16(2+h)^2 - 16(2)^2}{h} = \frac{16(4 + 4h + h^2) - 64}{h}$$

$$= \frac{64h + 16h^2}{h} = 64 + 16h.$$

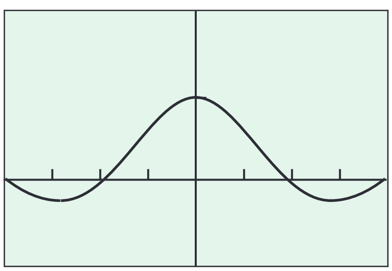
For values of h different from 0, the expressions on the right and left are equivalent and the average speed is $64 + 16h$ ft/sec. We can now see why the average speed has the limiting value $64 + 16(0) = 64$ ft/sec as h approaches 0. ***Now try Exercise 3.***

Definition of Limit

As in the preceding example, most limits of interest in the real world can be viewed as numerical limits of values of functions. And this is where a graphing utility and calculus come in. A calculator can suggest the limits, and calculus can give the mathematics for confirming the limits analytically.

Limits give us a language for describing how the outputs of a function behave as the inputs approach some particular value. In Example 2, the average speed was not defined at $h = 0$ but approached the limit 64 as h approached 0. We were able to see this numerically and to confirm it algebraically by eliminating h from the denominator. But we cannot always do that. For instance, we can see both graphically and numerically (Figure 2.1) that the values of $f(x) = (\sin x)/x$ approach 1 as x approaches 0.

We cannot eliminate the x from the denominator of $(\sin x)/x$ to confirm the observation algebraically. We need to use a theorem about limits to make that confirmation, as you will see in Exercise 75.



$[-2\pi, 2\pi]$ by $[-1, 2]$

(a)

X	Y1	
-.3	.98507	
-.2	.99335	
-.1	.99833	
0	ERROR	
.1	.99833	
.2	.99335	
.3	.98507	

Y1 = sin(X)/X

(b)

Figure 2.1 (a) A graph and (b) table of values for $f(x) = (\sin x)/x$ that suggest the limit of f as x approaches 0 is 1.

DEFINITION Limit

Assume f is defined in a neighborhood of c and let c and L be real numbers. The function ***f* has limit *L* as *x* approaches *c*** if, given any positive number ε, there is a positive number δ such that for all x,

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

We write

$$\lim_{x \to c} f(x) = L.$$

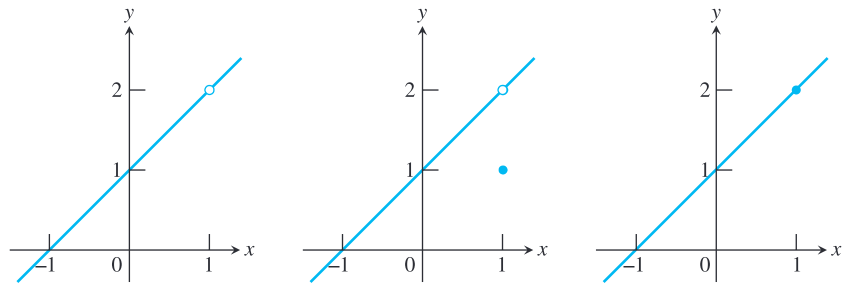
The sentence $\lim_{x \to c} f(x) = L$ is read, "The limit of f of x as x approaches c equals L." The notation means that the values $f(x)$ of the function f approach or equal L as the values of x approach *(but do not equal)* c. Appendix A3 provides practice applying the definition of limit.

We saw in Example 2 that $\lim_{h \to 0} (64 + 16h) = 64$.

As suggested in Figure 2.1,

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Figure 2.2 illustrates the fact that the existence of a limit as $x \to c$ never depends on how the function may or may not be defined at c. The function f has limit 2 as $x \to 1$ even though f is not defined at 1. The function g has limit 2 as $x \to 1$ even though $g(1) \neq 2$. The function h is the only one whose limit as $x \to 1$ equals its value at $x = 1$.



(a) $f(x) = \dfrac{x^2 - 1}{x - 1}$ (b) $g(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$ (c) $h(x) = x + 1$

Figure 2.2 $\lim_{x \to 1} f(x) = \lim_{x \to 1} g(x) = \lim_{x \to 1} h(x) = 2$

Properties of Limits

By applying six basic facts about limits, we can calculate many unfamiliar limits from limits we already know. For instance, from knowing that

$$\lim_{x \to c} (k) = k$$

and

$$\lim_{x \to c} (x) = c,$$

we can calculate the limits of all polynomial and rational functions. The facts are listed in Theorem 1.

THEOREM 1 Properties of Limits

If L, M, c, and k are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \text{ then}$$

1. *Sum Rule:* $\lim_{x \to c} (f(x) + g(x)) = L + M$

The limit of the sum of two functions is the sum of their limits.

2. *Difference Rule:* $\lim_{x \to c} (f(x) - g(x)) = L - M$

The limit of the difference of two functions is the difference of their limits.

3. *Product Rule:* $\lim_{x \to c} (f(x) \cdot g(x)) = L \cdot M$

The limit of a product of two functions is the product of their limits.

4. *Constant Multiple Rule:* $\lim_{x \to c} (k \cdot f(x)) = k \cdot L$

The limit of a constant times a function is the constant times the limit of the function.

5. *Quotient Rule:* $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \; M \neq 0$

The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.

continued

6. *Power Rule:* If r and s are integers, $s \neq 0$, then

$$\lim_{x \to c} (f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number.

The limit of a rational power of a function is that power of the limit of the function, provided the latter is a real number.

Here are some examples of how Theorem 1 can be used to find limits of polynomial and rational functions.

EXAMPLE 3 Using Properties of Limits

Use the observations $\lim_{x \to c} k = k$ and $\lim_{x \to c} x = c$, and the properties of limits to find the following limits.

(a) $\lim_{x \to c} (x^3 + 4x^2 - 3)$ **(b)** $\lim_{x \to c} \dfrac{x^4 + x^2 - 1}{x^2 + 5}$

SOLUTION

(a)
$$\lim_{x \to c} (x^3 + 4x^2 - 3) = \lim_{x \to c} x^3 + \lim_{x \to c} 4x^2 - \lim_{x \to c} 3$$
$$= c^3 + 4c^2 - 3$$

(b)
$$\lim_{x \to c} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{\lim_{x \to c} (x^4 + x^2 - 1)}{\lim_{x \to c} (x^2 + 5)}$$
$$= \frac{\lim_{x \to c} x^4 + \lim_{x \to c} x^2 - \lim_{x \to c} 1}{\lim_{x \to c} x^2 + \lim_{x \to c} 5}$$
$$= \frac{c^4 + c^2 - 1}{c^2 + 5}$$

Now try Exercises 5 and 6.

Example 3 shows the remarkable strength of Theorem 1. From the two simple observations that $\lim_{x \to c} k = k$ and $\lim_{x \to c} x = c$, we can immediately work our way to limits of polynomial functions and most rational functions using substitution.

THEOREM 2 Polynomial and Rational Functions

1. If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ is any polynomial function and c is any real number, then

$$\lim_{x \to c} f(x) = f(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0.$$

2. If $f(x)$ and $g(x)$ are polynomials and c is any real number, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)}, \quad \text{provided that } g(c) \neq 0.$$

EXAMPLE 4 Using Theorem 2

(a) $\lim_{x \to 3} [x^2(2 - x)] = (3)^2(2 - 3) = -9$

(b) $\lim_{x \to 2} \frac{x^2 + 2x + 4}{x + 2} = \frac{(2)^2 + 2(2) + 4}{2 + 2} = \frac{12}{4} = 3$

Now try Exercises 9 and 11.

As with polynomials, limits of many familiar functions can be found by substitution at points where they are defined. This includes trigonometric functions, exponential and logarithmic functions, and composites of these functions. Feel free to use these properties.

EXAMPLE 5 Using the Product Rule

Determine $\lim_{x \to 0} \frac{\tan x}{x}$.

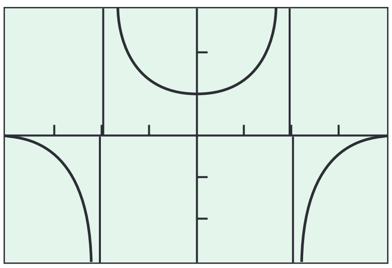
SOLUTION

Solve Graphically The graph of $f(x) = (\tan x)/x$ in Figure 2.3 suggests that the limit exists and is about 1.

Confirm Analytically Using the analytic result of Exercise 75, we have

$$\begin{aligned}\lim_{x \to 0} \frac{\tan x}{x} &= \lim_{x \to 0} \left(\frac{\sin x}{x} \cdot \frac{1}{\cos x} \right) \\ &= \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{1}{\cos x} \\ &= 1 \cdot \frac{1}{\cos 0} = 1 \cdot \frac{1}{1} = 1.\end{aligned}$$

Now try Exercise 27.



$[-\pi, \pi]$ by $[-3, 3]$

Figure 2.3 The graph of

$f(x) = (\tan x)/x$

suggests that $f(x) \to 1$ as $x \to 0$. (Example 5)

Sometimes we can use a graph to discover that limits do not exist, as illustrated by Example 6.

EXAMPLE 6 Exploring a Nonexistent Limit

Use a graph to show that

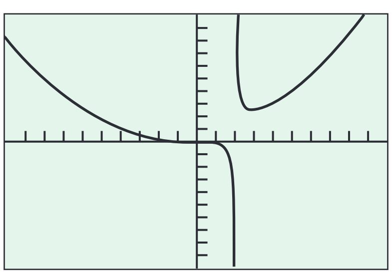
$$\lim_{x \to 2} \frac{x^3 - 1}{x - 2}$$

does not exist.

SOLUTION

Notice that the denominator is 0 when x is replaced by 2, so we cannot use substitution to determine the limit. The graph in Figure 2.4 of $f(x) = (x^3 - 1)/(x - 2)$ strongly suggests that as $x \to 2$ from either side, the absolute values of the function values get very large. This, in turn, suggests that the limit does not exist.

Now try Exercise 29.



$[-10, 10]$ by $[-100, 100]$

Figure 2.4 The graph of

$f(x) = (x^3 - 1)/(x - 2)$

obtained using parametric graphing to produce a more accurate graph. (Example 6)

One-sided and Two-sided Limits

Sometimes the values of a function f tend to different limits as x approaches a number c from opposite sides. When this happens, we call the limit of f as x approaches c from the right the **right-hand limit** of f at c and the limit as x approaches c from the left the **left-hand limit** of f at c. Here is the notation we use:

right-hand: $\lim_{x\to c^+} f(x)$ *The limit of f as x approaches c from the right.*

left-hand: $\lim_{x\to c^-} f(x)$ *The limit of f as x approaches c from the left.*

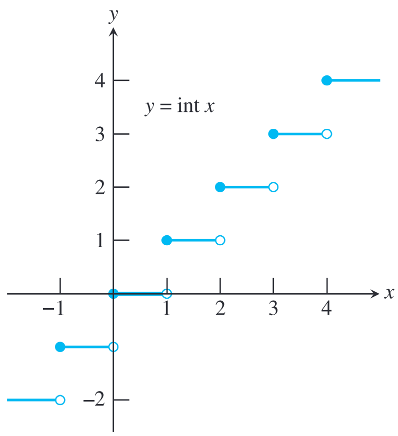


Figure 2.5 At each integer, the greatest integer function $y = \text{int } x$ has different right-hand and left-hand limits. (Example 7)

EXAMPLE 7 Function Values Approach Two Numbers

The greatest integer function $f(x) = \text{int } x$ has different right-hand and left-hand limits at each integer, as we can see in Figure 2.5. For example,

$$\lim_{x\to 3^+} \text{int } x = 3 \quad \text{and} \quad \lim_{x\to 3^-} \text{int } x = 2.$$

The limit of int x as x approaches an integer n from the right is n, while the limit as x approaches n from the left is $n - 1$.

Now try Exercises 31 and 32.

We sometimes call $\lim_{x\to c} f(x)$ the **two-sided limit** of f at c to distinguish it from the *one-sided* right-hand and left-hand limits of f at c. Theorem 3 shows how these limits are related.

On the Far Side

If f is not defined to the left of $x = c$, then f does not have a left-hand limit at c. Similarly, if f is not defined to the right of $x = c$, then f does not have a right-hand limit at c.

THEOREM 3 One-sided and Two-sided Limits

A function $f(x)$ has a limit as x approaches c if and only if the right-hand and left-hand limits at c exist and are equal. In symbols,

$$\lim_{x\to c} f(x) = L \Leftrightarrow \lim_{x\to c^+} f(x) = L \quad \text{and} \quad \lim_{x\to c^-} f(x) = L.$$

Thus, the greatest integer function $f(x) = \text{int } x$ of Example 7 does not have a limit as $x \to 3$ even though each one-sided limit exists.

EXAMPLE 8 Exploring Right- and Left-Hand Limits

All the following statements about the function $y = f(x)$ graphed in Figure 2.6 are true.

At $x = 0$: $\lim_{x\to 0^+} f(x) = 1.$

At $x = 1$: $\lim_{x\to 1^-} f(x) = 0$ even though $f(1) = 1$,

$\lim_{x\to 1^+} f(x) = 1,$

f has no limit as $x \to 1$. (The right- and left-hand limits at 1 are not equal, so $\lim_{x\to 1} f(x)$ does not exist.)

At $x = 2$: $\lim_{x\to 2^-} f(x) = 1,$

$\lim_{x\to 2^+} f(x) = 1,$

$\lim_{x\to 2} f(x) = 1$ even though $f(2) = 2$.

At $x = 3$: $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = 2 = f(3) = \lim_{x\to 3} f(x).$

At $x = 4$: $\lim_{x\to 4^-} f(x) = 1.$

At noninteger values of c between 0 and 4, f has a limit as $x \to c$.

Now try Exercise 37.

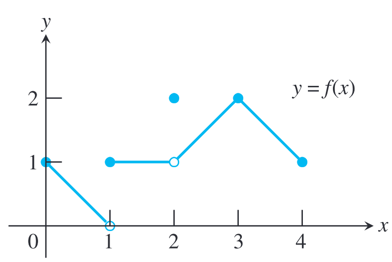


Figure 2.6 The graph of the function

$$f(x) = \begin{cases} -x + 1, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & x = 2 \\ x - 1, & 2 < x \le 3 \\ -x + 5, & 3 < x \le 4. \end{cases}$$

(Example 8)

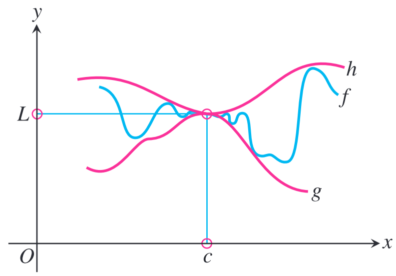


Figure 2.7 Sandwiching *f* between *g* and *h* forces the limiting value of *f* to be between the limiting values of *g* and *h*.

Sandwich Theorem

If we cannot find a limit directly, we may be able to find it indirectly with the Sandwich Theorem. The theorem refers to a function *f* whose values are sandwiched between the values of two other functions, *g* and *h*. If *g* and *h* have the same limit as $x \to c$, then *f* has that limit too, as suggested by Figure 2.7.

THEOREM 4 The Sandwich Theorem

If $g(x) \le f(x) \le h(x)$ for all $x \ne c$ in some interval about *c*, and

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L,$$

then

$$\lim_{x \to c} f(x) = L.$$

EXAMPLE 9 Using the Sandwich Theorem

Show that $\lim_{x \to 0} [x^2 \sin (1/x)] = 0$.

SOLUTION

We know that the values of the sine function lie between –1 and 1. So, it follows that

$$\left| x^2 \sin \frac{1}{x} \right| = |x^2| \cdot \left| \sin \frac{1}{x} \right| \le |x^2| \cdot 1 = x^2$$

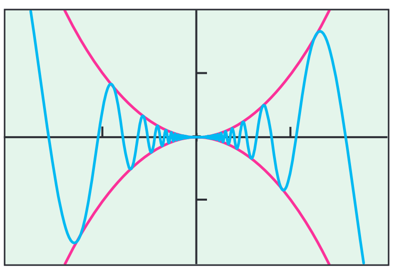
and

$$-x^2 \le x^2 \sin \frac{1}{x} \le x^2.$$

Because $\lim_{x \to 0} (-x^2) = \lim_{x \to 0} x^2 = 0$, the Sandwich Theorem gives

$$\lim_{x \to 0} \left(x^2 \sin \frac{1}{x} \right) = 0.$$

The graphs in Figure 2.8 support this result.



[–0.2, 0.2] by [–0.02, 0.02]

Figure 2.8 The graphs of $y_1 = x^2$, $y_2 = x^2 \sin (1/x)$, and $y_3 = -x^2$. Notice that $y_3 \le y_2 \le y_1$. (Example 9)

Quick Review 2.1 *(For help, go to Section 1.2.)*

In Exercises 1–4, find $f(2)$.

1. $f(x) = 2x^3 - 5x^2 + 4$

2. $f(x) = \dfrac{4x^2 - 5}{x^3 + 4}$

3. $f(x) = \sin \left(\pi \dfrac{x}{2} \right)$

4. $f(x) = \begin{cases} 3x - 1, & x < 2 \\ \dfrac{1}{x^2 - 1}, & x \ge 2 \end{cases}$

In Exercises 5–8, write the inequality in the form $a < x < b$.

5. $|x| < 4$

6. $|x| < c^2$

7. $|x - 2| < 3$

8. $|x - c| < d^2$

In Exercises 9 and 10, write the fraction in reduced form.

9. $\dfrac{x^2 - 3x - 18}{x + 3}$

10. $\dfrac{2x^2 - x}{2x^2 + x - 1}$

Section 2.1 Exercises

In Exercises 1–4, an object dropped from rest from the top of a tall building falls $y = 16t^2$ feet in the first t seconds.

1. Find the average speed during the first 3 seconds of fall.

2. Find the average speed during the first 4 seconds of fall.

3. Find the speed of the object at $t = 3$ seconds and confirm your answer algebraically.

4. Find the speed of the object at $t = 4$ seconds and confirm your answer algebraically.

In Exercises 5 and 6, use $\lim_{x \to c} k = k$, $\lim_{x \to c} x = c$, and the properties of limits to find the limit.

5. $\lim_{x \to c} (2x^3 - 3x^2 + x - 1)$

6. $\lim_{x \to c} \dfrac{x^4 - x^3 + 1}{x^2 + 9}$

In Exercises 7–14, determine the limit by substitution. Support graphically.

7. $\lim_{x \to -1/2} 3x^2(2x - 1)$

8. $\lim_{x \to -4} (x + 3)^{1998}$

9. $\lim_{x \to 1} (x^3 + 3x^2 - 2x - 17)$

10. $\lim_{y \to 2} \dfrac{y^2 + 5y + 6}{y + 2}$

11. $\lim_{y \to -3} \dfrac{y^2 + 4y + 3}{y^2 - 3}$

12. $\lim_{x \to 1/2} \text{int } x$

13. $\lim_{x \to -2} (x - 6)^{2/3}$

14. $\lim_{x \to 2} \sqrt{x + 3}$

In Exercises 15–18, explain why you cannot use substitution to determine the limit. Find the limit if it exists.

15. $\lim_{x \to -2} \sqrt{x - 2}$

16. $\lim_{x \to 0} \dfrac{1}{x^2}$

17. $\lim_{x \to 0} \dfrac{|x|}{x}$

18. $\lim_{x \to 0} \dfrac{(4 + x)^2 - 16}{x}$

In Exercises 19–28, determine the limit graphically. Confirm algebraically.

19. $\lim_{x \to 1} \dfrac{x - 1}{x^2 - 1}$

20. $\lim_{t \to 2} \dfrac{t^2 - 3t + 2}{t^2 - 4}$

21. $\lim_{x \to 0} \dfrac{5x^3 + 8x^2}{3x^4 - 16x^2}$

22. $\lim_{x \to 0} \dfrac{\dfrac{1}{2 + x} - \dfrac{1}{2}}{x}$

23. $\lim_{x \to 0} \dfrac{(2 + x)^3 - 8}{x}$

24. $\lim_{x \to 0} \dfrac{\sin 2x}{x}$

25. $\lim_{x \to 0} \dfrac{\sin x}{2x^2 - x}$

26. $\lim_{x \to 0} \dfrac{x + \sin x}{x}$

27. $\lim_{x \to 0} \dfrac{\sin^2 x}{x}$

28. $\lim_{x \to 0} \dfrac{3 \sin 4x}{\sin 3x}$

In Exercises 29 and 30, use a graph to show that the limit does not exist.

29. $\lim_{x \to 1} \dfrac{x^2 - 4}{x - 1}$

30. $\lim_{x \to 2} \dfrac{x + 1}{x^2 - 4}$

In Exercises 31–36, determine the limit.

31. $\lim_{x \to 0^+} \text{int } x$

32. $\lim_{x \to 0^-} \text{int } x$

33. $\lim_{x \to 0.01} \text{int } x$

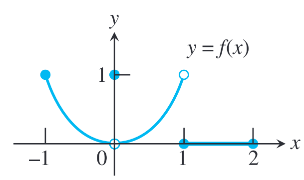
34. $\lim_{x \to 2^-} \text{int } x$

35. $\lim_{x \to 0^+} \dfrac{x}{|x|}$

36. $\lim_{x \to 0^-} \dfrac{x}{|x|}$

In Exercises 37 and 38, which of the statements are true about the function $y = f(x)$ graphed there, and which are false?

37.



(a) $\lim_{x \to -1^+} f(x) = 1$

(b) $\lim_{x \to 0^-} f(x) = 0$

(c) $\lim_{x \to 0^-} f(x) = 1$

(d) $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$

(e) $\lim_{x \to 0} f(x)$ exists

(f) $\lim_{x \to 0} f(x) = 0$

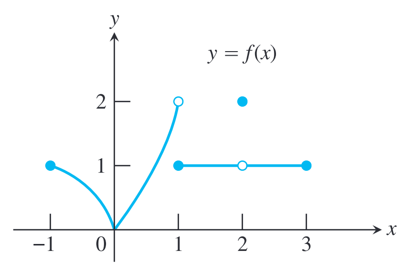
(g) $\lim_{x \to 0} f(x) = 1$

(h) $\lim_{x \to 1} f(x) = 1$

(i) $\lim_{x \to 1} f(x) = 0$

(j) $\lim_{x \to 2^-} f(x) = 2$

38.



(a) $\lim_{x \to -1^+} f(x) = 1$

(b) $\lim_{x \to 2} f(x)$ does not exist.

(c) $\lim_{x \to 2} f(x) = 2$

(d) $\lim_{x \to 1^-} f(x) = 2$

(e) $\lim_{x \to 1^+} f(x) = 1$

(f) $\lim_{x \to 1} f(x)$ does not exist.

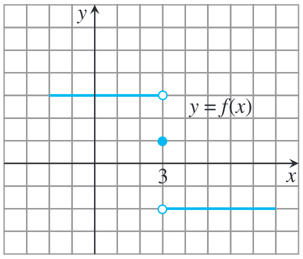
(g) $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x)$

(h) $\lim_{x \to c} f(x)$ exists at every c in $(-1, 1)$.

(i) $\lim_{x \to c} f(x)$ exists at every c in $(1, 3)$.

In Exercises 39–44, use the graph to estimate the limits and value of the function, or explain why the limits do not exist.

39.



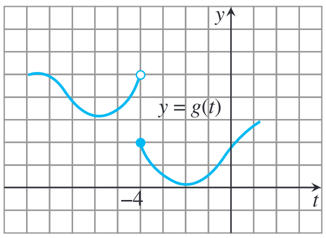
(a) $\lim_{x\to3^-} f(x)$

(b) $\lim_{x\to3^+} f(x)$

(c) $\lim_{x\to3} f(x)$

(d) $f(3)$

40.



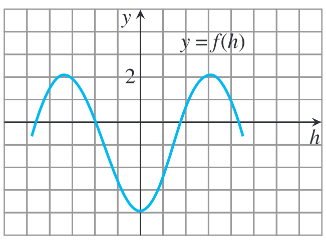
(a) $\lim_{t\to-4^-} g(t)$

(b) $\lim_{t\to-4^+} g(t)$

(c) $\lim_{t\to-4} g(t)$

(d) $g(-4)$

41.



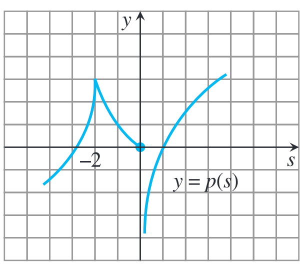
(a) $\lim_{h\to0^-} f(h)$

(b) $\lim_{h\to0^+} f(h)$

(c) $\lim_{h\to0} f(h)$

(d) $f(0)$

42.



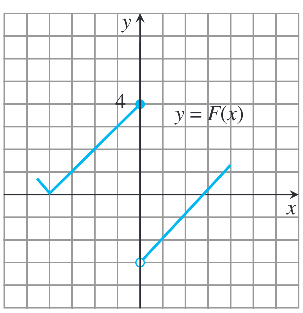
(a) $\lim_{s\to-2^-} p(s)$

(b) $\lim_{s\to-2^+} p(s)$

(c) $\lim_{s\to-2} p(s)$

(d) $p(-2)$

43.



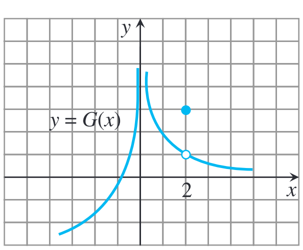
(a) $\lim_{x\to0^-} F(x)$

(b) $\lim_{x\to0^+} F(x)$

(c) $\lim_{x\to0} F(x)$

(d) $F(0)$

44.



(a) $\lim_{x\to2^-} G(x)$

(b) $\lim_{x\to2^+} G(x)$

(c) $\lim_{x\to2} G(x)$

(d) $G(2)$

In Exercises 45–48, match the function with the table.

45. $y_1 = \dfrac{x^2 + x - 2}{x - 1}$ **46.** $y_1 = \dfrac{x^2 - x - 2}{x - 1}$

47. $y_1 = \dfrac{x^2 - 2x + 1}{x - 1}$ **48.** $y_1 = \dfrac{x^2 + x - 2}{x + 1}$

X	Y1
.7	−.4765
.8	−.3111
.9	−.1526
1	0
1.1	.14762
1.2	.29091
1.3	.43043

X = .7

(a)

X	Y1
.7	7.3667
.8	10.8
.9	20.9
1	ERROR
1.1	−18.9
1.2	−8.8
1.3	−5.367

X = .7

(b)

X	Y1
.7	2.7
.8	2.8
.9	2.9
1	ERROR
1.1	3.1
1.2	3.2
1.3	3.3

X = .7

(c)

X	Y1
.7	−.3
.8	−.2
.9	−.1
1	ERROR
1.1	.1
1.2	.2
1.3	.3

X = .7

(d)

In Exercises 49 and 50, determine the limit.

49. Assume that $\lim_{x\to4} f(x) = 0$ and $\lim_{x\to4} g(x) = 3$.

(a) $\lim_{x\to4} (g(x) + 3)$ **(b)** $\lim_{x\to4} x f(x)$

(c) $\lim_{x\to4} g^2(x)$ **(d)** $\lim_{x\to4} \dfrac{g(x)}{f(x) - 1}$

50. Assume that $\lim_{x\to b} f(x) = 7$ and $\lim_{x\to b} g(x) = -3$.

(a) $\lim_{x\to b} (f(x) + g(x))$ **(b)** $\lim_{x\to b} (f(x) \cdot g(x))$

(c) $\lim_{x\to b} 4\,g(x)$ **(d)** $\lim_{x\to b} \dfrac{f(x)}{g(x)}$

In Exercises 51–54, complete parts (a), (b), and (c) for the piecewise-defined function.

(a) Draw the graph of f.

(b) Determine $\lim_{x\to c^+} f(x)$ and $\lim_{x\to c^-} f(x)$.

(c) **Writing to Learn** Does $\lim_{x\to c} f(x)$ exist? If so, what is it? If not, explain.

51. $c = 2, f(x) = \begin{cases} 3 - x, & x < 2 \\ \dfrac{x}{2} + 1, & x > 2 \end{cases}$

52. $c = 2, f(x) = \begin{cases} 3 - x, & x < 2 \\ 2, & x = 2 \\ x/2, & x > 2 \end{cases}$

53. $c = 1, f(x) = \begin{cases} \dfrac{1}{x - 1}, & x < 1 \\ x^3 - 2x + 5, & x \ge 1 \end{cases}$

54. $c = -1, f(x) = \begin{cases} 1 - x^2, & x \ne -1 \\ 2, & x = -1 \end{cases}$

In Exercises 55–58, complete parts (a)–(d) for the piecewise-defined function.

(a) Draw the graph of f.

(b) At what points c in the domain of f does $\lim_{x\to c} f(x)$ exist?

(c) At what points c does only the left-hand limit exist?

(d) At what points c does only the right-hand limit exist?

55. $f(x) = \begin{cases} \sin x, & -2\pi \le x < 0 \\ \cos x, & 0 \le x \le 2\pi \end{cases}$

56. $f(x) = \begin{cases} \cos x, & -\pi \le x < 0 \\ \sec x, & 0 \le x \le \pi \end{cases}$

57. $f(x) = \begin{cases} \sqrt{1 - x^2}, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & x = 2 \end{cases}$

58. $f(x) = \begin{cases} x, & -1 \le x < 0, \text{ or } 0 < x \le 1 \\ 1, & x = 0 \\ 0, & x < -1, \text{ or } x > 1 \end{cases}$

In Exercises 59–62, find the limit graphically. Use the Sandwich Theorem to confirm your answer.

59. $\lim_{x\to 0} x \sin x$

60. $\lim_{x\to 0} x^2 \sin x$

61. $\lim_{x\to 0} x^2 \sin \dfrac{1}{x^2}$

62. $\lim_{x\to 0} x^2 \cos \dfrac{1}{x^2}$

63. ***Free Fall*** A water balloon dropped from a window high above the ground falls $y = 4.9t^2$ m in t sec. Find the balloon's

(a) average speed during the first 3 sec of fall.

(b) speed at the instant $t = 3$.

64. ***Free Fall on a Small Airless Planet*** A rock released from rest to fall on a small airless planet falls $y = gt^2$ m in t sec, g a constant. Suppose that the rock falls to the bottom of a crevasse 20 m below and reaches the bottom in 4 sec.

(a) Find the value of g.

(b) Find the average speed for the fall.

(c) With what speed did the rock hit the bottom?

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

65. True or False If $\lim_{x\to c^-} f(x) = 2$ and $\lim_{x\to c^+} f(x) = 2$, then $\lim_{x\to c} f(x) = 2$. Justify your answer.

66. True or False $\lim_{x\to 0} \dfrac{x + \sin x}{x} = 2$. Justify your answer.

In Exercises 67–70, use the following function.

$$f(x) = \begin{cases} 2 - x, & x \le 1 \\ \dfrac{x}{2} + 1, & x > 1 \end{cases}$$

67. Multiple Choice What is the value of $\lim_{x\to 1^-} f(x)$?

(A) 5/2 **(B)** 3/2 **(C)** 1 **(D)** 0 **(E)** does not exist

68. Multiple Choice What is the value of $\lim_{x\to 1^+} f(x)$?

(A) 5/2 **(B)** 3/2 **(C)** 1 **(D)** 0 **(E)** does not exist

69. Multiple Choice What is the value of $\lim_{x\to 1} f(x)$?

(A) 5/2 **(B)** 3/2 **(C)** 1 **(D)** 0 **(E)** does not exist

70. Multiple Choice What is the value of $f(1)$?

(A) 5/2 **(B)** 3/2 **(C)** 1 **(D)** 0 **(E)** does not exist

Explorations

In Exercises 71–74, complete the following tables and state what you believe $\lim_{x\to 0} f(x)$ to be.

(a)

x	-0.1	-0.01	-0.001	-0.0001	…
$f(x)$	?	?	?	?	

(b)

x	0.1	0.01	0.001	0.0001	…
$f(x)$	?	?	?	?	

71. $f(x) = x \sin \dfrac{1}{x}$

72. $f(x) = \sin \dfrac{1}{x}$

73. $f(x) = \dfrac{10^x - 1}{x}$

74. $f(x) = x \sin (\ln |x|)$

75. Group Activity To prove that $\lim_{\theta\to 0} (\sin \theta)/\theta = 1$ when θ is measured in radians, the plan is to show that the right- and left-hand limits are both 1.

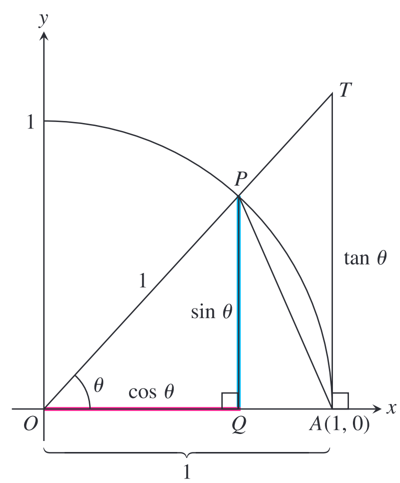
(a) To show that the right-hand limit is 1, explain why we can restrict our attention to $0 < \theta < \pi/2$.

(b) Use the figure to show that

$$\text{area of } \triangle OAP = \frac{1}{2}\sin\theta,$$

$$\text{area of sector } OAP = \frac{\theta}{2},$$

$$\text{area of } \triangle OAT = \frac{1}{2}\tan\theta.$$



(c) Use part (b) and the figure to show that for $0 < \theta < \pi/2$,

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta.$$

(d) Show that for $0 < \theta < \pi/2$ the inequality of part (c) can be written in the form

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

(e) Show that for $0 < \theta < \pi/2$ the inequality of part (d) can be written in the form

$$\cos\theta < \frac{\sin\theta}{\theta} < 1.$$

(f) Use the Sandwich Theorem to show that

$$\lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} = 1.$$

(g) Show that $(\sin\theta)/\theta$ is an even function.

(h) Use part (g) to show that

$$\lim_{\theta\to 0^-} \frac{\sin\theta}{\theta} = 1.$$

(i) Finally, show that

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$$

Extending the Ideas

76. ***Controlling Outputs*** Let $f(x) = \sqrt{3x - 2}$.

(a) Show that $\lim_{x\to 2} f(x) = 2 = f(2)$.

(b) Use a graph to estimate values for a and b so that $1.8 < f(x) < 2.2$ provided $a < x < b$.

(c) Use a graph to estimate values for a and b so that $1.99 < f(x) < 2.01$ provided $a < x < b$.

77. ***Controlling Outputs*** Let $f(x) = \sin x$.

(a) Find $f(\pi/6)$.

(b) Use a graph to estimate an interval (a, b) about $x = \pi/6$ so that $0.3 < f(x) < 0.7$ provided $a < x < b$.

(c) Use a graph to estimate an interval (a, b) about $x = \pi/6$ so that $0.49 < f(x) < 0.51$ provided $a < x < b$.

78. ***Limits and Geometry*** Let $P(a, a^2)$ be a point on the parabola $y = x^2$, $a > 0$. Let O be the origin and $(0, b)$ the y-intercept of the perpendicular bisector of line segment OP. Find $\lim_{P\to O} b$.

2.2 Limits Involving Infinity

What you'll learn about

- Finite Limits as $x \to \pm\infty$
- Sandwich Theorem Revisited
- Infinite Limits as $x \to a$
- End Behavior Models
- "Seeing" Limits as $x \to \pm\infty$

. . . and why

Limits can be used to describe the behavior of functions for numbers large in absolute value.

Finite Limits as $x \to \pm\infty$

The symbol for infinity (∞) does not represent a real number. We use ∞ to describe the behavior of a function when the values in its domain or range outgrow all finite bounds. For example, when we say "the limit of f as x approaches infinity" we mean the limit of f as x moves increasingly far to the right on the number line. When we say "the limit of f as x approaches negative infinity ($-\infty$)" we mean the limit of f as x moves increasingly far to the left. (The limit in each case may or may not exist.)

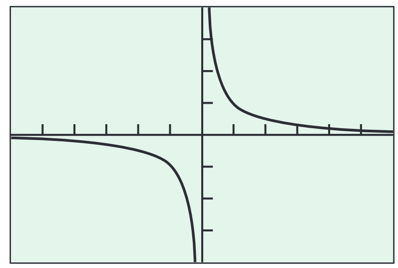
Looking at $f(x) = 1/x$ (Figure 2.9), we observe

(a) as $x \to \infty$, $(1/x) \to 0$ and we write

$$\lim_{x\to\infty} (1/x) = 0,$$

(b) as $x \to -\infty$, $(1/x) \to 0$ and we write

$$\lim_{x\to-\infty} (1/x) = 0.$$



[–6, 6] by [–4, 4]

Figure 2.9 The graph of $f(x) = 1/x$

We say that the line $y = 0$ is a *horizontal asymptote* of the graph of f.

DEFINITION Horizontal Asymptote

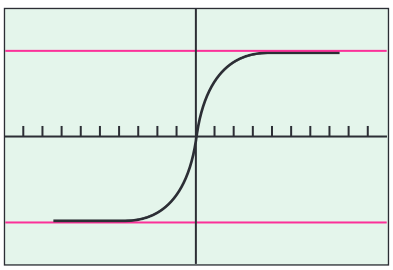
The line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b.$$

The graph of $f(x) = 2 + (1/x)$ has the single horizontal asymptote $y = 2$ because

$$\lim_{x\to\infty}\left(2 + \frac{1}{x}\right) = 2 \quad \text{and} \quad \lim_{x\to-\infty}\left(2 + \frac{1}{x}\right) = 2.$$

A function can have more than one horizontal asymptote, as Example 1 demonstrates.



[–10, 10] by [–1.5, 1.5]

(a)

X	Y1
0	0
1	.7071
2	.8944
3	.9487
4	.9701
5	.9806
6	.9864

Y1 = X/√(X² + 1)

X	Y1
-6	-.9864
-5	-.9806
-4	-.9701
-3	-.9487
-2	-.8944
-1	-.7071
0	0

Y1 = X/√(X² + 1)

(b)

Figure 2.10 (a) The graph of $f(x) = x/\sqrt{x^2+1}$ has two horizontal asymptotes, $y = -1$ and $y = 1$. (b) Selected values of f. (Example 1)

EXAMPLE 1 Looking for Horizontal Asymptotes

Use graphs and tables to find $\lim_{x\to\infty} f(x)$, $\lim_{x\to-\infty} f(x)$, and identify all horizontal asymptotes of $f(x) = x/\sqrt{x^2+1}$.

SOLUTION

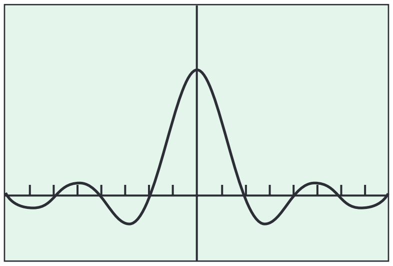
Solve Graphically Figure 2.10a shows the graph for $-10 \le x \le 10$. The graph climbs rapidly toward the line $y = 1$ as x moves away from the origin to the right. On our calculator screen, the graph soon becomes indistinguishable from the line. Thus $\lim_{x\to\infty} f(x) = 1$. Similarly, as x moves away from the origin to the left, the graph drops rapidly toward the line $y = -1$ and soon appears to overlap the line. Thus $\lim_{x\to-\infty} f(x) = -1$. The horizontal asymptotes are $y = 1$ and $y = -1$.

continued

Confirm Numerically The table in Figure 2.10b confirms the rapid approach of $f(x)$ toward 1 as $x \to \infty$. Since f is an odd function of x, we can expect its values to approach -1 in a similar way as $x \to -\infty$. *Now try Exercise 5.*

Sandwich Theorem Revisited

The Sandwich Theorem also holds for limits as $x \to \pm\infty$.



$[-4\pi, 4\pi]$ by $[-0.5, 1.5]$

(a)

X	Y1	
100	−.0051	
200	−.0044	
300	−.0033	
400	−.0021	
500	−9E−4	
600	7.4E−5	
700	7.8E−4	

Y1 = sin(X)/X

(b)

Figure 2.11 (a) The graph of $f(x) = (\sin x)/x$ oscillates about the x-axis. The amplitude of the oscillations decreases toward zero as $x \to \pm\infty$. (b) A table of values for f that suggests $f(x) \to 0$ as $x \to \infty$. (Example 2)

EXAMPLE 2 Finding a Limit as x Approaches ∞

Find $\lim_{x \to \infty} f(x)$ for $f(x) = \dfrac{\sin x}{x}$.

SOLUTION

Solve Graphically and Numerically The graph and table of values in Figure 2.11 suggest that $y = 0$ is the horizontal asymptote of f.

Confirm Analytically We know that $-1 \le \sin x \le 1$. So, for $x > 0$ we have

$$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}.$$

Therefore, by the Sandwich Theorem,

$$0 = \lim_{x \to \infty}\left(-\frac{1}{x}\right) = \lim_{x \to \infty} \frac{\sin x}{x} = \lim_{x \to \infty} \frac{1}{x} = 0.$$

Since $(\sin x)/x$ is an even function of x, we can also conclude that

$$\lim_{x \to \infty} \frac{\sin x}{x} = 0.$$

Now try Exercise 9.

Limits at infinity have properties similar to those of finite limits.

THEOREM 5 Properties of Limits as $x \to \pm\infty$

If L, M, and k are real numbers and

$$\lim_{x \to \pm\infty} f(x) = L \quad \text{and} \quad \lim_{x \to \pm\infty} g(x) = M, \text{ then}$$

1. *Sum Rule:* $\lim_{x \to \pm\infty} (f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x \to \pm\infty} (f(x) - g(x)) = L - M$
3. *Product Rule:* $\lim_{x \to \pm\infty} (f(x) \cdot g(x)) = L \cdot M$
4. *Constant Multiple Rule:* $\lim_{x \to \pm\infty} (k \cdot f(x)) = k \cdot L$
5. *Quotient Rule:* $\lim_{x \to \pm\infty} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, $M \ne 0$
6. *Power Rule:* If r and s are integers, $s \ne 0$, then

$$\lim_{x \to \pm\infty} (f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number.

We can use Theorem 5 to find limits at infinity of functions with complicated expressions, as illustrated in Example 3.

EXAMPLE 3 Using Theorem 5

Find $\lim_{x\to\infty} \dfrac{5x + \sin x}{x}$.

SOLUTION

Notice that

$$\frac{5x + \sin x}{x} = \frac{5x}{x} + \frac{\sin x}{x} = 5 + \frac{\sin x}{x}.$$

So,

$$\lim_{x\to\infty} \frac{5x + \sin x}{x} = \lim_{x\to\infty} 5 + \lim_{x\to\infty} \frac{\sin x}{x}$$
$$= 5 + 0 = 5.$$

Now try Exercise 25.

EXPLORATION 1 Exploring Theorem 5

We must be careful how we apply Theorem 5.

1. (Example 3 again) Let $f(x) = 5x + \sin x$ and $g(x) = x$. Do the limits as $x\to\infty$ of f and g exist? Can we apply the Quotient Rule to $\lim_{x\to\infty} f(x)/g(x)$? Explain. Does the limit of the quotient exist?
2. Let $f(x) = \sin^2 x$ and $g(x) = \cos^2 x$. Describe the behavior of f and g as $x\to\infty$. Can we apply the Sum Rule to $\lim_{x\to\infty} (f(x) + g(x))$? Explain. Does the limit of the sum exist?
3. Let $f(x) = \ln(2x)$ and $g(x) = \ln(x + 1)$. Find the limits as $x\to\infty$ of f and g. Can we apply the Difference Rule to $\lim_{x\to\infty} (f(x) - g(x))$? Explain. Does the limit of the difference exist?
4. Based on parts 1–3, what advice might you give about applying Theorem 5?

Infinite Limits as $x\to a$

If the values of a function $f(x)$ outgrow all positive bounds as x approaches a finite number a, we say that $\lim_{x\to a} f(x) = \infty$. If the values of f become large and negative, exceeding all negative bounds as $x\to a$, we say that $\lim_{x\to a} f(x) = -\infty$.

Looking at $f(x) = 1/x$ (Figure 2.9, page 70), we observe that

$$\lim_{x\to 0^+} 1/x = \infty \quad \text{and} \quad \lim_{x\to 0^-} 1/x = -\infty.$$

We say that the line $x = 0$ is a *vertical asymptote* of the graph of f.

DEFINITION Vertical Asymptote

The line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = \pm\infty$$

EXAMPLE 4 Finding Vertical Asymptotes

Find the vertical asymptotes of $f(x) = \dfrac{1}{x^2}$. Describe the behavior to the left and right of each vertical asymptote.

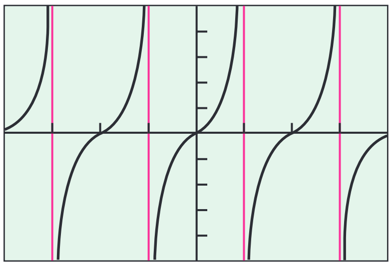
SOLUTION

The values of the function approach ∞ on either side of $x = 0$.

$$\lim_{x\to 0^+} \frac{1}{x^2} = \infty \quad \text{and} \quad \lim_{x\to 0^-} \frac{1}{x^2} = \infty.$$

The line $x = 0$ is the only vertical asymptote. ***Now try Exercise 27.***

We can also say that $\lim_{x\to 0} (1/x^2) = \infty$. We can make no such statement about $1/x$.



$[-2\pi, 2\pi]$ by $[-5, 5]$

Figure 2.12 The graph of $f(x) = \tan x$ has a vertical asymptote at $\dots, -\dfrac{3\pi}{2}, -\dfrac{\pi}{2}, \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dots$. (Example 5)

EXAMPLE 5 Finding Vertical Asymptotes

The graph of $f(x) = \tan x = (\sin x)/(\cos x)$ has infinitely many vertical asymptotes, one at each point where the cosine is zero. If a is an odd multiple of $\pi/2$, then

$$\lim_{x\to a^+} \tan x = -\infty \quad \text{and} \quad \lim_{x\to a^-} \tan x = \infty,$$

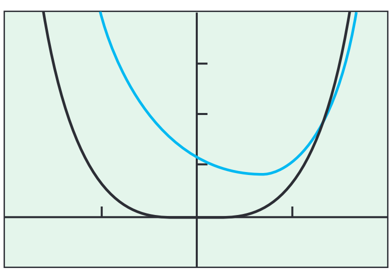
as suggested by Figure 2.12. ***Now try Exercise 31.***

You might think that the graph of a quotient always has a vertical asymptote where the denominator is zero, but that need not be the case. For example, we observed in Section 2.1 that $\lim_{x\to 0} (\sin x)/x = 1$.

End Behavior Models

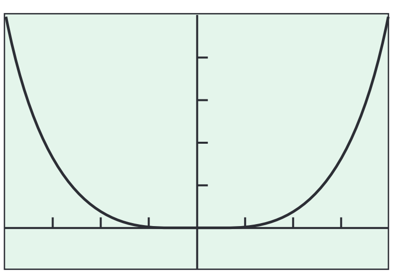
For numerically large values of x, we can sometimes model the behavior of a complicated function by a simpler one that acts virtually in the same way.

$y = 3x^4 - 2x^3 + 3x^2 - 5x + 6$



$[-2, 2]$ by $[-5, 20]$

(a)



$[-20, 20]$ by $[-100000, 500000]$

(b)

Figure 2.13 The graphs of f and g, (a) distinct for $|x|$ small, are (b) nearly identical for $|x|$ large. (Example 6)

EXAMPLE 6 Modeling Functions For $|x|$ Large

Let $f(x) = 3x^4 - 2x^3 + 3x^2 - 5x + 6$ and $g(x) = 3x^4$. Show that while f and g are quite different for numerically small values of x, they are virtually identical for $|x|$ large.

SOLUTION

Solve Graphically The graphs of f and g (Figure 2.13a), quite different near the origin, are virtually identical on a larger scale (Figure 2.13b).

Confirm Analytically We can test the claim that g models f for numerically large values of x by examining the ratio of the two functions as $x \to \pm\infty$. We find that

$$\begin{aligned}\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\pm\infty} \frac{3x^4 - 2x^3 + 3x^2 - 5x + 6}{3x^4} \\ &= \lim_{x\to\pm\infty} \left(1 - \frac{2}{3x} + \frac{1}{x^2} - \frac{5}{3x^3} + \frac{2}{x^4}\right) \\ &= 1,\end{aligned}$$

convincing evidence that f and g behave alike for $|x|$ large. ***Now try Exercise 39.***

DEFINITION End Behavior Model

The function g is

(a) a **right end behavior model** for f if and only if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$

(b) a **left end behavior model** for f if and only if $\lim_{x\to-\infty} \frac{f(x)}{g(x)} = 1.$

If one function provides both a left and right end behavior model, it is simply called an **end behavior model.** Thus, $g(x) = 3x^4$ is an end behavior model for $f(x) = 3x^4 - 2x^3 + 3x^2 - 5x + 6$ (Example 6).

In general, $g(x) = a_n x^n$ is an end behavior model for the polynomial function $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, $a_n \neq 0$. Overall, the end behavior of all polynomials behave like the end behavior of monomials. This is the key to the end behavior of rational functions, as illustrated in Example 7.

EXAMPLE 7 Finding End Behavior Models

Find an end behavior model for

(a) $f(x) = \dfrac{2x^5 + x^4 - x^2 + 1}{3x^2 - 5x + 7}$ **(b)** $g(x) = \dfrac{2x^3 - x^2 + x - 1}{5x^3 + x^2 + x - 5}$

SOLUTION

(a) Notice that $2x^5$ is an end behavior model for the numerator of f, and $3x^2$ is one for the denominator. This makes

$$\frac{2x^5}{3x^2} = \frac{2}{3}x^3$$

an end behavior model for f.

(b) Similarly, $2x^3$ is an end behavior model for the numerator of g, and $5x^3$ is one for the denominator of g. This makes

$$\frac{2x^3}{5x^3} = \frac{2}{5}$$

an end behavior model for g. *Now try Exercise 43.*

Notice in Example 7b that the end behavior model for g, $y = 2/5$, is also a horizontal asymptote of the graph of g, while in 7a, the graph of f does not have a horizontal asymptote. We can use the end behavior model of a rational function to identify any horizontal asymptote.

We can see from Example 7 that a rational function always has a simple power function as an end behavior model.

A function's right and left end behavior models need not be the same function.

EXAMPLE 8 Finding End Behavior Models

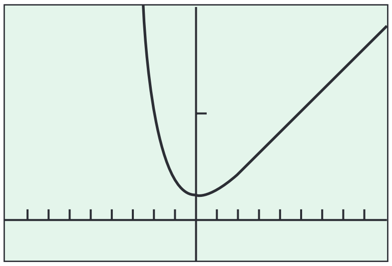
Let $f(x) = x + e^{-x}$. Show that $g(x) = x$ is a right end behavior model for f while $h(x) = e^{-x}$ is a left end behavior model for f.

SOLUTION

On the right,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x + e^{-x}}{x} = \lim_{x\to\infty} \left(1 + \frac{e^{-x}}{x}\right) = 1 \text{ because } \lim_{x\to\infty} \frac{e^{-x}}{x} = 0.$$

continued



[–9, 9] by [–2, 10]

Figure 2.14 The graph of $f(x) = x + e^{-x}$ looks like the graph of $g(x) = x$ to the right of the y-axis, and like the graph of $h(x) = e^{-x}$ to the left of the y-axis. (Example 8)

On the left,

$$\lim_{x\to-\infty} \frac{f(x)}{h(x)} = \lim_{x\to-\infty} \frac{x + e^{-x}}{e^{-x}} = \lim_{x\to-\infty} \left(\frac{x}{e^{-x}} + 1\right) = 1 \text{ because } \lim_{x\to-\infty} \frac{x}{e^{-x}} = 0.$$

The graph of f in Figure 2.14 supports these end behavior conclusions.

Now try Exercise 45.

"Seeing" Limits as $x \to \pm\infty$

We can investigate the graph of $y = f(x)$ as $x \to \pm\infty$ by investigating the graph of $y = f(1/x)$ as $x \to 0$.

EXAMPLE 9 Using Substitution

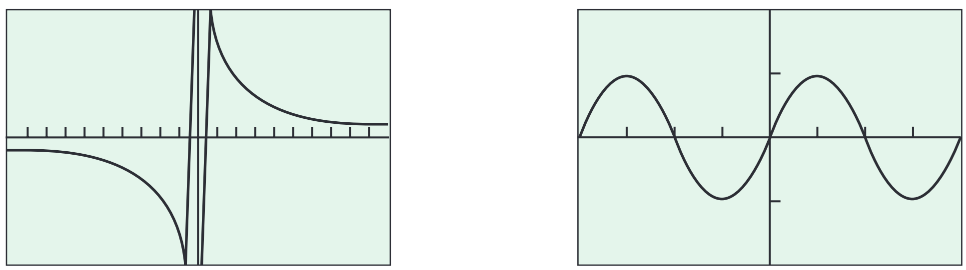
Find $\lim_{x\to\infty} \sin(1/x)$.

SOLUTION

Figure 2.15a suggests that the limit is 0. Indeed, replacing $\lim_{x\to\infty} \sin(1/x)$ by the equivalent $\lim_{x\to 0^+} \sin x = 0$ (Figure 2.15b), we find

$$\lim_{x\to\infty} \sin 1/x = \lim_{x\to 0^+} \sin x = 0.$$

Now try Exercise 49.



[–10, 10] by [–1, 1]

(a)

[-2π, 2π] by [–2, 2]

(b)

Figure 2.15 The graphs of (a) $f(x) = \sin(1/x)$ and (b) $g(x) = f(1/x) = \sin x$. (Example 9)

Quick Review 2.2 *(For help, go to Section 1.2 and 1.5.)*

In Exercises 1–4, find f^{-1} and graph f, f^{-1}, and $y = x$ in the same square viewing window.

1. $f(x) = 2x - 3$

2. $f(x) = e^x$

3. $f(x) = \tan^{-1} x$

4. $f(x) = \cot^{-1} x$

In Exercises 5 and 6, find the quotient $q(x)$ and remainder $r(x)$ when $f(x)$ is divided by $g(x)$.

5. $f(x) = 2x^3 - 3x^2 + x - 1$, $g(x) = 3x^3 + 4x - 5$

6. $f(x) = 2x^5 - x^3 + x - 1$, $g(x) = x^3 - x^2 + 1$

In Exercises 7–10, write a formula for **(a)** $f(-x)$ and **(b)** $f(1/x)$. Simplify where possible.

7. $f(x) = \cos x$

8. $f(x) = e^{-x}$

9. $f(x) = \dfrac{\ln x}{x}$

10. $f(x) = \left(x + \dfrac{1}{x}\right) \sin x$

Section 2.2 Exercises

In Exercises 1–8, use graphs and tables to find **(a)** $\lim_{x\to\infty} f(x)$ and **(b)** $\lim_{x\to-\infty} f(x)$ **(c)** Identify all horizontal asymptotes.

1. $f(x) = \cos\left(\dfrac{1}{x}\right)$

2. $f(x) = \dfrac{\sin 2x}{x}$

3. $f(x) = \dfrac{e^{-x}}{x}$

4. $f(x) = \dfrac{3x^3 - x + 1}{x + 3}$

5. $f(x) = \dfrac{3x + 1}{|x| + 2}$

6. $f(x) = \dfrac{2x - 1}{|x| - 3}$

7. $f(x) = \dfrac{x}{|x|}$

8. $f(x) = \dfrac{|x|}{|x| + 1}$

In Exercises 9–12, find the limit and confirm your answer using the Sandwich Theorem.

9. $\lim\limits_{x\to\infty} \dfrac{1 - \cos x}{x^2}$

10. $\lim\limits_{x\to-\infty} \dfrac{1 - \cos x}{x^2}$

11. $\lim\limits_{x\to-\infty} \dfrac{\sin x}{x}$

12. $\lim\limits_{x\to\infty} \dfrac{\sin(x^2)}{x}$

In Exercises 13–20, use graphs and tables to find the limits.

13. $\lim\limits_{x\to2^+} \dfrac{1}{x - 2}$

14. $\lim\limits_{x\to2^-} \dfrac{x}{x - 2}$

15. $\lim\limits_{x\to-3^-} \dfrac{1}{x + 3}$

16. $\lim\limits_{x\to-3^+} \dfrac{x}{x + 3}$

17. $\lim\limits_{x\to0^+} \dfrac{\text{int } x}{x}$

18. $\lim\limits_{x\to0^-} \dfrac{\text{int } x}{x}$

19. $\lim\limits_{x\to0^+} \csc x$

20. $\lim\limits_{x\to(\pi/2)^+} \sec x$

In Exercises 21–26, find $\lim_{x\to\infty} y$ and $\lim_{x\to-\infty} y$.

21. $y = \left(2 - \dfrac{x}{x + 1}\right)\left(\dfrac{x^2}{5 + x^2}\right)$

22. $y = \left(\dfrac{2}{x} + 1\right)\left(\dfrac{5x^2 - 1}{x^2}\right)$

23. $y = \dfrac{\cos(1/x)}{1 + (1/x)}$

24. $y = \dfrac{2x + \sin x}{x}$

25. $y = \dfrac{\sin x}{2x^2 + x}$

26. $y = \dfrac{x \sin x + 2 \sin x}{2x^2}$

In Exercises 27–34, **(a)** find the vertical asymptotes of the graph of $f(x)$. **(b)** Describe the behavior of $f(x)$ to the left and right of each vertical asymptote.

27. $f(x) = \dfrac{1}{x^2 - 4}$

28. $f(x) = \dfrac{x^2 - 1}{2x + 4}$

29. $f(x) = \dfrac{x^2 - 2x}{x + 1}$

30. $f(x) = \dfrac{1 - x}{2x^2 - 5x - 3}$

31. $f(x) = \cot x$

32. $f(x) = \sec x$

33. $f(x) = \dfrac{\tan x}{\sin x}$

34. $f(x) = \dfrac{\cot x}{\cos x}$

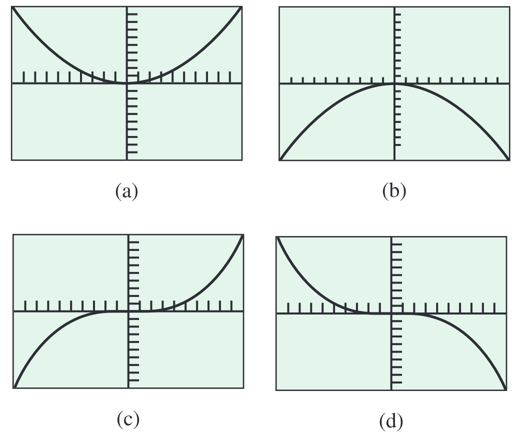
In Exercises 35–38, match the function with the graph of its end behavior model.

35. $y = \dfrac{2x^3 - 3x^2 + 1}{x + 3}$

36. $y = \dfrac{x^5 - x^4 + x + 1}{2x^2 + x - 3}$

37. $y = \dfrac{2x^4 - x^3 + x^2 - 1}{2 - x}$

38. $y = \dfrac{x^4 - 3x^3 + x^2 - 1}{1 - x^2}$



In Exercises 39–44, **(a)** find a power function end behavior model for *f*. **(b)** Identify any horizontal asymptotes.

39. $f(x) = 3x^2 - 2x + 1$

40. $f(x) = -4x^3 + x^2 - 2x - 1$

41. $f(x) = \dfrac{x - 2}{2x^2 + 3x - 5}$

42. $f(x) = \dfrac{3x^2 - x + 5}{x^2 - 4}$

43. $f(x) = \dfrac{4x^3 - 2x + 1}{x - 2}$

44. $f(x) = \dfrac{-x^4 + 2x^2 + x - 3}{x^2 - 4}$

In Exercises 45–48, find **(a)** a simple basic function as a right end behavior model and **(b)** a simple basic function as a left end behavior model for the function.

45. $y = e^x - 2x$

46. $y = x^2 + e^{-x}$

47. $y = x + \ln|x|$

48. $y = x^2 + \sin x$

In Exercises 49–52, use the graph of $y = f(1/x)$ to find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

49. $f(x) = xe^x$

50. $f(x) = x^2e^{-x}$

51. $f(x) = \dfrac{\ln|x|}{x}$

52. $f(x) = x \sin\dfrac{1}{x}$

In Exercises 53 and 54, find the limit of $f(x)$ as **(a)** $x\to-\infty$, **(b)** $x\to\infty$, **(c)** $x\to0^-$, and **(d)** $x\to0^+$.

53. $f(x) = \begin{cases} 1/x, & x < 0 \\ -1, & x \ge 0 \end{cases}$

54. $f(x) = \begin{cases} \dfrac{x - 2}{x - 1}, & x \le 0 \\ 1/x^2, & x > 0 \end{cases}$

Group Activity In Exercises 55 and 56, sketch a graph of a function $y = f(x)$ that satisfies the stated conditions. Include any asymptotes.

55. $\lim\limits_{x\to1} f(x) = 2$, $\lim\limits_{x\to5^-} f(x) = \infty$, $\lim\limits_{x\to5^+} f(x) = \infty$,
$\lim\limits_{x\to\infty} f(x) = -1$, $\lim\limits_{x\to-2^+} f(x) = -\infty$,
$\lim\limits_{x\to-2^-} f(x) = \infty$, $\lim\limits_{x\to-\infty} f(x) = 0$

56. $\lim\limits_{x\to2} f(x) = -1$, $\lim\limits_{x\to4^+} f(x) = -\infty$, $\lim\limits_{x\to4^-} f(x) = \infty$,
$\lim\limits_{x\to\infty} f(x) = \infty$, $\lim\limits_{x\to-\infty} f(x) = 2$

57. Group Activity ***End Behavior Models*** Suppose that $g_1(x)$ is a right end behavior model for $f_1(x)$ and that $g_2(x)$ is a right end behavior model for $f_2(x)$. Explain why this makes $g_1(x)/g_2(x)$ a right end behavior model for $f_1(x)/f_2(x)$.

58. Writing to Learn Let L be a real number, $\lim_{x\to c} f(x) = L$, and $\lim_{x\to c} g(x) = \infty$ or $-\infty$. Can $\lim_{x\to c} (f(x) + g(x))$ be determined? Explain.

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

59. True or False It is possible for a function to have more than one horizontal asymptote. Justify your answer.

60. True or False If $f(x)$ has a vertical asymptote at $x = c$, then either $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = \infty$ or $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = -\infty$. Justify your answer.

61. Multiple Choice $\lim_{x\to 2^-} \dfrac{x}{x-2} =$

(A) $-\infty$ **(B)** ∞ **(C)** 1 **(D)** $-1/2$ **(E)** -1

62. Multiple Choice $\lim_{x\to 0} \dfrac{\cos(2x)}{x} =$

(A) 1/2 **(B)** 1 **(C)** 2 **(D)** cos 2 **(E)** does not exist

63. Multiple Choice $\lim_{x\to 0} \dfrac{\sin(3x)}{x} =$

(A) 1/3 **(B)** 1 **(C)** 3 **(D)** sin 3 **(E)** does not exist

64. Multiple Choice Which of the following is an end behavior for

$$f(x) = \frac{2x^3 - x^2 + x + 1}{x^3 - 1}?$$

(A) x^3 **(B)** $2x^3$ **(C)** $1/x^3$ **(D)** 2 **(E)** 1/2

Exploration

65. ***Exploring Properties of Limits*** Find the limits of f, g, and fg as $x\to c$.

(a) $f(x) = \dfrac{1}{x}, \quad g(x) = x, \quad c = 0$

(b) $f(x) = -\dfrac{2}{x^3}, \quad g(x) = 4x^3, \quad c = 0$

(c) $f(x) = \dfrac{3}{x-2}, \quad g(x) = (x-2)^3, \quad c = 2$

(d) $f(x) = \dfrac{5}{(3-x)^4}, \quad g(x) = (x-3)^2, \quad c = 3$

(e) Writing to Learn Suppose that $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = \infty$. Based on your observations in parts (a)–(d), what can you say about $\lim_{x\to c} (f(x) \cdot g(x))$?

Extending the Ideas

66. ***The Greatest Integer Function***

(a) Show that

$$\frac{x-1}{x} < \frac{\text{int } x}{x} \le 1 \ (x > 0) \ \text{ and } \ \frac{x-1}{x} > \frac{\text{int } x}{x} \ge 1 \ (x < 0).$$

(b) Determine $\lim_{x\to\infty} \dfrac{\text{int } x}{x}$.

(c) Determine $\lim_{x\to -\infty} \dfrac{\text{int } x}{x}$.

67. ***Sandwich Theorem*** Use the Sandwich Theorem to confirm the limit as $x\to\infty$ found in Exercise 3.

68. Writing to Learn Explain why there is no value L for which $\lim_{x\to\infty} \sin x = L$.

In Exercises 69–71, find the limit. Give a convincing argument that the value is correct.

69. $\lim_{x\to\infty} \dfrac{\ln x^2}{\ln x}$

70. $\lim_{x\to\infty} \dfrac{\ln x}{\log x}$

71. $\lim_{x\to\infty} \dfrac{\ln(x+1)}{\ln x}$

Quick Quiz for AP* Preparation: Sections 2.1 and 2.2

You should solve the following problems without using a graphing calculator.

1. Multiple Choice Find $\lim_{x\to 3} \dfrac{x^2 - x - 6}{x - 3}$, if it exists.

(A) -1 **(B)** 1 **(C)** 2 **(D)** 5 **(E)** does not exist

2. Multiple Choice Find $\lim_{x\to 2^+} f(x)$, if it exists, where

$$f(x) = \begin{cases} 3x + 1, & x < 2 \\ \dfrac{5}{x+1}, & x \ge 2 \end{cases}$$

(A) 5/3 **(B)** 13/3 **(C)** 7 **(D)** ∞ **(E)** does not exist

3. Multiple Choice Which of the following lines is a horizontal asymptote for

$$f(x) = \frac{3x^3 - x^2 + x - 7}{2x^3 + 4x - 5}?$$

(A) $y = \dfrac{3}{2}x$ **(B)** $y = 0$ **(C)** $y = 2/3$ **(D)** $y = 7/5$ **(E)** $y = 3/2$

4. Free Response Let $f(x) = \dfrac{\cos x}{x}$.

(a) Find the domain and range of f.

(b) Is f even, odd, or neither? Justify your answer.

(c) Find $\lim_{x\to\infty} f(x)$.

(d) Use the Sandwich Theorem to justify your answer to part (c).

2.3 Continuity

What you'll learn about

- Continuity at a Point
- Continuous Functions
- Algebraic Combinations
- Composites
- Intermediate Value Theorem for Continuous Functions

. . . and why

Continuous functions are used to describe how a body moves through space and how the speed of a chemical reaction changes with time.

Continuity at a Point

When we plot function values generated in the laboratory or collected in the field, we often connect the plotted points with an unbroken curve to show what the function's values are likely to have been at the times we did not measure (Figure 2.16). In doing so, we are assuming that we are working with a *continuous function,* a function whose outputs vary continuously with the inputs and do not jump from one value to another without taking on the values in between. Any function $y = f(x)$ whose graph can be sketched in one continuous motion without lifting the pencil is an example of a continuous function.

Continuous functions are the functions we use to find a planet's closest point of approach to the sun or the peak concentration of antibodies in blood plasma. They are also the functions we use to describe how a body moves through space or how the speed of a chemical reaction changes with time. In fact, so many physical processes proceed continuously that throughout the eighteenth and nineteenth centuries it rarely occurred to anyone to look for any other kind of behavior. It came as a surprise when the physicists of the 1920s discovered that light comes in particles and that heated atoms emit light at discrete frequencies (Figure 2.17). As a result of these and other discoveries, and because of the heavy use of discontinuous functions in computer science, statistics, and mathematical modeling, the issue of continuity has become one of practical as well as theoretical importance.

To understand continuity, we need to consider a function like the one in Figure 2.18, whose limits we investigated in Example 8, Section 2.1.

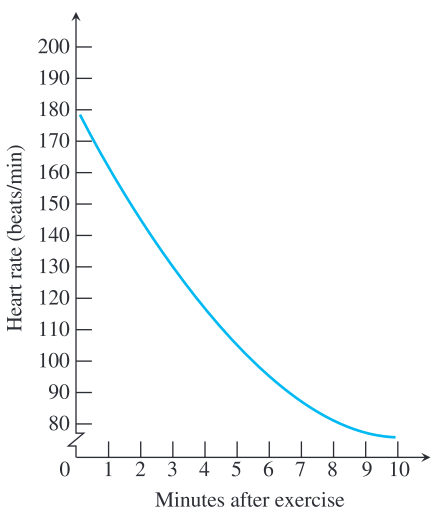


Figure 2.16 How the heartbeat returns to a normal rate after running.

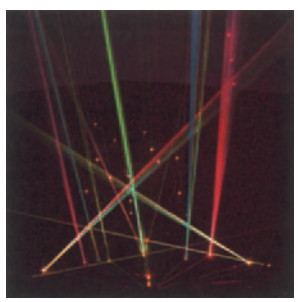


Figure 2.17 The laser was developed as a result of an understanding of the nature of the atom.

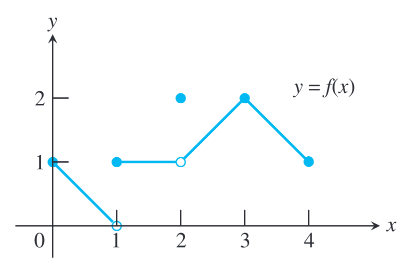


Figure 2.18 The function is continuous on [0, 4] except at $x = 1$ and $x = 2$. (Example 1)

EXAMPLE 1 Investigating Continuity

Find the points at which the function f in Figure 2.18 is continuous, and the points at which f is discontinuous.

SOLUTION

The function f is continuous at every point in its domain [0, 4] except at $x = 1$ and $x = 2$. At these points there are breaks in the graph. Note the relationship between the limit of f and the value of f at each point of the function's domain.

Points at which f is continuous:

At $x = 0$, $\quad \lim_{x\to 0^+} f(x) = f(0)$.

At $x = 4$, $\quad \lim_{x\to 4^-} f(x) = f(4)$.

At $0 < c < 4$, $c \neq 1, 2$, $\quad \lim_{x\to c} f(x) = f(c)$.

continued

Points at which f is discontinuous:

At $x = 1$, $\lim_{x\to 1} f(x)$ does not exist.

At $x = 2$, $\lim_{x\to 2} f(x) = 1$, but $1 \neq f(2)$.

At $c < 0, c > 4$, these points are not in the domain of f.

Now try Exercise 5.

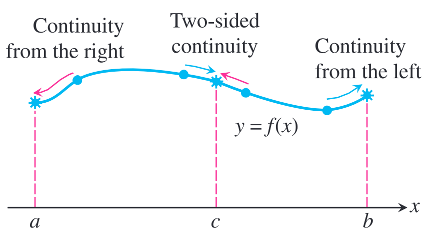


Figure 2.19 Continuity at points a, b, and c for a function $y = f(x)$ that is continuous on the interval $[a, b]$.

To define continuity at a point in a function's domain, we need to define continuity at an interior point (which involves a two-sided limit) and continuity at an endpoint (which involves a one-sided limit). (Figure 2.19)

DEFINITION Continuity at a Point

Interior Point: A function $y = f(x)$ is **continuous at an interior point c** of its domain if

$$\lim_{x\to c} f(x) = f(c).$$

Endpoint: A function $y = f(x)$ is **continuous at a left endpoint a** or is **continuous at a right endpoint b** of its domain if

$$\lim_{x\to a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x\to b^-} f(x) = f(b), \quad \text{respectively.}$$

If a function f is not continuous at a point c, we say that f is **discontinuous** at c and c is a **point of discontinuity** of f. Note that c need not be in the domain of f.

EXAMPLE 2 Finding Points of Continuity and Discontinuity

Find the points of continuity and the points of discontinuity of the greatest integer function (Figure 2.20).

SOLUTION

For the function to be continuous at $x = c$, the limit as $x\to c$ must exist and must equal the value of the function at $x = c$. The greatest integer function is discontinuous at every integer. For example,

$$\lim_{x\to 3^-} \text{int } x = 2 \quad \text{and} \quad \lim_{x\to 3^+} \text{int } x = 3$$

so the limit as $x\to 3$ does not exist. Notice that int $3 = 3$. In general, if n is any integer,

$$\lim_{x\to n^-} \text{int } x = n - 1 \quad \text{and} \quad \lim_{x\to n^+} \text{int } x = n,$$

so the limit as $x\to n$ does not exist.

The greatest integer function is continuous at every other real number. For example,

$$\lim_{x\to 1.5} \text{int } x = 1 = \text{int } 1.5.$$

In general, if $n - 1 < c < n$, n an integer, then

$$\lim_{x\to c} \text{int } x = n - 1 = \text{int } c.$$

Now try Exercise 7.

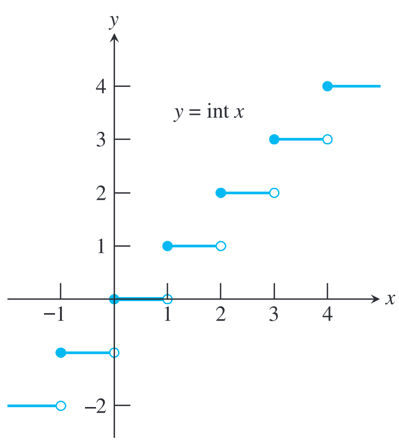


Figure 2.20 The function int x is continuous at every noninteger point. (Example 2)

Shirley Ann Jackson

(1946–)

Distinguished scientist, Shirley Jackson credits her interest in science to her parents and excellent mathematics and science teachers in high school. She studied physics, and in 1973, became the first African American woman to earn a Ph.D. at the Massachusetts Institute of Technology. Since then, Dr. Jackson has done research on topics relating to theoretical material sciences, has received numerous scholarships and honors, and has published more than one hundred scientific articles.

Figure 2.21 is a catalog of discontinuity types. The function in (a) is continuous at $x = 0$. The function in (b) would be continuous if it had $f(0) = 1$. The function in (c) would be continuous if $f(0)$ were 1 instead of 2. The discontinuities in (b) and (c) are **removable.** Each function has a limit as $x \to 0$, and we can remove the discontinuity by setting $f(0)$ equal to this limit.

The discontinuities in (d)–(f) of Figure 2.21 are more serious: $\lim_{x \to 0} f(x)$ does not exist and there is no way to improve the situation by changing f at 0. The step function in (d) has a **jump discontinuity:** the one-sided limits exist but have different values. The function $f(x) = 1/x^2$ in (e) has an **infinite discontinuity.** The function in (f) has an **oscillating discontinuity:** it oscillates and has no limit as $x \to 0$.

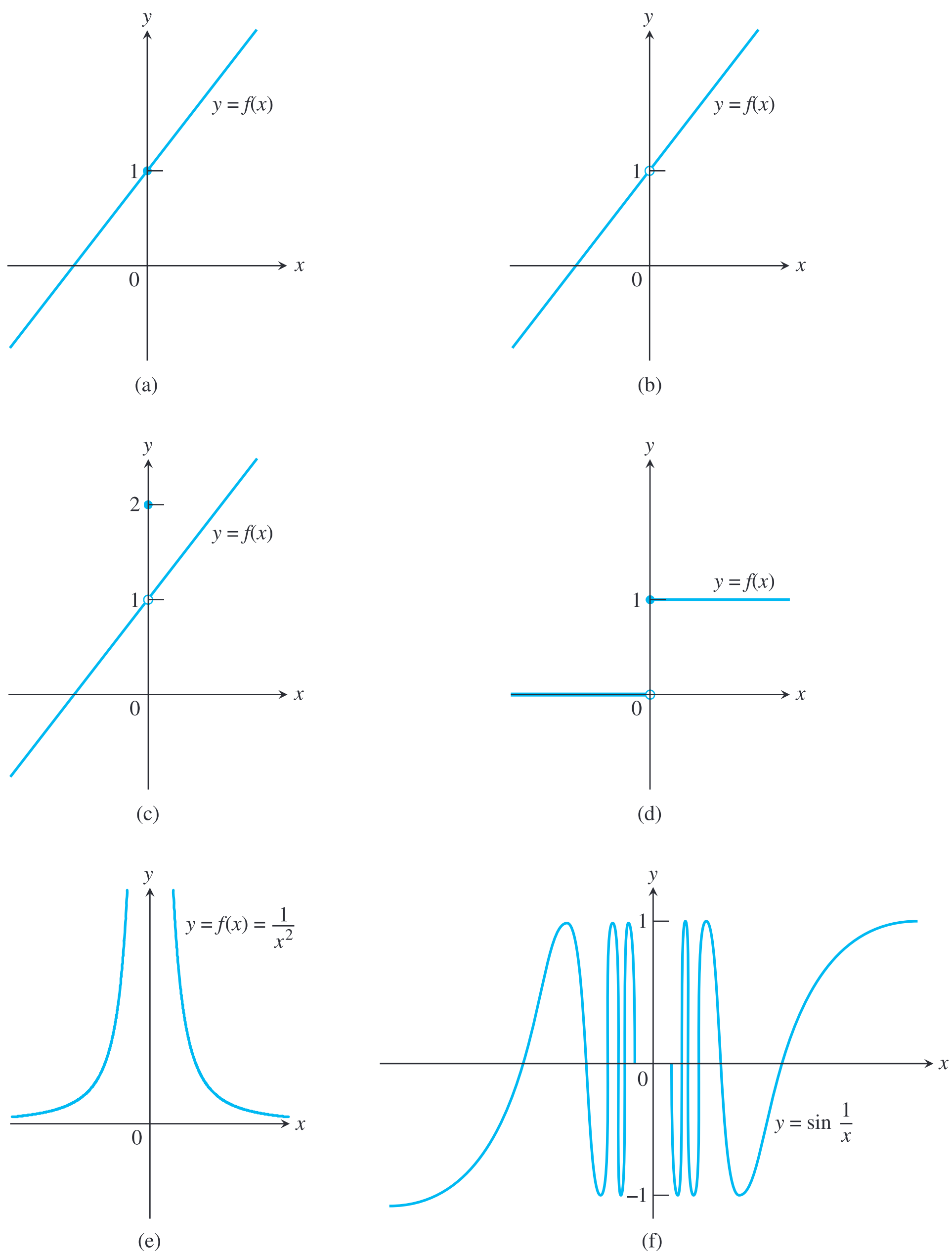


Figure 2.21 The function in part (a) is continuous at $x = 0$. The functions in parts (b)–(f) are not.

EXPLORATION 1 **Removing a Discontinuity**

Let $f(x) = \dfrac{x^3 - 7x - 6}{x^2 - 9}$.

1. Factor the denominator. What is the domain of f?
2. Investigate the graph of f around $x = 3$ to see that f has a removable discontinuity at $x = 3$.
3. How should f be defined at $x = 3$ to remove the discontinuity? Use zoom-in and tables as necessary.
4. Show that $(x - 3)$ is a factor of the numerator of f, and remove all common factors. Now compute the limit as $x \to 3$ of the reduced form for f.
5. Show that the *extended function*

$$g(x) = \begin{cases} \dfrac{x^3 - 7x - 6}{x^2 - 9}, & x \neq 3 \\ 10/3, & x = 3 \end{cases}$$

is continuous at $x = 3$. The function g is the **continuous extension** of the original function f to include $x = 3$.

Now try Exercise 25.

Continuous Functions

A function is **continuous on an interval** if and only if it is continuous at every point of the interval. A **continuous function** is one that is continuous at every point of its domain. A continuous function need not be continuous on every interval. For example, $y = 1/x$ is not continuous on $[-1, 1]$.

EXAMPLE 3 Identifying Continuous Functions

The reciprocal function $y = 1/x$ (Figure 2.22) is a continuous function because it is continuous at every point of its domain. However, it has a point of discontinuity at $x = 0$ because it is not defined there.

Now try Exercise 31.

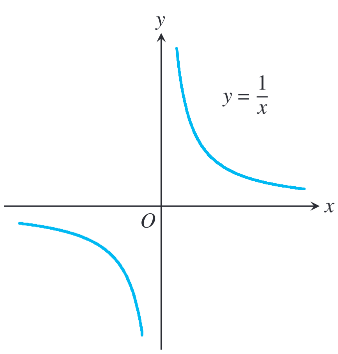


Figure 2.22 The function $y = 1/x$ is continuous at every value of x except $x = 0$. It has a point of discontinuity at $x = 0$. (Example 3)

Polynomial functions f are continuous at every real number c because $\lim_{x\to c} f(x) = f(c)$. Rational functions are continuous at every point of their domains. They have points of discontinuity at the zeros of their denominators. The absolute value function $y = |x|$ is continuous at every real number. The exponential functions, logarithmic functions, trigonometric functions, and radical functions like $y = \sqrt[n]{x}$ (n a positive integer greater than 1) are continuous at every point of their domains. All of these functions are continuous functions.

Algebraic Combinations

As you may have guessed, algebraic combinations of continuous functions are continuous wherever they are defined.

THEOREM 6 Properties of Continuous Functions

If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. *Sums:* $f + g$
2. *Differences:* $f - g$
3. *Products:* $f \cdot g$
4. *Constant multiples:* $k \cdot f$, for any number k
5. *Quotients:* f/g, provided $g(c) \neq 0$

Composites

All composites of continuous functions are continuous. This means composites like

$$y = \sin(x^2) \quad \text{and} \quad y = |\cos x|$$

are continuous at every point at which they are defined. The idea is that if $f(x)$ is continuous at $x = c$ and $g(x)$ is continuous at $x = f(c)$, then $g \circ f$ is continuous at $x = c$ (Figure 2.23). In this case, the limit as $x \to c$ is $g(f(c))$.

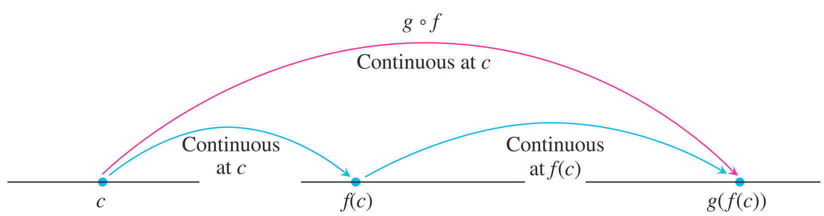
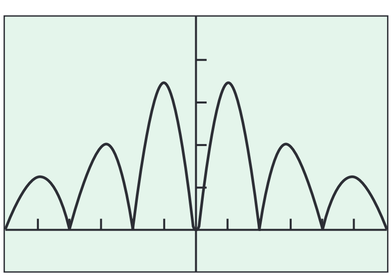


Figure 2.23 Composites of continuous functions are continuous.

THEOREM 7 Composite of Continuous Functions

If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.



$[-3\pi, 3\pi]$ by $[-0.1, 0.5]$

Figure 2.24 The graph suggests that $y = |(x \sin x)/(x^2 + 2)|$ is continuous. (Example 4)

EXAMPLE 4 Using Theorem 7

Show that $y = \left|\dfrac{x \sin x}{x^2 + 2}\right|$ is continuous.

SOLUTION

The graph (Figure 2.24) of $y = |(x \sin x)/(x^2 + 2)|$ suggests that the function is continuous at every value of x. By letting

$$g(x) = |x| \quad \text{and} \quad f(x) = \frac{x \sin x}{x^2 + 2},$$

we see that y is the composite $g \circ f$.

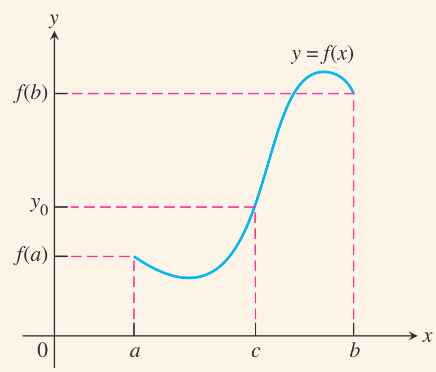
We know that the absolute value function g is continuous. The function f is continuous by Theorem 6. Their composite is continuous by Theorem 7. ***Now try Exercise 33.***

Intermediate Value Theorem for Continuous Functions

Functions that are continuous on intervals have properties that make them particularly useful in mathematics and its applications. One of these is the *intermediate value property.* A function is said to have the **intermediate value property** if it never takes on two values without taking on all the values in between.

THEOREM 8 The Intermediate Value Theorem for Continuous Functions

A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



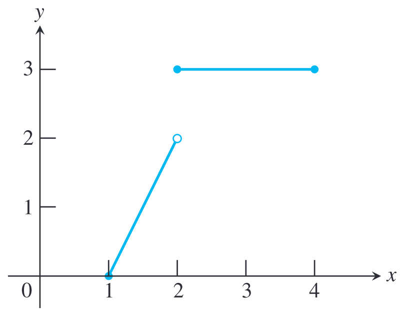


Figure 2.25 The function

$$f(x) = \begin{cases} 2x - 2, & 1 \le x < 2 \\ 3, & 2 \le x \le 4 \end{cases}$$

does not take on all values between $f(1) = 0$ and $f(4) = 3$; it misses all the values between 2 and 3.

The continuity of f on the interval is essential to Theorem 8. If f is discontinuous at even one point of the interval, the theorem's conclusion may fail, as it does for the function graphed in Figure 2.25.

A Consequence for Graphing: Connectivity Theorem 8 is the reason why the graph of a function continuous on an interval cannot have any breaks. The graph will be **connected,** a single, unbroken curve, like the graph of sin x. It will not have jumps like those in the graph of the greatest integer function int x, or separate branches like we see in the graph of $1/x$.

Most graphers can plot points *(dot mode)*. Some can turn on pixels between plotted points to suggest an unbroken curve *(connected mode).* For functions, the connected format basically assumes that outputs *vary continuously* with inputs and do not jump from one value to another without taking on all values in between.

Grapher Failure

In connected mode, a grapher may conceal a function's discontinuities by portraying the graph as a connected curve when it is not. To see what we mean, graph $y = \text{int}\,(x)$ in a $[-10, 10]$ by $[-10, 10]$ window in both connected and dot modes. A knowledge of where to expect discontinuities will help you recognize this form of grapher failure.

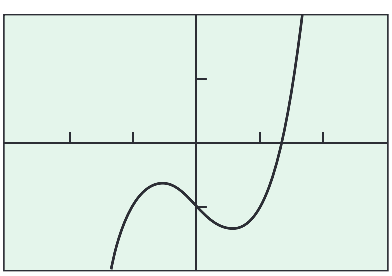
EXAMPLE 5 Using Theorem 8

Is any real number exactly 1 less than its cube?

SOLUTION

We answer this question by applying the Intermediate Value Theorem in the following way. Any such number must satisfy the equation $x = x^3 - 1$ or, equivalently, $x^3 - x - 1 = 0$. Hence, we are looking for a zero value of the continuous function $f(x) = x^3 - x - 1$ (Figure 2.26). The function changes sign between 1 and 2, so there must be a point c between 1 and 2 where $f(c) = 0$.

Now try Exercise 46.



[–3, 3] by [–2, 2]

Figure 2.26 The graph of $f(x) = x^3 - x - 1$. (Example 5)

Quick Review 2.3 *(For help, go to Sections 1.2 and 2.1.)*

1. Find $\lim_{x \to -1} \dfrac{3x^2 - 2x + 1}{x^3 + 4}$.

2. Let $f(x) = \text{int } x$. Find each limit.

(a) $\lim_{x \to -1^-} f(x)$ **(b)** $\lim_{x \to -1^+} f(x)$ **(c)** $\lim_{x \to -1} f(x)$ **(d)** $f(-1)$

3. Let $f(x) = \begin{cases} x^2 - 4x + 5, & x < 2 \\ 4 - x, & x \ge 2. \end{cases}$

Find each limit.

(a) $\lim_{x \to 2^-} f(x)$ **(b)** $\lim_{x \to 2^+} f(x)$ **(c)** $\lim_{x \to 2} f(x)$ **(d)** $f(2)$

In Exercises 4–6, find the remaining functions in the list of functions: $f, g, f \circ g, g \circ f$.

4. $f(x) = \dfrac{2x - 1}{x + 5}, \quad g(x) = \dfrac{1}{x} + 1$

5. $f(x) = x^2, \quad (g \circ f)(x) = \sin x^2, \quad \text{domain of } g = [0, \infty)$

6. $g(x) = \sqrt{x - 1}, \quad (g \circ f)(x) = 1/x, \quad x > 0$

7. Use factoring to solve $2x^2 + 9x - 5 = 0$.

8. Use graphing to solve $x^3 + 2x - 1 = 0$.

In Exercises 9 and 10, let

$$f(x) = \begin{cases} 5 - x, & x \le 3 \\ -x^2 + 6x - 8, & x > 3. \end{cases}$$

9. Solve the equation $f(x) = 4$.

10. Find a value of c for which the equation $f(x) = c$ has no solution.

Section 2.3 Exercises

In Exercises 1–10, find the points of continuity and the points of discontinuity of the function. Identify each type of discontinuity.

1. $y = \dfrac{1}{(x + 2)^2}$

2. $y = \dfrac{x + 1}{x^2 - 4x + 3}$

3. $y = \dfrac{1}{x^2 + 1}$

4. $y = |x - 1|$

5. $y = \sqrt{2x + 3}$

6. $y = \sqrt[3]{2x - 1}$

7. $y = |x|/x$

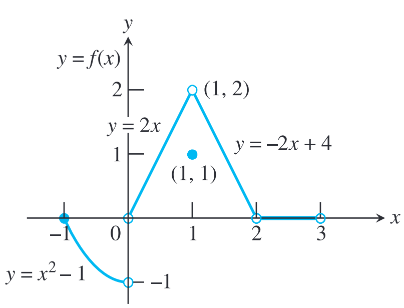
8. $y = \cot x$

9. $y = e^{1/x}$

10. $y = \ln(x + 1)$

In Exercises 11–18, use the function f defined and graphed below to answer the questions.

$$f(x) = \begin{cases} x^2 - 1, & -1 \le x < 0 \\ 2x, & 0 < x < 1 \\ 1, & x = 1 \\ -2x + 4, & 1 < x < 2 \\ 0, & 2 < x < 3 \end{cases}$$



11. (a) Does $f(-1)$ exist?

(b) Does $\lim_{x \to -1^+} f(x)$ exist?

(c) Does $\lim_{x \to -1^+} f(x) = f(-1)$?

(d) Is f continuous at $x = -1$?

12. (a) Does $f(1)$ exist?

(b) Does $\lim_{x \to 1} f(x)$ exist?

(c) Does $\lim_{x \to 1} f(x) = f(1)$?

(d) Is f continuous at $x = 1$?

13. (a) Is f defined at $x = 2$? (Look at the definition of f.)

(b) Is f continuous at $x = 2$?

14. At what values of x is f continuous?

15. What value should be assigned to $f(2)$ to make the extended function continuous at $x = 2$?

16. What new value should be assigned to $f(1)$ to make the new function continuous at $x = 1$?

17. Writing to Learn Is it possible to extend f to be continuous at $x = 0$? If so, what value should the extended function have there? If not, why not?

18. Writing to Learn Is it possible to extend f to be continuous at $x = 3$? If so, what value should the extended function have there? If not, why not?

In Exercises 19–24, **(a)** find each point of discontinuity. **(b)** Which of the discontinuities are removable? not removable? Give reasons for your answers.

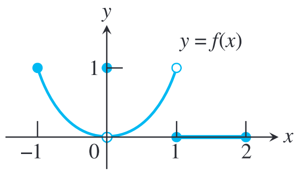
19. $f(x) = \begin{cases} 3 - x, & x < 2 \\ \dfrac{x}{2} + 1, & x > 2 \end{cases}$

20. $f(x) = \begin{cases} 3 - x, & x < 2 \\ 2, & x = 2 \\ x/2, & x > 2 \end{cases}$

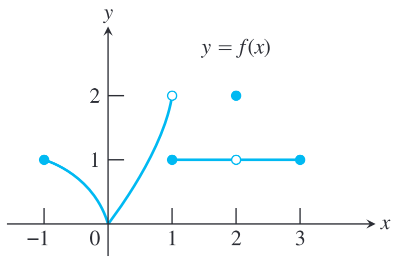
21. $f(x) = \begin{cases} \dfrac{1}{x - 1}, & x < 1 \\ x^3 - 2x + 5, & x \ge 1 \end{cases}$

22. $f(x) = \begin{cases} 1 - x^2, & x \ne -1 \\ 2, & x = -1 \end{cases}$

23.



24.



In Exercises 25–30, give a formula for the extended function that is continuous at the indicated point.

25. $f(x) = \dfrac{x^2 - 9}{x + 3}, \quad x = -3$

26. $f(x) = \dfrac{x^3 - 1}{x^2 - 1}, \quad x = 1$

27. $f(x) = \dfrac{\sin x}{x}, \quad x = 0$

28. $f(x) = \dfrac{\sin 4x}{x}, \quad x = 0$

29. $f(x) = \dfrac{x - 4}{\sqrt{x} - 2}, \quad x = 4$

30. $f(x) = \dfrac{x^3 - 4x^2 - 11x + 30}{x^2 - 4}, \quad x = 2$

In Exercises 31 and 32, explain why the given function is continuous.

31. $f(x) = \dfrac{1}{x - 3}$

32. $g(x) = \dfrac{1}{\sqrt{x - 1}}$

In Exercises 33–36, use Theorem 7 to show that the given function is continuous.

33. $f(x) = \sqrt{\left(\dfrac{x}{x + 1}\right)}$

34. $f(x) = \sin(x^2 + 1)$

35. $f(x) = \cos(\sqrt[3]{1 - x})$

36. $f(x) = \tan\left(\dfrac{x^2}{x^2 + 4}\right)$

Group Activity In Exercises 37–40, verify that the function is continuous and state its domain. Indicate which theorems you are using, and which functions you are assuming to be continuous.

37. $y = \dfrac{1}{\sqrt{x + 2}}$

38. $y = x^2 + \sqrt[3]{4 - x}$

39. $y = |x^2 - 4x|$

40. $y = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1 \\ 2, & x = 1 \end{cases}$

In Exercises 41–44, sketch a possible graph for a function f that has the stated properties.

41. $f(3)$ exists but $\lim_{x \to 3} f(x)$ does not.

42. $f(-2)$ exists, $\lim_{x \to -2^+} f(x) = f(-2)$, but $\lim_{x \to -2} f(x)$ does not exist.

43. $f(4)$ exists, $\lim_{x \to 4} f(x)$ exists, but f is not continuous at $x = 4$.

44. $f(x)$ is continuous for all x except $x = 1$, where f has a nonremovable discontinuity.

45. ***Solving Equations*** Is any real number exactly 1 less than its fourth power? Give any such values accurate to 3 decimal places.

46. ***Solving Equations*** Is any real number exactly 2 more than its cube? Give any such values accurate to 3 decimal places.

47. ***Continuous Function*** Find a value for a so that the function

$$f(x) = \begin{cases} x^2 - 1, & x < 3 \\ 2ax, & x \geq 3 \end{cases}$$

is continuous.

48. ***Continuous Function*** Find a value for a so that the function

$$f(x) = \begin{cases} 2x + 3, & x \leq 2 \\ ax + 1, & x > 2 \end{cases}$$

is continuous.

49. ***Continuous Function*** Find a value for a so that the function

$$f(x) = \begin{cases} 4 - x^2, & x < -1 \\ ax^2 - 1, & x \geq -1 \end{cases}$$

is continuous.

50. ***Continuous Function*** Find a value for a so that the function

$$f(x) = \begin{cases} x^2 + x + a, & x < 1 \\ x^3, & x \geq 1 \end{cases}$$

is continuous.

51. **Writing to Learn** Explain why the equation $e^{-x} = x$ has at least one solution.

52. ***Salary Negotiation*** A welder's contract promises a 3.5% salary increase each year for 4 years and Luisa has an initial salary of $36,500.

(a) Show that Luisa's salary is given by

$$y = 36{,}500(1.035)^{\text{int } t},$$

where t is the time, measured in years, since Luisa signed the contract.

(b) Graph Luisa's salary function. At what values of t is it continuous?

53. ***Airport Parking*** Valuepark charge $1.10 per hour or fraction of an hour for airport parking. The maximum charge per day is $7.25.

(a) Write a formula that gives the charge for x hours with $0 \leq x \leq 24$. (*Hint:* See Exercise 52.)

(b) Graph the function in part (a). At what values of x is it continuous?

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

54. **True or False** A continuous function cannot have a point of discontinuity. Justify your answer.

55. **True or False** It is possible to extend the definition of a function f at a jump discontinuity $x = a$ so that f is continuous at $x = a$. Justify your answer.

56. **Multiple Choice** On which of the following intervals is $f(x) = \dfrac{1}{\sqrt{x}}$ not continuous?

(A) $(0, \infty)$ **(B)** $[0, \infty)$ **(C)** $(0, 2)$
(D) $(1, 2)$ **(E)** $[1, \infty)$

57. **Multiple Choice** Which of the following points is not a point of discontinuity of $f(x) = \sqrt{x-1}$?

(A) $x = -1$ **(B)** $x = -1/2$ **(C)** $x = 0$
(D) $x = 1/2$ **(E)** $x = 1$

58. **Multiple Choice** Which of the following statements about the function

$$f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 1, & x = 1 \\ -x + 3, & 1 < x < 2 \end{cases}$$

is not true?

(A) $f(1)$ does not exist.
(B) $\lim_{x \to 0^+} f(x)$ exists.
(C) $\lim_{x \to 2^-} f(x)$ exists.
(D) $\lim_{x \to 1} f(x)$ exists.
(E) $\lim_{x \to 1} f(x) = f(1)$

59. **Multiple Choice** Which of the following points of discontinuity of

$$f(x) = \frac{x(x-1)(x-2)^2(x+1)^2(x-3)^2}{x(x-1)(x-2)(x+1)^2(x-3)^3}$$

is not removable?

(A) $x = -1$ **(B)** $x = 0$ **(C)** $x = 1$
(D) $x = 2$ **(E)** $x = 3$

Exploration

60. Let $f(x) = \left(1 + \dfrac{1}{x}\right)^x$.

(a) Find the domain of f. **(b)** Draw the graph of f.

(c) **Writing to Learn** Explain why $x = -1$ and $x = 0$ are points of discontinuity of f.

(d) **Writing to Learn** Are either of the discontinuities in part (c) removable? Explain.

(e) Use graphs and tables to estimate $\lim_{x \to \infty} f(x)$.

Extending the Ideas

61. ***Continuity at a Point*** Show that $f(x)$ is continuous at $x = a$ if and only if

$$\lim_{h \to 0} f(a + h) = f(a).$$

62. ***Continuity on Closed Intervals*** Let f be continuous and never zero on $[a, b]$. Show that either $f(x) > 0$ for all x in $[a, b]$ or $f(x) < 0$ for all x in $[a, b]$.

63. ***Properties of Continuity*** Prove that if f is continuous on an interval, then so is $|f|$.

64. ***Everywhere Discontinuous*** Give a convincing argument that the following function is not continuous at any real number.

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

2.4 Rates of Change and Tangent Lines

What you'll learn about

- Average Rates of Change
- Tangent to a Curve
- Slope of a Curve
- Normal to a Curve
- Speed Revisited

. . . and why

The tangent line determines the direction of a body's motion at every point along its path.

Average Rates of Change

We encounter average rates of change in such forms as average speed (in miles per hour), growth rates of populations (in percent per year), and average monthly rainfall (in inches per month). The **average rate of change** of a quantity over a period of time is the amount of change divided by the time it takes. In general, the *average rate of change* of a function over an interval is the amount of change divided by the length of the interval.

EXAMPLE 1 Finding Average Rate of Change

Find the average rate of change of $f(x) = x^3 - x$ over the interval [1, 3].

SOLUTION

Since $f(1) = 0$ and $f(3) = 24$, the average rate of change over the interval [1, 3] is

$$\frac{f(3) - f(1)}{3 - 1} = \frac{24 - 0}{2} = 12.$$

Now try Exercise 1.

Experimental biologists often want to know the rates at which populations grow under controlled laboratory conditions. Figure 2.27 shows how the number of fruit flies *(Drosophila)* grew in a controlled 50-day experiment. The graph was made by counting flies at regular intervals, plotting a point for each count, and drawing a smooth curve through the plotted points.

Secant to a Curve

A line through two points on a curve is a **secant to the curve.**

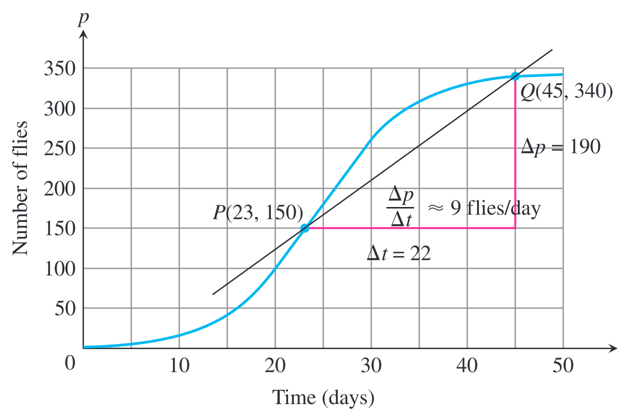


Figure 2.27 Growth of a fruit fly population in a controlled experiment.
Source: Elements of Mathematical Biology. (Example 2)

Marjorie Lee Browne (1914–1979)

When Marjorie Browne graduated from the University of Michigan in 1949, she was one of the first two African American women to be awarded a Ph.D. in Mathematics. Browne went on to become chairperson of the mathematics department at North Carolina Central University, and succeeded in obtaining grants for retraining high school mathematics teachers.

EXAMPLE 2 Growing Drosophila in a Laboratory

Use the points $P(23, 150)$ and $Q(45, 340)$ in Figure 2.27 to compute the average rate of change and the slope of the secant line PQ.

SOLUTION

There were 150 flies on day 23 and 340 flies on day 45. This gives an increase of $340 - 150 = 190$ flies in $45 - 23 = 22$ days.

The average rate of change in the population p from day 23 to day 45 was

$$\textit{Average rate of change:}\ \frac{\Delta p}{\Delta t} = \frac{340 - 150}{45 - 23} = \frac{190}{22} \approx 8.6 \text{ flies/day,}$$

or about 9 flies per day.

continued

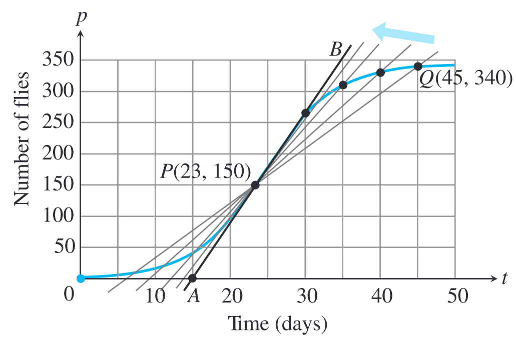
This average rate of change is also the slope of the secant line through the two points P and Q on the population curve. We can calculate the slope of the secant PQ from the coordinates of P and Q.

$$\textit{Secant slope:}\quad \frac{\Delta p}{\Delta t} = \frac{340 - 150}{45 - 23} = \frac{190}{22} \approx 8.6 \text{ flies/day}$$

Now try Exercise 7.

As suggested by Example 2, *we can always think of an average rate of change as the slope of a secant line.*

In addition to knowing the average rate at which the population grew from day 23 to day 45, we may also want to know how fast the population was growing on day 23 itself. To find out, we can watch the slope of the secant PQ change as we back Q along the curve toward P. The results for four positions of Q are shown in Figure 2.28.



(a)

Q	Slope of $PQ = \Delta p/\Delta t$ (flies/day)
(45, 340)	(340 – 150)/(45 – 23) ≈ 8.6
(40, 330)	(330 – 150)/(40 – 23) ≈ 10.6
(35, 310)	(310 – 150)/(35 – 23) ≈ 13.3
(30, 265)	(265 – 150)/(30 – 23) ≈ 16.4

(b)

Figure 2.28 (a) Four secants to the fruit fly graph of Figure 2.27, through the point $P(23, 150)$. (b) The slopes of the four secants.

In terms of geometry, what we see as Q approaches P along the curve is this: The secant PQ approaches the tangent line AB that we drew by eye at P. This means that within the limitations of our drawing, the slopes of the secants approach the slope of the tangent, which we calculate from the coordinates of A and B to be

$$\frac{350 - 0}{35 - 15} = 17.5 \text{ flies/day.}$$

In terms of population, what we see as Q approaches P is this: The average growth rates for increasingly smaller time intervals approach the slope of the tangent to the curve at P (17.5 flies per day). The slope of the tangent line is therefore the number we take as the rate at which the fly population was growing on day $t = 23$.

Tangent to a Curve

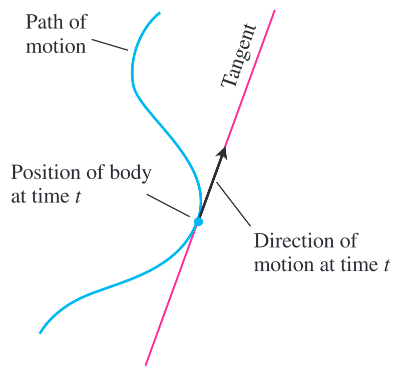
The moral of the fruit fly story would seem to be that we should define the rate at which the value of the function $y = f(x)$ is changing with respect to x at any particular value $x = a$ to be the slope of the tangent to the curve $y = f(x)$ at $x = a$. But how are we to define the tangent line at an arbitrary point P on the curve and find its slope from the formula $y = f(x)$? The problem here is that we know only one point. Our usual definition of slope requires two points.

The solution that mathematician Pierre Fermat found in 1629 proved to be one of that century's major contributions to calculus. We still use his method of defining tangents to produce formulas for slopes of curves and rates of change:

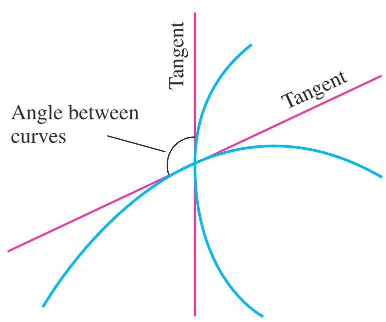
1. We start with what we can calculate, namely, the slope of a secant through P and a point Q nearby on the curve.

Why Find Tangents to Curves?

In mechanics, the tangent determines the direction of a body's motion at every point along its path.



In geometry, the tangents to two curves at a point of intersection determine the angle at which the curves intersect.



In optics, the tangent determines the angle at which a ray of light enters a curved lens (more about this in Section 3.7). The problem of how to find a tangent to a curve became the dominant mathematical problem of the early seventeenth century and it is hard to overestimate how badly the scientists of the day wanted to know the answer. Descartes went so far as to say that the problem was the most useful and most general problem not only that he knew but that he had any desire to know.

2. We find the limiting value of the secant slope (if it exists) as Q approaches P along the curve.

3. We define the *slope of the curve* at P to be this number and define the *tangent to the curve* at P to be the line through P with this slope.

EXAMPLE 3 Finding Slope and Tangent Line

Find the slope of the parabola $y = x^2$ at the point $P(2, 4)$. Write an equation for the tangent to the parabola at this point.

SOLUTION

We begin with a secant line through $P(2, 4)$ and a nearby point $Q(2 + h, (2 + h)^2)$ on the curve (Figure 2.29).

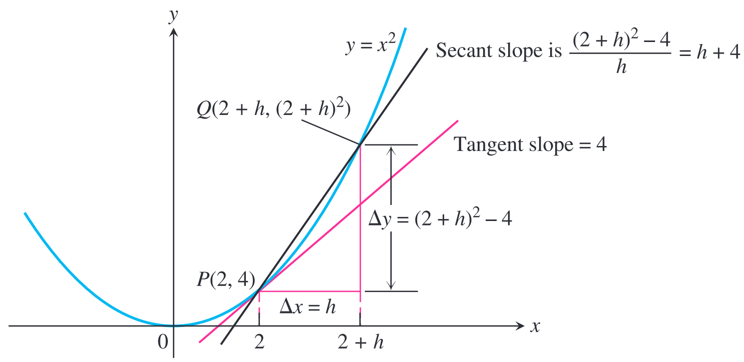


Figure 2.29 The slope of the tangent to the parabola $y = x^2$ at $P(2, 4)$ is 4.

We then write an expression for the slope of the secant line and find the limiting value of this slope as Q approaches P along the curve.

$$\text{Secant slope} = \frac{\Delta y}{\Delta x} = \frac{(2+h)^2 - 4}{h}$$
$$= \frac{h^2 + 4h + 4 - 4}{h}$$
$$= \frac{h^2 + 4h}{h} = h + 4$$

The limit of the secant slope as Q approaches P along the curve is

$$\lim_{Q \to P} (\text{secant slope}) = \lim_{h \to 0} (h + 4) = 4.$$

Thus, the slope of the parabola at P is 4.

The tangent to the parabola at P is the line through $P(2, 4)$ with slope $m = 4$.

$$y - 4 = 4(x - 2)$$
$$y = 4x - 8 + 4$$
$$y = 4x - 4$$

Now try Exercise 11 (a, b).

Slope of a Curve

To find the tangent to a curve $y = f(x)$ at a point $P(a, f(a))$ we use the same dynamic procedure. We calculate the slope of the secant line through P and a point $Q(a + h, f(a + h))$. We then investigate the limit of the slope as $h \to 0$ (Figure 2.30). If the limit exists, it is the slope of the curve at P and we define the tangent at P to be the line through P having this slope.

Pierre de Fermat
(1601–1665)

The dynamic approach to tangency, invented by Fermat in 1629, proved to be one of the seventeenth century's major contributions to calculus.

Fermat, a skilled linguist and one of his century's greatest mathematicians, tended to confine his writing to professional correspondence and to papers written for personal friends. He rarely wrote completed descriptions of his work, even for his personal use. His name slipped into relative obscurity until the late 1800s, and it was only from a four-volume edition of his works published at the beginning of this century that the true importance of his many achievements became clear.

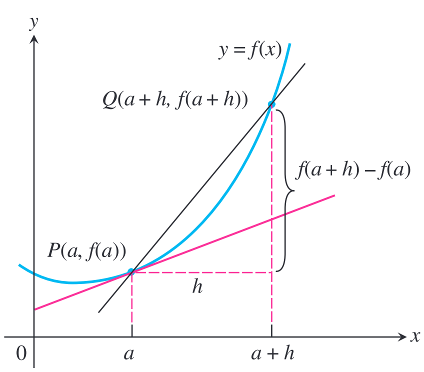


Figure 2.30 The tangent slope is

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

DEFINITION Slope of a Curve at a Point

The **slope of the curve** $y = f(x)$ at the point $P(a, f(a))$ is the number

$$m = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h},$$

provided the limit exists.

The **tangent line to the curve** at P is the line through P with this slope.

EXAMPLE 4 Exploring Slope and Tangent

Let $f(x) = 1/x$.

(a) Find the slope of the curve at $x = a$.
(b) Where does the slope equal $-1/4$?
(c) What happens to the tangent to the curve at the point $(a, 1/a)$ for different values of a?

SOLUTION

(a) The slope at $x = a$ is

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{a+h} - \dfrac{1}{a}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h} \cdot \frac{a - (a+h)}{a(a+h)}$$

$$= \lim_{h\to 0} \cdot \frac{-h}{ha(a+h)}$$

$$= \lim_{h\to 0} \frac{-1}{a(a+h)} = -\frac{1}{a^2}.$$

(b) The slope will be $-1/4$ if

$$-\frac{1}{a^2} = -\frac{1}{4}$$

$$a^2 = 4$$

$$a = \pm 2.$$

The curve has the slope $-1/4$ at the two points $(2, 1/2)$ and $(-2, -1/2)$ (Figure 2.31).
(c) The slope $-1/a^2$ is always negative. As $a \to 0^+$, the slope approaches $-\infty$ and the tangent becomes increasingly steep. We see this again as $a \to 0^-$. As a moves away from the origin in either direction, the slope approaches 0 and the tangent becomes increasingly horizontal. ***Now try Exercise 19.***

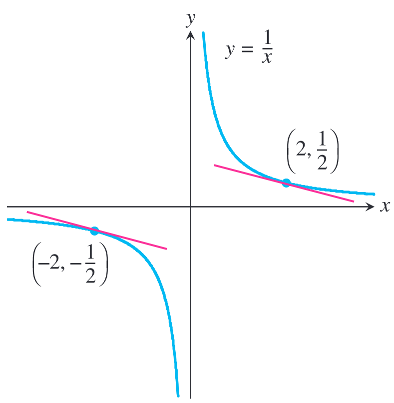


Figure 2.31 The two tangent lines to $y = 1/x$ having slope $-1/4$. (Example 4)

The expression

$$\frac{f(a+h) - f(a)}{h}$$

is the **difference quotient of *f* at *a*.** Suppose the difference quotient has a limit as h approaches zero. If we interpret the difference quotient as a secant slope, the limit is the slope of both the curve and the tangent to the curve at the point $x = a$. If we interpret the difference quotient as an average rate of change, the limit is the function's rate of change with respect to x at the point $x = a$. This limit is one of the two most important mathematical objects considered in calculus. We will begin a thorough study of it in Chapter 3.

All of these are the same:

1. the slope of $y = f(x)$ at $x = a$
2. the slope of the tangent to $y = f(x)$ at $x = a$
3. the (instantaneous) rate of change of $f(x)$ with respect to x at $x = a$
4. $\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$

About the Word *Normal*

When analytic geometry was developed in the seventeenth century, European scientists still wrote about their work and ideas in Latin, the one language that all educated Europeans could read and understand. The Latin word *normalis,* which scholars used for *perpendicular,* became *normal* when they discussed geometry in English.

Normal to a Curve

The **normal line** to a curve at a point is the line perpendicular to the tangent at that point.

EXAMPLE 5 Finding a Normal Line

Write an equation for the normal to the curve $f(x) = 4 - x^2$ at $x = 1$.

SOLUTION

The slope of the tangent to the curve at $x = 1$ is

$$\begin{aligned}\lim_{h\to 0}\frac{f(1+h)-f(1)}{h} &= \lim_{h\to 0}\frac{4-(1+h)^2-3}{h}\\ &= \lim_{h\to 0}\frac{4-1-2h-h^2-3}{h}\\ &= \lim_{h\to 0}\frac{-h(2+h)}{h} = -2.\end{aligned}$$

Thus, the slope of the normal is $1/2$, the negative reciprocal of -2. The normal to the curve at $(1, f(1)) = (1, 3)$ is the line through $(1, 3)$ with slope $m = 1/2$.

$$\begin{aligned}y - 3 &= \frac{1}{2}(x-1)\\ y &= \frac{1}{2}x - \frac{1}{2} + 3\\ y &= \frac{1}{2}x + \frac{5}{2}\end{aligned}$$

You can support this result by drawing the graphs in a square viewing window.

Now try Exercise 11 (c, d).

Speed Revisited

The function $y = 16t^2$ that gave the distance fallen by the rock in Example 1, Section 2.1, was the rock's *position function.* A body's average speed along a coordinate axis (here, the y-axis) for a given period of time is the average rate of change of its *position* $y = f(t)$. Its **instantaneous speed** at any time t is the **instantaneous rate of change** of position with respect to time at time t, or

$$\lim_{h\to 0}\frac{f(t+h)-f(t)}{h}.$$

We saw in Example 1, Section 2.1, that the rock's instantaneous speed at $t = 2$ sec was 64 ft/sec.

Particle Motion

We only have considered objects moving in one direction in this chapter. In Chapter 3, we will deal with more complicated motion.

EXAMPLE 6 Investigating Free Fall

Find the speed of the falling rock in Example 1, Section 2.1, at $t = 1$ sec.

SOLUTION

The position function of the rock is $f(t) = 16t^2$. The average speed of the rock over the interval between $t = 1$ and $t = 1 + h$ sec was

$$\frac{f(1+h)-f(1)}{h} = \frac{16(1+h)^2-16(1)^2}{h} = \frac{16(h^2+2h)}{h} = 16(h+2).$$

The rock's speed at the instant $t = 1$ was

$$\lim_{h\to 0} 16(h+2) = 32 \text{ ft/sec}.$$

Now try Exercise 27.

Quick Review 2.4 *(For help, go to Section 1.1.)*

In Exercises 1 and 2, find the increments Δx and Δy from point A to point B.

1. $A(-5, 2)$, $B(3, 5)$ **2.** $A(1, 3)$, $B(a, b)$

In Exercises 3 and 4, find the slope of the line determined by the points.

3. $(-2, 3)$, $(5, -1)$ **4.** $(-3, -1)$, $(3, 3)$

In Exercises 5–9, write an equation for the specified line.

5. through $(-2, 3)$ with slope $= 3/2$

6. through $(1, 6)$ and $(4, -1)$

7. through $(1, 4)$ and parallel to $y = -\frac{3}{4}x + 2$

8. through $(1, 4)$ and perpendicular to $y = -\frac{3}{4}x + 2$

9. through $(-1, 3)$ and parallel to $2x + 3y = 5$

10. For what value of b will the slope of the line through $(2, 3)$ and $(4, b)$ be $5/3$?

Section 2.4 Exercises

In Exercises 1–6, find the average rate of change of the function over each interval.

1. $f(x) = x^3 + 1$
 (a) $[2, 3]$ **(b)** $[-1, 1]$

2. $f(x) = \sqrt{4x + 1}$
 (a) $[0, 2]$ **(b)** $[10, 12]$

3. $f(x) = e^x$
 (a) $[-2, 0]$ **(b)** $[1, 3]$

4. $f(x) = \ln x$
 (a) $[1, 4]$ **(b)** $[100, 103]$

5. $f(x) = \cot t$
 (a) $[\pi/4, 3\pi/4]$ **(b)** $[\pi/6, \pi/2]$

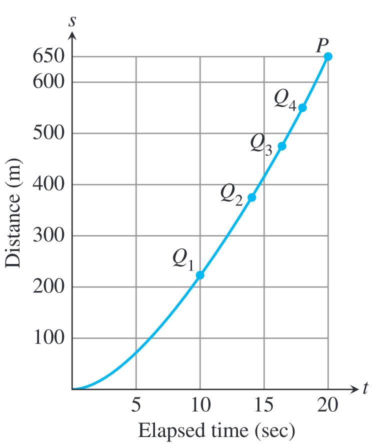
6. $f(x) = 2 + \cos t$
 (a) $[0, \pi]$ **(b)** $[-\pi, \pi]$

In Exercises 7 and 8, a distance-time graph is shown.

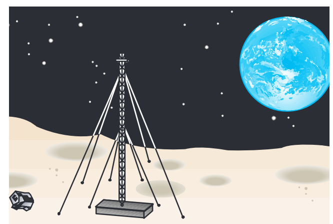
(a) Estimate the slopes of the secants PQ_1, PQ_2, PQ_3, and PQ_4, arranging them in order in a table. What is the appropriate unit for these slopes?

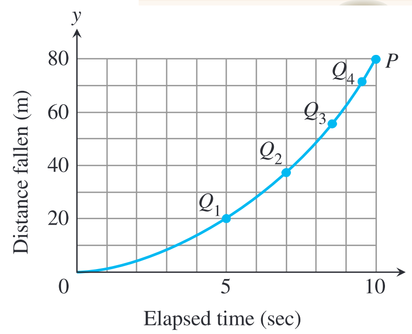
(b) Estimate the speed at point P.

7. ***Accelerating from a Standstill*** The figure shows the distance-time graph for a 1994 Ford® Mustang Cobra™ accelerating from a standstill.



8. ***Lunar Data*** The accompanying figure shows a distance-time graph for a wrench that fell from the top platform of a communication mast on the moon to the station roof 80 m below.





In Exercises 9–12, at the indicated point find

(a) the slope of the curve,

(b) an equation of the tangent, and

(c) an equation of the normal.

(d) Then draw a graph of the curve, tangent line, and normal line in the same square viewing window.

9. $y = x^2$ at $x = -2$

10. $y = x^2 - 4x$ at $x = 1$

11. $y = \dfrac{1}{x - 1}$ at $x = 2$

12. $y = x^2 - 3x - 1$ at $x = 0$

In Exercises 13 and 14, find the slope of the curve at the indicated point.

13. $f(x) = |x|$ at **(a)** $x = 2$ **(b)** $x = -3$

14. $f(x) = |x - 2|$ at $x = 1$

In Exercises 15–18, determine whether the curve has a tangent at the indicated point. If it does, give its slope. If not, explain why not.

15. $f(x) = \begin{cases} 2 - 2x - x^2, & x < 0 \\ 2x + 2, & x \geq 0 \end{cases}$ at $x = 0$

16. $f(x) = \begin{cases} -x, & x < 0 \\ x^2 - x, & x \geq 0 \end{cases}$ at $x = 0$

17. $f(x) = \begin{cases} 1/x, & x \le 2 \\ \dfrac{4-x}{4}, & x > 2 \end{cases}$ at $x = 2$

18. $f(x) = \begin{cases} \sin x, & 0 \le x < 3\pi/4 \\ \cos x, & 3\pi/4 \le x \le 2\pi \end{cases}$ at $x = 3\pi/4$

In Exercises 19–22, **(a)** find the slope of the curve at $x = a$.
(b) Writing to Learn Describe what happens to the tangent at $x = a$ as a changes.

19. $y = x^2 + 2$

20. $y = 2/x$

21. $y = \dfrac{1}{x-1}$

22. $y = 9 - x^2$

23. ***Free Fall*** An object is dropped from the top of a 100-m tower. Its height above ground after t sec is $100 - 4.9t^2$ m. How fast is it falling 2 sec after it is dropped?

24. ***Rocket Launch*** At t sec after lift-off, the height of a rocket is $3t^2$ ft. How fast is the rocket climbing after 10 sec?

25. ***Area of Circle*** What is the rate of change of the area of a circle with respect to the radius when the radius is $r = 3$ in.?

26. ***Volume of Sphere*** What is the rate of change of the volume of a sphere with respect to the radius when the radius is $r = 2$ in.?

27. ***Free Fall on Mars*** The equation for free fall at the surface of Mars is $s = 1.86t^2$ m with t in seconds. Assume a rock is dropped from the top of a 200-m cliff. Find the speed of the rock at $t = 1$ sec.

28. ***Free Fall on Jupiter*** The equation for free fall at the surface of Jupiter is $s = 11.44t^2$ m with t in seconds. Assume a rock is dropped from the top of a 500-m cliff. Find the speed of the rock at $t = 2$ sec.

29. ***Horizontal Tangent*** At what point is the tangent to $f(x) = x^2 + 4x - 1$ horizontal?

30. ***Horizontal Tangent*** At what point is the tangent to $f(x) = 3 - 4x - x^2$ horizontal?

31. ***Finding Tangents and Normals***

(a) Find an equation for each tangent to the curve $y = 1/(x - 1)$ that has slope -1. (See Exercise 21.)

(b) Find an equation for each normal to the curve $y = 1/(x - 1)$ that has slope 1.

32. ***Finding Tangents*** Find the equations of all lines tangent to $y = 9 - x^2$ that pass through the point (1, 12).

33. Table 2.2 gives the amount of federal spending in billions of dollars for national defense for several years.

Table 2.2 National Defense Spending

Year	National Defense Spending ($ billions)
1990	299.3
1995	272.1
1999	274.9
2000	294.5
2001	305.5
2002	348.6
2003	404.9

Source: U.S. Census Bureau, *Statistical Abstract of the United States, 2004–2005.*

(a) Find the average rate of change in spending from 1990 to 1995.

(b) Find the average rate of change in spending from 2000 to 2001.

(c) Find the average rate of change in spending from 2002 to 2003.

(d) Let $x = 0$ represent 1990, $x = 1$ represent 1991, and so forth. Find the quadratic regression equation for the data and superimpose its graph on a scatter plot of the data.

(e) Compute the average rates of change in parts (a), (b), and (c) using the regression equation.

(f) Use the regression equation to find how fast the spending was growing in 2003.

(g) Writing to Learn Explain why someone might be hesitant to make predictions about the rate of change of national defense spending based on this equation.

34. Table 2.3 gives the amount of federal spending in billions of dollars for agriculture for several years.

Table 2.3 Agriculture Spending

Year	Agriculture Spending ($ billions)
1990	12.0
1995	9.8
1999	23.0
2000	36.6
2001	26.4
2002	22.0
2003	22.6

Source: U.S. Census Bureau, *Statistical Abstract of the United States, 2004–2005.*

(a) Let $x = 0$ represent 1990, $x = 1$ represent 1991, and so forth. Make a scatter plot of the data.

(b) Let P represent the point corresponding to 2003, Q_1 the point corresponding to 2000, Q_2 the point corresponding to 2001, and Q_3 the point corresponding to 2002. Find the slope of the secant line PQ_i for $i = 1, 2, 3$.

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

35. True or False If the graph of a function has a tangent line at $x = a$, then the graph also has a normal line at $x = a$. Justify your answer.

36. True or False The graph of $f(x)=|x|$ has a tangent line at $x = 0$. Justify your answer.

37. Multiple Choice If the line L tangent to the graph of a function f at the point (2, 5) passes through the point (−1, −3), what is the slope of L?

(A) −3/8 **(B)** 3/8 **(C)** −8/3 **(D)** 8/3 **(E)** undefined

38. Multiple Choice Find the average rate of change of $f(x) = x^2 + x$ over the interval [1, 3].

(A) −5 **(B)** 1/5 **(C)** 1/4 **(D)** 4 **(E)** 5

39. Multiple Choice Which of the following is an equation of the tangent to the graph of $f(x) = 2/x$ at $x = 1$?

(A) $y = -2x$ **(B)** $y = 2x$ **(C)** $y = -2x + 4$

(D) $y = -x + 3$ **(E)** $y = x + 3$

40. Multiple Choice Which of the following is an equation of the normal to the graph of $f(x) = 2/x$ at $x = 1$?

(A) $y = \frac{1}{2}x + \frac{3}{2}$ **(B)** $y = -\frac{1}{2}x$ **(C)** $y = \frac{1}{2}x + 2$

(D) $y = -\frac{1}{2}x + 2$ **(E)** $y = 2x + 5$

Explorations

In Exercises 41 and 42, complete the following for the function.

(a) Compute the difference quotient

$$\frac{f(1 + h) - f(1)}{h}.$$

(b) Use graphs and tables to estimate the limit of the difference quotient in part (a) as $h \to 0$.

(c) Compare your estimate in part (b) with the given number.

(d) Writing to Learn Based on your computations, do you think the graph of f has a tangent at $x = 1$? If so, estimate its slope. If not, explain why not.

41. $f(x) = e^x$, e

42. $f(x) = 2^x$, $\ln 4$

Group Activity In Exercises 43–46, the curve $y = f(x)$ has a **vertical tangent** at $x = a$ if

$$\lim_{h\to 0} \frac{f(a + h) - f(a)}{h} = \infty$$

or if

$$\lim_{h\to 0} \frac{f(a + h) - f(a)}{h} = -\infty.$$

In each case, the right- and left-hand limits are required to be the same: both $+\infty$ or both $-\infty$.

Use graphs to investigate whether the curve has a vertical tangent at $x = 0$.

43. $y = x^{2/5}$

44. $y = x^{3/5}$

45. $y = x^{1/3}$

46. $y = x^{2/3}$

Extending the Ideas

In Exercises 47 and 48, determine whether the graph of the function has a tangent at the origin. Explain your answer.

47. $f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

48. $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

49. *Sine Function* Estimate the slope of the curve $y = \sin x$ at $x = 1$. (*Hint:* See Exercises 41 and 42.)

Quick Quiz for AP* Preparation: Sections 2.3 and 2.4

You may use a calculator with these problems.

1. Multiple Choice Which of the following values is the average rate of $f(x) = \sqrt{x + 1}$ over the interval (0, 3)?

(A) −3 **(B)** −1 **(C)** −1/3 **(D)** 1/3 **(E)** 3

2. Multiple Choice Which of the following statements is false for the function

$$f(x) = \begin{cases} \frac{3}{4}x, & 0 \le x < 4 \\ 2, & x = 4 \\ -x + 7, & 4 < x \le 6 \\ 1, & 6 < x < 8? \end{cases}$$

(A) $\lim_{x\to 4} f(x)$ exists **(B)** $f(4)$ exists

(C) $\lim_{x\to 6} f(x)$ exists **(D)** $\lim_{x\to 8^-} f(x)$ exists

(E) f is continuous at $x = 4$

3. Multiple Choice Which of the following is an equation for the tangent line to $f(x) = 9 - x^2$ at $x = 2$?

(A) $y = \frac{1}{4}x + \frac{9}{2}$ **(B)** $y = -4x + 13$

(C) $y = -4x - 3$ **(D)** $y = 4x - 3$

(E) $y = 4x + 13$

4. Free Response Let $f(x) = 2x - x^2$.

(a) Find $f(3)$. **(b)** Find $f(3 + h)$.

(c) Find $\frac{f(3 + h) - f(3)}{h}$.

(d) Find the instantaneous rate of change of f at $x = 3$.

Chapter 2 Key Terms

average rate of change (p. 87)
average speed (p. 59)
connected graph (p. 83)
Constant Multiple Rule for Limits (p. 61)
continuity at a point (p. 78)
continuous at an endpoint (p. 79)
continuous at an interior point (p. 79)
continuous extension (p. 81)
continuous function (p. 81)
continuous on an interval (p. 81)
difference quotient (p. 90)
Difference Rule for Limits (p. 61)
discontinuous (p. 79)
end behavior model (p. 74)
free fall (p. 91)
horizontal asymptote (p. 70)
infinite discontinuity (p. 80)
instantaneous rate of change (p. 91)
instantaneous speed (p. 91)
intermediate value property (p. 83)
Intermediate Value Theorem for Continuous Functions (p. 83)
jump discontinuity (p. 80)
left end behavior model (p. 74)
left-hand limit (p. 64)
limit of a function (p. 60)
normal to a curve (p. 91)
oscillating discontinuity (p. 80)
point of discontinuity (p. 79)
Power Rule for Limits (p. 71)
Product Rule for Limits (p. 61)
Properties of Continuous Functions (p. 82)
Quotient Rule for Limits (p. 61)
removable discontinuity (p. 80)
right end behavior model (p. 74)
right-hand limit (p. 64)
Sandwich Theorem (p. 65)
secant to a curve (p. 87)
slope of a curve (p. 89)
Sum Rule for Limits (p. 61)
tangent line to a curve (p. 88)
two-sided limit (p. 64)
vertical asymptote (p. 72)
vertical tangent (p. 94)

Chapter 2 Review Exercises

The collection of exercises marked in red could be used as a chapter test.

In Exercises 1–14, find the limits.

1. $\lim_{x\to -2} (x^3 - 2x^2 + 1)$

2. $\lim_{x\to -2} \dfrac{x^2 + 1}{3x^2 - 2x + 5}$

3. $\lim_{x\to 4} \sqrt{1 - 2x}$

4. $\lim_{x\to 5} \sqrt[4]{9 - x^2}$

5. $\lim_{x\to 0} \dfrac{\dfrac{1}{2+x} - \dfrac{1}{2}}{x}$

6. $\lim_{x\to \pm\infty} \dfrac{2x^2 + 3}{5x^2 + 7}$

7. $\lim_{x\to \pm\infty} \dfrac{x^4 + x^3}{12x^3 + 128}$

8. $\lim_{x\to 0} \dfrac{\sin 2x}{4x}$

9. $\lim_{x\to 0} \dfrac{x \csc x + 1}{x \csc x}$

10. $\lim_{x\to 0} e^x \sin x$

11. $\lim_{x\to 7/2^+} \operatorname{int}(2x - 1)$

12. $\lim_{x\to 7/2^-} \operatorname{int}(2x - 1)$

13. $\lim_{x\to \infty} e^{-x} \cos x$

14. $\lim_{x\to \infty} \dfrac{x + \sin x}{x + \cos x}$

In Exercises 15–20, determine whether the limit exists on the basis of the graph of $y = f(x)$. The domain of f is the set of real numbers.

15. $\lim_{x\to d} f(x)$

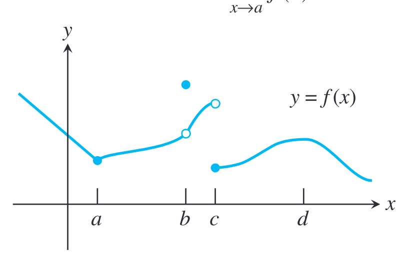
16. $\lim_{x\to c^+} f(x)$

17. $\lim_{x\to c^-} f(x)$

18. $\lim_{x\to c} f(x)$

19. $\lim_{x\to b} f(x)$

20. $\lim_{x\to a} f(x)$



In Exercises 21–24, determine whether the function f used in Exercises 15–20 is continuous at the indicated point.

21. $x = a$

22. $x = b$

23. $x = c$

24. $x = d$

In Exercises 25 and 26, use the graph of the function with domain $-1 \le x \le 3$.

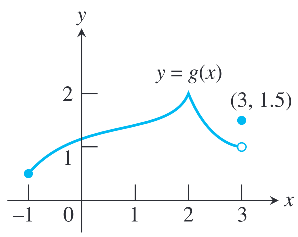
25. Determine

(a) $\lim_{x \to 3^-} g(x)$. **(b)** $g(3)$.

(c) whether $g(x)$ is continuous at $x = 3$.

(d) the points of discontinuity of $g(x)$.

(e) Writing to Learn whether any points of discontinuity are removable. If so, describe the new function. If not, explain why not.



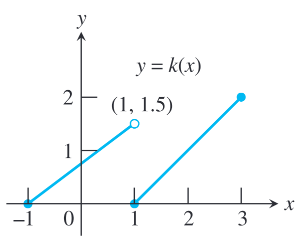
26. Determine

(a) $\lim_{x \to 1^-} k(x)$. **(b)** $\lim_{x \to 1^+} k(x)$. **(c)** $k(1)$.

(d) whether $k(x)$ is continuous at $x = 1$.

(e) the points of discontinuity of $k(x)$.

(f) Writing to Learn whether any points of discontinuity are removable. If so, describe the new function. If not, explain why not.



In Exercises 27 and 28, **(a)** find the vertical asymptotes of the graph of $y = f(x)$, and **(b)** describe the behavior of $f(x)$ to the left and right of any vertical asymptote.

27. $f(x) = \dfrac{x+3}{x+2}$ **28.** $f(x) = \dfrac{x-1}{x^2(x+2)}$

In Exercises 29 and 30, answer the questions for the piecewise-defined function.

29. $f(x) = \begin{cases} 1, & x \le -1 \\ -x, & -1 < x < 0 \\ 1, & x = 0 \\ -x, & 0 < x < 1 \\ 1, & x \ge 1 \end{cases}$

(a) Find the right-hand and left-hand limits of f at $x = -1$, 0, and 1.

(b) Does f have a limit as x approaches -1? 0? 1? If so, what is it? If not, why not?

(c) Is f continuous at $x = -1$? 0? 1? Explain.

30. $f(x) = \begin{cases} |x^3 - 4x|, & x < 1 \\ x^2 - 2x - 2, & x \ge 1 \end{cases}$

(a) Find the right-hand and left-hand limits of f at $x = 1$.

(b) Does f have a limit as $x \to 1$? If so, what is it? If not, why not?

(c) At what points is f continuous?

(d) At what points is f discontinuous?

In Exercises 31 and 32, find all points of discontinuity of the function.

31. $f(x) = \dfrac{x+1}{4-x^2}$ **32.** $g(x) = \sqrt[3]{3x+2}$

In Exercises 33–36, find **(a)** a power function end behavior model and **(b)** any horizontal asymptotes.

33. $f(x) = \dfrac{2x+1}{x^2-2x+1}$ **34.** $f(x) = \dfrac{2x^2+5x-1}{x^2+2x}$

35. $f(x) = \dfrac{x^3-4x^2+3x+3}{x-3}$ **36.** $f(x) = \dfrac{x^4-3x^2+x-1}{x^3-x+1}$

In Exercises 37 and 38, find **(a)** a right end behavior model and **(b)** a left end behavior model for the function.

37. $f(x) = x + e^x$ **38.** $f(x) = \ln|x| + \sin x$

Group Activity In Exercises 39 and 40, what value should be assigned to k to make f a continuous function?

39. $f(x) = \begin{cases} \dfrac{x^2+2x-15}{x-3}, & x \ne 3 \\ k, & x = 3 \end{cases}$

40. $f(x) = \begin{cases} \dfrac{\sin x}{2x}, & x \ne 0 \\ k, & x = 0 \end{cases}$

Group Activity In Exercises 41 and 42, sketch a graph of a function f that satisfies the given conditions.

41. $\lim_{x \to \infty} f(x) = 3$, $\lim_{x \to -\infty} f(x) = \infty$,

$\lim_{x \to 3^+} f(x) = \infty$, $\lim_{x \to 3^-} f(x) = -\infty$

42. $\lim_{x \to 2} f(x)$ does not exist, $\lim_{x \to 2^+} f(x) = f(2) = 3$

43. ***Average Rate of Change*** Find the average rate of change of $f(x) = 1 + \sin x$ over the interval $[0, \pi/2]$.

44. ***Rate of Change*** Find the instantaneous rate of change of the volume $V = (1/3)\pi r^2 H$ of a cone with respect to the radius r at $r = a$ if the height H does not change.

45. ***Rate of Change*** Find the instantaneous rate of change of the surface area $S = 6x^2$ of a cube with respect to the edge length x at $x = a$.

46. ***Slope of a Curve*** Find the slope of the curve $y = x^2 - x - 2$ at $x = a$.

47. ***Tangent and Normal*** Let $f(x) = x^2 - 3x$ and $P = (1, f(1))$. Find **(a)** the slope of the curve $y = f(x)$ at P, **(b)** an equation of the tangent at P, and **(c)** an equation of the normal at P.

48. ***Horizontal Tangents*** At what points, if any, are the tangents to the graph of $f(x) = x^2 - 3x$ horizontal? (See Exercise 47.)

49. ***Bear Population*** The number of bears in a federal wildlife reserve is given by the population equation

$$p(t) = \frac{200}{1 + 7e^{-0.1t}},$$

where t is in years.

(a) **Writing to Learn** Find $p(0)$. Give a possible interpretation of this number.

(b) Find $\lim_{t\to\infty} p(t)$.

(c) **Writing to Learn** Give a possible interpretation of the result in part (b).

50. ***Taxi Fares*** Bluetop Cab charges \$3.20 for the first mile and \$1.35 for each additional mile or part of a mile.

(a) Write a formula that gives the charge for x miles with $0 \le x \le 20$.

(b) Graph the function in (a). At what values of x is it discontinuous?

51. Table 2.4 gives the population of Florida for several years.

Table 2.4 Population of Florida

Year	Population (in thousands)
1998	15,487
1999	15,759
2000	15,983
2001	16,355
2002	16,692
2003	17,019

Source: U.S. Census Bureau, *Statistical Abstract of the United States; 2004–2005.*

(a) Let $x = 0$ represent 1990, $x = 1$ represent 1991, and so forth. Make a scatter plot for the data.

(b) Let P represent the point corresponding to 2003, Q_1 the point corresponding to 1998, Q_2 the point corresponding to 1999, . . . , and Q_5 the point corresponding to 2002. Find the slope of the secant the PQ_i for $i = 1, 2, 3, 4, 5$.

(c) Predict the rate of change of population in 2003.

(d) Find a linear regression equation for the data, and use it to calculate the rate of the population in 2003.

52. ***Limit Properties*** Assume that

$$\lim_{x\to c} [f(x) + g(x)] = 2,$$

$$\lim_{x\to c} [f(x) - g(x)] = 1,$$

and that $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist. Find $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$.

AP* Examination Preparation

You should solve the following problems without using a graphing calculation.

53. **Free Response** Let $f(x) = \dfrac{x}{|x^2 - 9|}$.

(a) Find the domain of f.

(b) Write an equation for each vertical asymptote of the graph of f.

(c) Write an equation for each horizontal asymptote of the graph of f.

(d) Is f odd, even, or neither? Justify your answer.

(e) Find all values of x for which f is discontinuous and classify each discontinuity as removable or nonremovable.

54. **Free Response** Let $f(x) = \begin{cases} x^2 - a^2x & \text{if } x < 2, \\ 4 - 2x^2 & \text{if } x \ge 2. \end{cases}$

(a) Find $\lim_{x\to 2^-} f(x)$.

(b) Find $\lim_{x\to 2^+} f(x)$.

(c) Find all values of a that make f continuous at 2. Justify your answer.

55. **Free Response** Let $f(x) = \dfrac{x^3 - 2x^2 + 1}{x^2 + 3}$.

(a) Find all zeros of f.

(b) Find a right end behavior model $g(x)$ for f.

(c) Determine $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} \dfrac{f(x)}{g(x)}$.

Chapter 3 Derivatives

Shown here is the pain reliever acetaminophen in crystalline form, photographed under a transmitted light microscope. While acetaminophen relieves pain with few side effects, it is toxic in large doses. One study found that only 30% of parents who gave acetaminophen to their children could accurately calculate and measure the correct dose.

One rule for calculating the dosage (mg) of acetaminophen for children ages 1 to 12 years old is $D(t) = 750t/(t + 12)$, where t is age in years. What is an expression for the rate of change of a child's dosage with respect to the child's age? How does the rate of change of the dosage relate to the growth rate of children? This problem can be solved with the information covered in Section 3.4.

Chapter 3 Overview

In Chapter 2, we learned how to find the slope of a tangent to a curve as the limit of the slopes of secant lines. In Example 4 of Section 2.4, we derived a formula for the slope of the tangent at an arbitrary point $(a, 1/a)$ on the graph of the function $f(x) = 1/x$ and showed that it was $-1/a^2$.

This seemingly unimportant result is more powerful than it might appear at first glance, as it gives us a simple way to calculate the instantaneous rate of change of f at any point. The study of rates of change of functions is called *differential calculus,* and the formula $-1/a^2$ was our first look at a *derivative.* The derivative was the 17th-century breakthrough that enabled mathematicians to unlock the secrets of planetary motion and gravitational attraction—of objects changing position over time. We will learn many uses for derivatives in Chapter 4, but first we will concentrate in this chapter on understanding what derivatives are and how they work.

3.1 Derivative of a Function

What you'll learn about

- Definition of Derivative
- Notation
- Relationships between the Graphs of f and f'
- Graphing the Derivative from Data
- One-sided Derivatives

. . . and why

The derivative gives the value of the slope of the tangent line to a curve at a point.

Definition of Derivative

In Section 2.4, we defined the slope of a curve $y = f(x)$ at the point where $x = a$ to be

$$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

When it exists, this limit is called the **derivative of f at a.** In this section, we investigate the derivative as a *function* derived from f by considering the limit at each point of the domain of f.

DEFINITION Derivative

The **derivative** of the function f with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}, \qquad (1)$$

provided the limit exists.

The domain of f', the set of points in the domain of f for which the limit exists, may be smaller than the domain of f. If $f'(x)$ exists, we say that f **has a derivative (is differentiable)** at x. A function that is differentiable at every point of its domain is a **differentiable function.**

EXAMPLE 1 Applying the Definition

Differentiate (that is, find the derivative of) $f(x) = x^3$.

continued

SOLUTION

Applying the definition, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \\ &= \lim_{h\to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h} \\ &= \lim_{h\to 0} \frac{(3x^2 + 3xh + h^2)h}{h} \\ &= \lim_{h\to 0} (3x^2 + 3xh + h^2) = 3x^2. \end{aligned}$$

Now try Exercise 1.

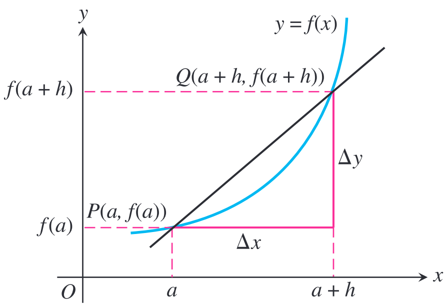


Figure 3.1 The slope of the secant line PQ is

$$\frac{\Delta y}{\Delta x} = \frac{f(a+h)-f(a)}{(a+h)-a} = \frac{f(a+h)-f(a)}{h}.$$

The derivative of $f(x)$ at a point where $x = a$ is found by taking the limit as $h \to 0$ of slopes of secant lines, as shown in Figure 3.1.

By relabeling the picture as in Figure 3.2, we arrive at a useful alternate formula for calculating the derivative. This time, the limit is taken as x approaches a.

DEFINITION (ALTERNATE) Derivative at a Point

The **derivative** of the function ***f* at the point *x* = *a*** is the limit

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}, \tag{2}$$

provided the limit exists.

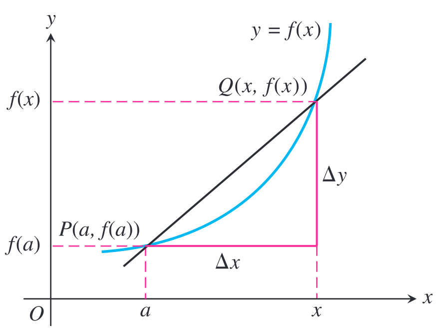


Figure 3.2 The slope of the secant line PQ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x)-f(a)}{x-a}.$$

After we find the derivative of f at a point $x = a$ using the alternate form, we can find the derivative of f as a function by applying the resulting formula to an arbitrary x in the domain of f.

EXAMPLE 2 Applying the Alternate Definition

Differentiate $f(x) = \sqrt{x}$ using the alternate definition.

SOLUTION

At the point $x = a$,

$$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \\ &= \lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{x-a} \\ &= \lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{x-a} \cdot \frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}} \\ &= \lim_{x\to a} \frac{x-a}{(x-a)(\sqrt{x}+\sqrt{a})} \\ &= \lim_{x\to a} \frac{1}{\sqrt{x}+\sqrt{a}} \\ &= \frac{1}{2\sqrt{a}}. \end{aligned}$$

Applying this formula to an arbitrary $x > 0$ in the domain of f identifies the derivative as the function $f'(x) = 1/(2\sqrt{x})$ with domain $(0, \infty)$.

Now try Exercise 5.

Why all the notation?

The "prime" notations y' and f' come from notations that Newton used for derivatives. The d/dx notations are similar to those used by Leibniz. Each has its advantages and disadvantages.

Notation

There are many ways to denote the derivative of a function $y = f(x)$. Besides $f'(x)$, the most common notations are these:

y'	"y prime"	Nice and brief, but does not name the independent variable.
$\dfrac{dy}{dx}$	"$dy\ dx$" or "the derivative of y with respect to x"	Names both variables and uses d for derivative.
$\dfrac{df}{dx}$	"$df\ dx$" or "the derivative of f with respect to x"	Emphasizes the function's name.
$\dfrac{d}{dx}f(x)$	"$d\ dx$ of f at x" or "the derivative of f at x"	Emphasizes the idea that differentiation is an operation performed on f.

Relationships between the Graphs of f and f'

When we have the explicit formula for $f(x)$, we can derive a formula for $f'(x)$ using methods like those in Examples 1 and 2. We have already seen, however, that functions are encountered in other ways: graphically, for example, or in tables of data.

Because we can think of the derivative at a point in graphical terms as *slope,* we can get a good idea of what the graph of the function f' looks like by *estimating the slopes* at various points along the graph of f.

EXAMPLE 3 GRAPHING f' from f

Graph the derivative of the function f whose graph is shown in Figure 3.3a. Discuss the behavior of f in terms of the signs and values of f'.

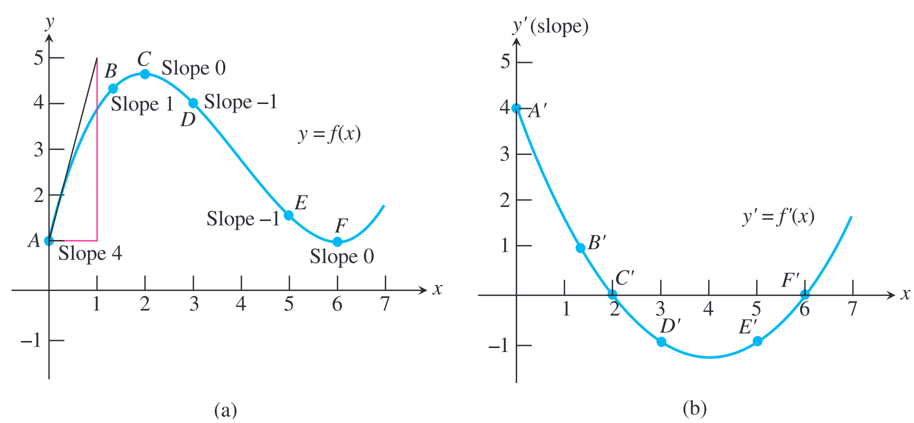


Figure 3.3 By plotting the slopes at points on the graph of $y = f(x)$, we obtain a graph of $y' = f'(x)$. The slope at point A of the graph of f in part (a) is the y-coordinate of point A' on the graph of f' in part (b), and so on. (Example 3)

SOLUTION

First, we draw a pair of coordinate axes, marking the horizontal axis in x-units and the vertical axis in slope units (Figure 3.3b). Next, we estimate the slope of the graph of f at various points, plotting the corresponding slope values using the new axes. At $A(0, f(0))$, the graph of f has slope 4, so $f'(0) = 4$. At B, the graph of f has slope 1, so $f' = 1$ at B', and so on.

continued

We complete our estimate of the graph of f' by connecting the plotted points with a smooth curve.

Although we do not have a formula for either f or f', the graph of each reveals important information about the behavior of the other. In particular, notice that f is decreasing where f' is negative and increasing where f' is positive. Where f' is zero, the graph of f has a horizontal tangent, changing from increasing to decreasing (point C) or from decreasing to increasing (point F). **Now try Exercise 23.**

EXPLORATION 1 Reading the Graphs

Suppose that the function f in Figure 3.3a represents the depth y (in inches) of water in a ditch alongside a dirt road as a function of time x (in days). How would you answer the following questions?

1. What does the graph in Figure 3.3b represent? What units would you use along the y'-axis?
2. Describe as carefully as you can what happened to the water in the ditch over the course of the 7-day period.
3. Can you describe the weather during the 7 days? When was it the wettest? When was it the driest?
4. How does the graph of the derivative help in finding when the weather was wettest or driest?
5. Interpret the significance of point C in terms of the water in the ditch. How does the significance of point C' reflect that in terms of rate of change?
6. It is tempting to say that it rains right up until the beginning of the second day, but that overlooks a fact about rainwater that is important in flood control. Explain.

Construct your own "real-world" scenario for the function in Example 3, and pose a similar set of questions that could be answered by considering the two graphs in Figure 3.3.

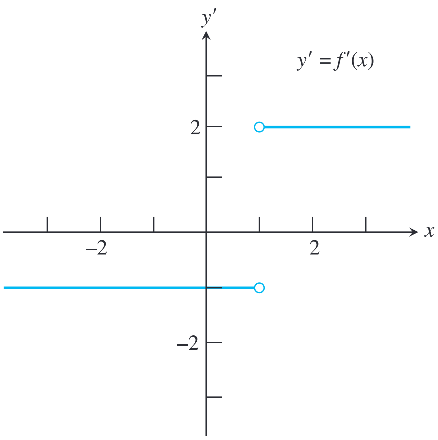


Figure 3.4 The graph of the derivative. (Example 4)

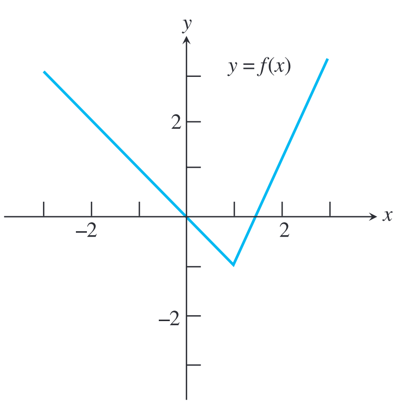


Figure 3.5 The graph of f, constructed from the graph of f' and two other conditions. (Example 4)

EXAMPLE 4 Graphing f from f'

Sketch the graph of a function f that has the following properties:

i. $f(0) = 0$;

ii. the graph of f', the derivative of f, is as shown in Figure 3.4;

iii. f is continuous for all x.

SOLUTION

To satisfy property (i), we begin with a point at the origin.

To satisfy property (ii), we consider what the graph of the derivative tells us about slopes. To the left of $x = 1$, the graph of f has a constant slope of -1; therefore we draw a line with slope -1 to the left of $x = 1$, making sure that it goes through the origin.

To the right of $x = 1$, the graph of f has a constant slope of 2, so it must be a line with slope 2. There are infinitely many such lines but only one—the one that meets the left side of the graph at $(1, -1)$—will satisfy the continuity requirement. The resulting graph is shown in Figure 3.5. **Now try Exercise 27.**

Graphing the Derivative from Data

Discrete points plotted from sets of data do not yield a continuous curve, but we have seen that the shape and pattern of the graphed points (called a scatter plot) can be meaningful nonetheless. It is often possible to fit a curve to the points using regression techniques. If the fit is good, we could use the curve to get a graph of the derivative visually, as in Example 3. However, it is also possible to get a scatter plot of the derivative numerically, directly from the data, by computing the slopes between successive points, as in Example 5.

What's happening at $x = 1$?

Notice that f in Figure 3.5 is defined at $x = 1$, while f' is not. It is the continuity of f that enables us to conclude that $f(1) = -1$. Looking at the graph of f, can you see why f' could not possibly be defined at $x = 1$? We will explore the reason for this in Example 6.

EXAMPLE 5 Estimating the Probability of Shared Birthdays

Suppose 30 people are in a room. What is the probability that two of them share the same birthday? Ignore the year of birth.

SOLUTION

It may surprise you to learn that the probability of a shared birthday among 30 people is at least 0.706, well above two-thirds! In fact, if we assume that no one day is more likely to be a birthday than any other day, the probabilities shown in Table 3.1 are not hard to determine (see Exercise 45).

Table 3.1 Probabilities of Shared Birthdays

People in Room (x)	Probability (y)
0	0
5	0.027
10	0.117
15	0.253
20	0.411
25	0.569
30	0.706
35	0.814
40	0.891
45	0.941
50	0.970
55	0.986
60	0.994
65	0.998
70	0.999

Table 3.2 Estimates of Slopes on the Probability Curve

Midpoint of Interval (x)	Change (slope $\Delta y/\Delta x$)
2.5	0.0054
7.5	0.0180
12.5	0.0272
17.5	0.0316
22.5	0.0316
27.5	0.0274
32.5	0.0216
37.5	0.0154
42.5	0.0100
47.5	0.0058
52.5	0.0032
57.5	0.0016
62.5	0.0008
67.5	0.0002

A scatter plot of the data in Table 3.1 is shown in Figure 3.6.

Notice that the probabilities grow slowly at first, then faster, then much more slowly past $x = 45$. At which x are they growing the fastest? To answer the question, we need the graph of the derivative.

Using the data in Table 3.1, we compute the slopes between successive points on the probability plot. For example, from $x = 0$ to $x = 5$ the slope is

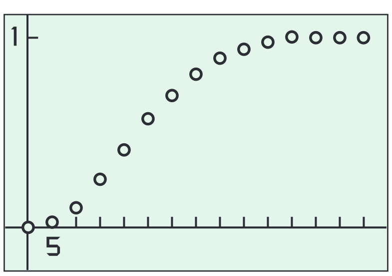
$$\frac{0.027 - 0}{5 - 0} = 0.0054.$$

We make a new table showing the slopes, beginning with slope 0.0054 on the interval $[0, 5]$ (Table 3.2). A logical x value to use to represent the interval is its midpoint 2.5.

continued

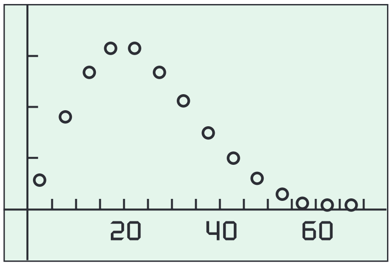
David H. Blackwell
(1919–)

By the age of 22, David Blackwell had earned a Ph.D. in Mathematics from the University of Illinois. He taught at Howard University, where his research included statistics, Markov chains, and sequential analysis. He then went on to teach and continue his research at the University of California at Berkeley. Dr. Blackwell served as president of the American Statistical Association and was the first African American mathematician of the National Academy of Sciences.



[–5, 75] by [–0.2, 1.1]

Figure 3.6 Scatter plot of the probabilities (y) of shared birthdays among x people, for $x = 0, 5, 10, \ldots, 70$. (Example 5)



[–5, 75] by [–0.01, 0.04]

Figure 3.7 A scatter plot of the derivative data in Table 3.2. (Example 5)

A scatter plot of the derivative data in Table 3.2 is shown in Figure 3.7.

From the derivative plot, we can see that the rate of change peaks near $x = 20$. You can impress your friends with your "psychic powers" by predicting a shared birthday in a room of just 25 people (since you will be right about 57% of the time), but the derivative warns you to be cautious: a few less people can make quite a difference. On the other hand, going from 40 people to 100 people will not improve your chances much at all.

Now try Exercise 29.

Generating shared birthday probabilities: If you know a little about probability, you might try generating the probabilities in Table 3.1. Extending the Idea Exercise 45 at the end of this section shows how to generate them on a calculator.

One-Sided Derivatives

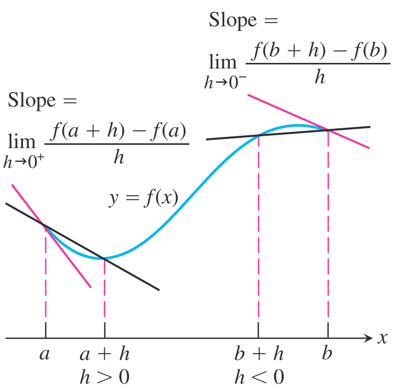


Figure 3.8 Derivatives at endpoints are one-sided limits.

A function $y = f(x)$ is **differentiable on a closed interval $[a, b]$** if it has a derivative at every interior point of the interval, and if the limits

$$\lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h} \quad \text{[the \textbf{right-hand derivative at } \boldsymbol{a}\text{]}}$$

$$\lim_{h\to 0^-} \frac{f(b+h)-f(b)}{h} \quad \text{[the \textbf{left-hand derivative at } \boldsymbol{b}\text{]}}$$

exist at the endpoints. In the right-hand derivative, h is positive and $a + h$ approaches a from the right. In the left-hand derivative, h is negative and $b + h$ approaches b from the left (Figure 3.8).

Right-hand and left-hand derivatives may be defined at any point of a function's domain.

The usual relationship between one-sided and two-sided limits holds for derivatives. Theorem 3, Section 2.1, allows us to conclude that a function has a (two-sided) derivative at a point if and only if the function's right-hand and left-hand derivatives are defined and equal at that point.

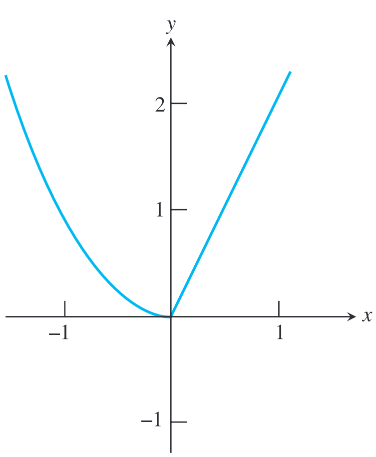


Figure 3.9 A function with different one-sided derivatives at $x = 0$. (Example 6)

EXAMPLE 6 One-Sided Derivatives can Differ at a Point

Show that the following function has left-hand and right-hand derivatives at $x = 0$, but no derivative there (Figure 3.9).

$$y = \begin{cases} x^2, & x \le 0 \\ 2x, & x > 0 \end{cases}$$

SOLUTION

We verify the existence of the left-hand derivative:

$$\lim_{h\to 0^-} \frac{(0+h)^2 - 0^2}{h} = \lim_{h\to 0^-} \frac{h^2}{h} = 0.$$

We verify the existence of the right-hand derivative:

$$\lim_{h\to 0^+} \frac{2(0+h) - 0^2}{h} = \lim_{h\to 0^+} \frac{2h}{h} = 2.$$

Since the left-hand derivative equals zero and the right-hand derivative equals 2, the derivatives are not equal at $x = 0$. The function does not have a derivative at 0.

Now try Exercise 31.

Quick Review 3.1 *(For help, go to Sections 2.1 and 2.4.)*

In Exercises 1–4, evaluate the indicated limit algebraically.

1. $\lim\limits_{h\to 0} \dfrac{(2+h)^2-4}{h}$

2. $\lim\limits_{x\to 2^+} \dfrac{x+3}{2}$

3. $\lim\limits_{y\to 0^-} \dfrac{|y|}{y}$

4. $\lim\limits_{x\to 4} \dfrac{2x-8}{\sqrt{x}-2}$

5. Find the slope of the line tangent to the parabola $y = x^2 + 1$ at its vertex.

6. By considering the graph of $f(x) = x^3 - 3x^2 + 2$, find the intervals on which f is increasing.

In Exercises 7–10, let

$$f(x) = \begin{cases} x+2, & x \le 1 \\ (x-1)^2, & x > 1. \end{cases}$$

7. Find $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1^-} f(x)$.

8. Find $\lim_{h\to 0^+} f(1+h)$.

9. Does $\lim_{x\to 1} f(x)$ exist? Explain.

10. Is f continuous? Explain.

Section 3.1 Exercises

In Exercises 1–4, use the definition

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

to find the derivative of the given function at the indicated point.

1. $f(x) = 1/x,\ a = 2$

2. $f(x) = x^2 + 4,\ a = 1$

3. $f(x) = 3 - x^2,\ a = -1$

4. $f(x) = x^3 + x,\ a = 0$

In Exercises 5–8, use the definition

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

to find the derivative of the given function at the indicated point.

5. $f(x) = 1/x,\ a = 2$

6. $f(x) = x^2 + 4,\ a = 1$

7. $f(x) = \sqrt{x+1},\ a = 3$

8. $f(x) = 2x + 3,\ a = -1$

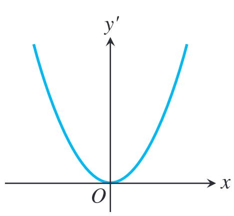
9. Find $f'(x)$ if $f(x) = 3x - 12$.

10. Find dy/dx if $y = 7x$.

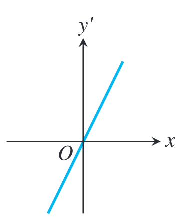
11. Find $\dfrac{d}{dx}(x^2)$.

12. Find $\dfrac{d}{dx} f(x)$ if $f(x) = 3x^2$.

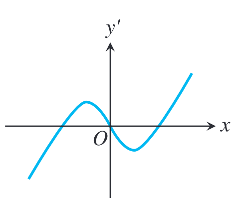
In Exercises 13–16, match the graph of the function with the graph of the derivative shown here:



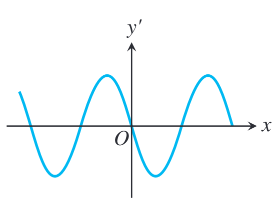
(a)



(b)

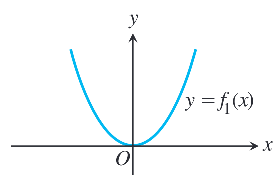


(c)

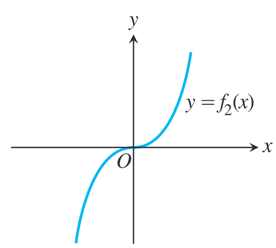


(d)

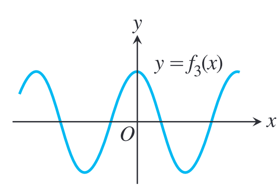
13.



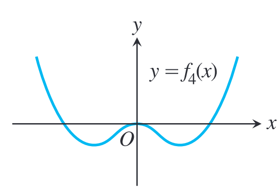
14.



15.



16.



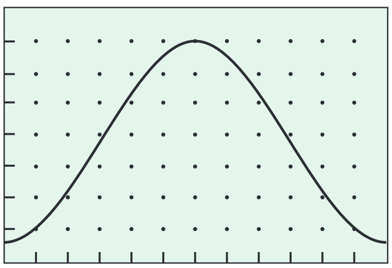
17. If $f(2) = 3$ and $f'(2) = 5$, find an equation of **(a)** the *tangent* line, and **(b)** the *normal* line to the graph of $y = f(x)$ at the point where $x = 2$.

18. Find the derivative of the function $y = 2x^2 - 13x + 5$ and use it to find an equation of the line tangent to the curve at $x = 3$.

19. Find the lines that are **(a)** tangent and **(b)** normal to the curve $y = x^3$ at the point (1, 1).

20. Find the lines that are **(a)** tangent and **(b)** normal to the curve $y = \sqrt{x}$ at $x = 4$.

21. ***Daylight in Fairbanks*** The viewing window below shows the number of hours of daylight in Fairbanks, Alaska, on each day for a typical 365-day period from January 1 to December 31. Answer the following questions by estimating slopes on the graph in hours per day. For the purposes of estimation, assume that each month has 30 days.



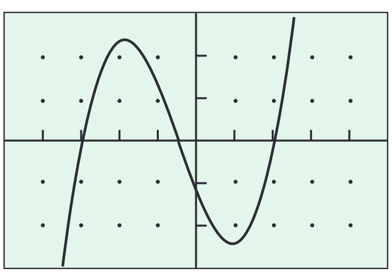
[0, 365] by [0, 24]

(a) On about what date is the amount of daylight increasing at the fastest rate? What is that rate?

(b) Do there appear to be days on which the rate of change in the amount of daylight is zero? If so, which ones?

(c) On what dates is the rate of change in the number of daylight hours positive? negative?

22. ***Graphing f′ from f*** Given the graph of the function *f* below, sketch a graph of the *derivative* of *f*.

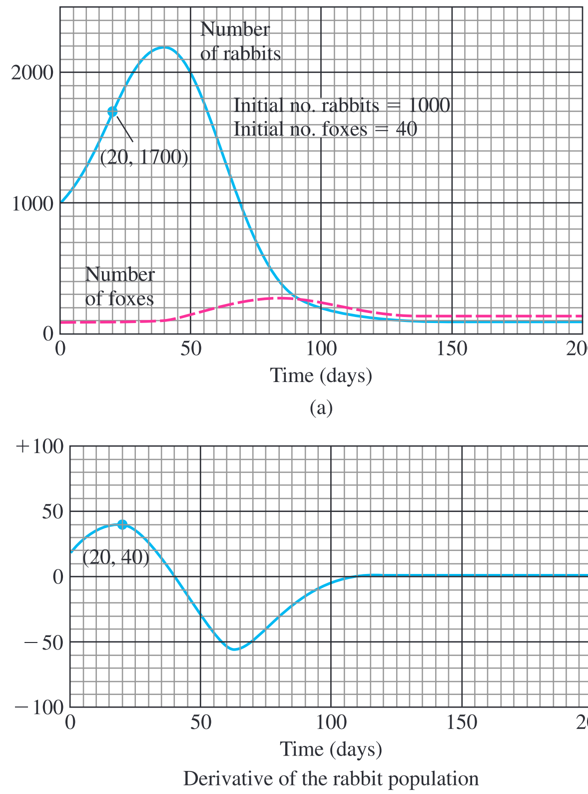


[–5, 5] by [–3, 3]

23. The graphs in Figure 3.10a show the numbers of rabbits and foxes in a small arctic population. They are plotted as functions of time for 200 days. The number of rabbits increases at first, as the rabbits reproduce. But the foxes prey on the rabbits and, as the number of foxes increases, the rabbit population levels off and then drops. Figure 3.10b shows the graph of the derivative of the rabbit population. We made it by plotting slopes, as in Example 3.

(a) What is the value of the derivative of the rabbit population in Figure 3.10 when the number of rabbits is largest? smallest?

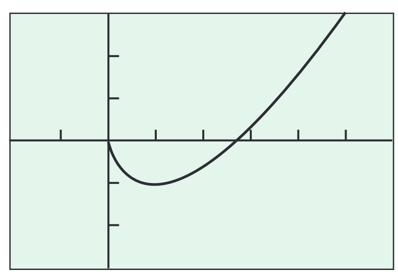
(b) What is the size of the rabbit population in Figure 3.10 when its derivative is largest? smallest?



(a)

Figure 3.10 Rabbits and foxes in an arctic predator-prey food chain. *Source: Differentiation* by W. U. Walton et al., Project CALC, Education Development Center, Inc., Newton, MA, 1975, p. 86.

24. Shown below is the graph of $f(x) = x \ln x - x$. From what you know about the graphs of functions (i) through (v), pick out the one that is the *derivative* of *f* for $x > 0$.



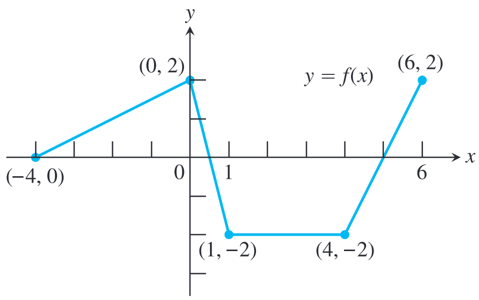
[–2, 6] by [–3, 3]

i. $y = \sin x$ **ii.** $y = \ln x$ **iii.** $y = \sqrt{x}$
iv. $y = x^2$ **v.** $y = 3x - 1$

25. From what you know about the graphs of functions (i) through (v), pick out the one that is *its own derivative.*

i. $y = \sin x$ **ii.** $y = x$ **iii.** $y = \sqrt{x}$
iv. $y = e^x$ **v.** $y = x^2$

26. The graph of the function $y = f(x)$ shown here is made of line segments joined end to end.



(a) Graph the function's derivative.

(b) At what values of x between $x = -4$ and $x = 6$ is the function not differentiable?

27. *Graphing f from f′* Sketch the graph of a continuous function f with $f(0) = -1$ and

$$f'(x) = \begin{cases} 1, & x < -1 \\ -2, & x > -1. \end{cases}$$

28. *Graphing f from f′* Sketch the graph of a continuous function f with $f(0) = 1$ and

$$f'(x) = \begin{cases} 2, & x < 2 \\ -1, & x > 2. \end{cases}$$

In Exercises 29 and 30, use the data to answer the questions.

29. *A Downhill Skier* Table 3.3 gives the approximate distance traveled by a downhill skier after t seconds for $0 \le t \le 10$. Use the method of Example 5 to sketch a graph of the derivative; then answer the following questions:

(a) What does the derivative represent?

(b) In what units would the derivative be measured?

(c) Can you guess an equation of the derivative by considering its graph?

Table 3.3 Skiing Distances

Time t (seconds)	Distance Traveled (feet)
0	0
1	3.3
2	13.3
3	29.9
4	53.2
5	83.2
6	119.8
7	163.0
8	212.9
9	269.5
10	332.7

30. *A Whitewater River* Bear Creek, a Georgia river known to kayaking enthusiasts, drops more than 770 feet over one stretch of 3.24 miles. By reading a contour map, one can estimate the elevations (y) at various distances (x) downriver from the start of the kayaking route (Table 3.4).

Table 3.4 Elevations along Bear Creek

Distance Downriver (miles)	River Elevation (feet)
0.00	1577
0.56	1512
0.92	1448
1.19	1384
1.30	1319
1.39	1255
1.57	1191
1.74	1126
1.98	1062
2.18	998
2.41	933
2.64	869
3.24	805

(a) Sketch a graph of elevation (y) as a function of distance downriver (x).

(b) Use the technique of Example 5 to get an approximate graph of the derivative, dy/dx.

(c) The average change in elevation over a given distance is called a *gradient*. In this problem, what units of measure would be appropriate for a gradient?

(d) In this problem, what units of measure would be appropriate for the derivative?

(e) How would you identify the most dangerous section of the river (ignoring rocks) by analyzing the graph in (a)? Explain.

(f) How would you identify the most dangerous section of the river by analyzing the graph in (b)? Explain.

31. Using one-sided derivatives, show that the function

$$f(x) = \begin{cases} x^2 + x, & x \le 1 \\ 3x - 2, & x > 1 \end{cases}$$

does not have a derivative at $x = 1$.

32. Using one-sided derivatives, show that the function

$$f(x) = \begin{cases} x^3, & x \le 1 \\ 3x, & x > 1 \end{cases}$$

does not have a derivative at $x = 1$.

33. Writing to Learn Graph $y = \sin x$ and $y = \cos x$ in the same viewing window. Which function could be the derivative of the other? Defend your answer in terms of the behavior of the graphs.

34. In Example 2 of this section we showed that the derivative of $y = \sqrt{x}$ is a function with domain $(0, \infty)$. However, the function $y = \sqrt{x}$ itself has domain $[0, \infty)$, so it could have a right-hand derivative at $x = 0$. Prove that it does not.

35. Writing to Learn Use the concept of the derivative to define what it might mean for two parabolas to be parallel. Construct equations for two such parallel parabolas and graph them. Are the parabolas "everywhere equidistant," and if so, in what sense?

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

36. True or False If $f(x) = x^2 + x$, then $f'(x)$ exists for every real number x. Justify your answer.

37. True or False If the left-hand derivative and the right-hand derivative of f exist at $x = a$, then $f'(a)$ exists. Justify your answer.

38. Multiple Choice Let $f(x) = 4 - 3x$. Which of the following is equal to $f'(-1)$?

(A) -7 **(B)** 7 **(C)** -3 **(D)** 3 **(E)** does not exist

39. Multiple Choice Let $f(x) = 1 - 3x^2$. Which of the following is equal to $f'(1)$?

(A) -6 **(B)** -5 **(C)** 5 **(D)** 6 **(E)** does not exist

In Exercises 40 and 41, let

$$f(x) = \begin{cases} x^2 - 1, & x < 0 \\ 2x - 1, & x \geq 0. \end{cases}$$

40. Multiple Choice Which of the following is equal to the left-hand derivative of f at $x = 0$?

(A) -2 **(B)** 0 **(C)** 2 **(D)** ∞ **(E)** $-\infty$

41. Multiple Choice Which of the following is equal to the right-hand derivative of f at $x = 0$?

(A) -2 **(B)** 0 **(C)** 2 **(D)** ∞ **(E)** $-\infty$

Explorations

42. Let $f(x) = \begin{cases} x^2, & x \leq 1 \\ 2x, & x > 1. \end{cases}$

(a) Find $f'(x)$ for $x < 1$.

(b) Find $f'(x)$ for $x > 1$.

(c) Find $\lim_{x \to 1^-} f'(x)$.

(d) Find $\lim_{x \to 1^+} f'(x)$.

(e) Does $\lim_{x \to 1} f'(x)$ exist? Explain.

(f) Use the definition to find the left-hand derivative of f at $x = 1$ if it exists.

(g) Use the definition to find the right-hand derivative of f at $x = 1$ if it exists.

(h) Does $f'(1)$ exist? Explain.

43. Group Activity Using graphing calculators, have each person in your group do the following:

(a) pick two numbers a and b between 1 and 10;

(b) graph the function $y = (x - a)(x + b)$;

(c) graph the *derivative* of your function (it will be a line with slope 2);

(d) find the y-intercept of your derivative graph.

(e) Compare your answers and determine a simple way to predict the y-intercept, given the values of a and b. Test your result.

Extending the Ideas

44. Find the unique value of k that makes the function

$$f(x) = \begin{cases} x^3, & x \leq 1 \\ 3x + k, & x > 1 \end{cases}$$

differentiable at $x = 1$.

45. ***Generating the Birthday Probabilities*** Example 5 of this section concerns the probability that, in a group of n people, at least two people will share a common birthday. You can generate these probabilities on your calculator for values of n from 1 to 365.

Step 1: Set the values of N and P to zero:

```
0→N:0→P
                    0
```

Step 2: Type in this single, multi-step command:

```
N+1→N:1–(1–P)(366
–N)/365→P: {N,P}
```

Now each time you press the ENTER key, the command will print a new value of N (the number of people in the room) alongside P (the probability that at least two of them share a common birthday):

```
            {1 0}
   {2 .002739726}
  {3 .0082041659}
  {4 .0163559125}
  {5 .0271355737}
  {6 .0404624836}
  {7 .0562357031}
```

If you have some experience with probability, try to answer the following questions without looking at the table:

(a) If there are three people in the room, what is the probability that they all have *different* birthdays? (Assume that there are 365 possible birthdays, all of them equally likely.)

(b) If there are three people in the room, what is the probability that at least two of them share a common birthday?

(c) Explain how you can use the answer in part (b) to find the probability of a shared birthday when there are *four* people in the room. (This is how the calculator statement in Step 2 generates the probabilities.)

(d) Is it reasonable to assume that all calendar dates are equally likely birthdays? Explain your answer.

3.2 Differentiability

What you'll learn about

- How $f'(a)$ Might Fail to Exist
- Differentiability Implies Local Linearity
- Derivatives on a Calculator
- Differentiability Implies Continuity
- Intermediate Value Theorem for Derivatives

. . . and why

Graphs of differentiable functions can be approximated by their tangent lines at points where the derivative exists.

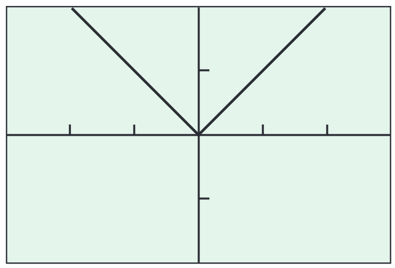
How $f'(a)$ Might Fail to Exist

A function will not have a derivative at a point $P(a, f(a))$ where the slopes of the secant lines,

$$\frac{f(x) - f(a)}{x - a},$$

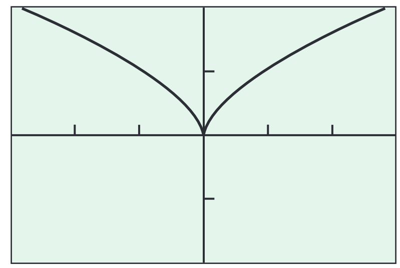
fail to approach a limit as x approaches a. Figures 3.11–3.14 illustrate four different instances where this occurs. For example, a function whose graph is otherwise smooth will fail to have a derivative at a point where the graph has

1. a *corner,* where the one-sided derivatives differ; Example: $f(x) = |x|$



[–3, 3] by [–2, 2]

Figure 3.11 There is a "corner" at $x = 0$.

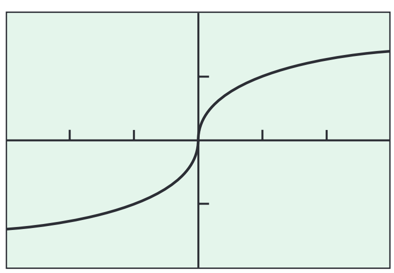


[–3, 3] by [–2, 2]

Figure 3.12 There is a "cusp" at $x = 0$.

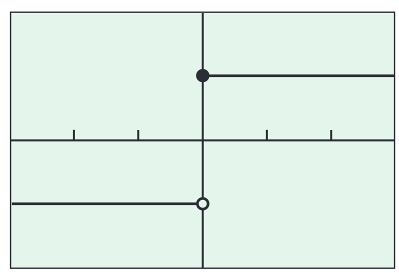
2. a *cusp,* where the slopes of the secant lines approach ∞ from one side and $-\infty$ from the other (an extreme case of a corner); Example: $f(x) = x^{2/3}$

3. a *vertical tangent,* where the slopes of the secant lines approach either ∞ or $-\infty$ from both sides (in this example, ∞); Example: $f(x) = \sqrt[3]{x}$



[–3, 3] by [–2, 2]

Figure 3.13 There is a vertical tangent line at $x = 0$.



[–3, 3] by [–2, 2]

Figure 3.14 There is a discontinuity at $x = 0$.

4. a *discontinuity* (which will cause one or both of the one-sided derivatives to be nonexistent). Example: The *Unit Step Function*

$$U(x) = \begin{cases} -1, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

In this example, the left-hand derivative fails to exist:

$$\lim_{h \to 0^-} \frac{(-1) - (1)}{h} = \lim_{h \to 0^-} \frac{-2}{h} = \infty.$$

How rough can the graph of a continuous function be?

The graph of the absolute value function fails to be differentiable at a single point. If you graph $y = \sin^{-1}(\sin(x))$ on your calculator, you will see a continuous function with an *infinite* number of points of nondifferentiability. But can a continuous function fail to be differentiable at *every* point?

The answer, surprisingly enough, is yes, as Karl Weierstrass showed in 1872. One of his formulas (there are many like it) was

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \cos(9^n \pi x),$$

a formula that expresses f as an infinite (but converging) sum of cosines with increasingly higher frequencies. By adding wiggles to wiggles infinitely many times, so to speak, the formula produces a function whose graph is too bumpy in the limit to have a tangent anywhere!

Later in this section we will prove a theorem that states that a function *must* be continuous at a to be differentiable at a. This theorem would provide a quick and easy verification that U is not differentiable at $x = 0$.

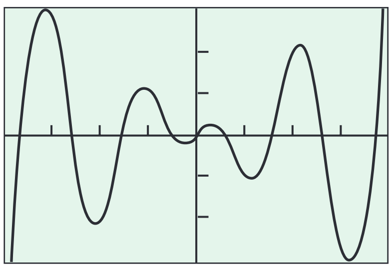
EXAMPLE 1 Finding Where a Function is not Differentiable

Find all points in the domain of $f(x) = |x - 2| + 3$ where f is not differentiable.

SOLUTION

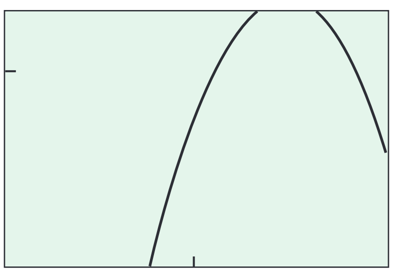
Think graphically! The graph of this function is the same as that of $y = |x|$, translated 2 units to the right and 3 units up. This puts the corner at the point (2, 3), so this function is not differentiable at $x = 2$.

At every other point, the graph is (locally) a straight line and f has derivative $+1$ or -1 (again, just like $y = |x|$). ***Now try Exercise 1.***



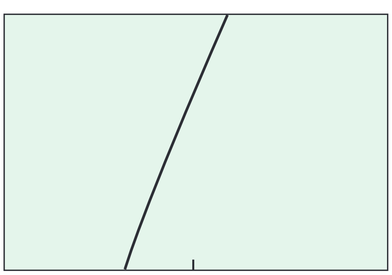
[–4, 4] by [–3, 3]

(a)



[1.7, 2.3] by [1.7, 2.1]

(b)



[1.93, 2.07] by [1.85, 1.95]

(c)

Figure 3.15 Three different views of the differentiable function $f(x) = x \cos(3x)$. We have zoomed in here at the point (2, 1.9).

Most of the functions we encounter in calculus are differentiable wherever they are defined, which means that they will *not* have corners, cusps, vertical tangent lines, or points of discontinuity within their domains. Their graphs will be unbroken and smooth, with a well-defined slope at each point. Polynomials are differentiable, as are rational functions, trigonometric functions, exponential functions, and logarithmic functions. Composites of differentiable functions are differentiable, and so are sums, products, integer powers, and quotients of differentiable functions, where defined. We will see why all of this is true as the chapter continues.

Differentiability Implies Local Linearity

A good way to think of differentiable functions is that they are **locally linear;** that is, a function that is differentiable at a closely resembles its own tangent line very close to a. In the jargon of graphing calculators, differentiable curves will "straighten out" when we zoom in on them at a point of differentiability. (See Figure 3.15.)

EXPLORATION 1 Zooming in to "See" Differentiability

Is either of these functions differentiable at $x = 0$?

(a) $f(x) = |x| + 1$ **(b)** $g(x) = \sqrt{x^2 + 0.0001} + 0.99$

1. We already know that f is not differentiable at $x = 0$; its graph has a corner there. Graph f and zoom in at the point (0, 1) several times. Does the corner show signs of straightening out?
2. Now do the same thing with g. Does the graph of g show signs of straightening out? We will learn a quick way to differentiate g in Section 3.6, but for now suffice it to say that it *is* differentiable at $x = 0$, and in fact has a horizontal tangent there.
3. How many zooms does it take before the graph of g looks exactly like a horizontal line?
4. Now graph f and g *together* in a standard square viewing window. They appear to be identical until you start zooming in. The differentiable function eventually straightens out, while the nondifferentiable function remains impressively unchanged.

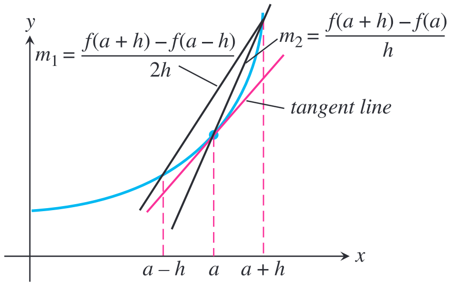


Figure 3.16 The symmetric difference quotient (slope m_1) usually gives a better approximation of the derivative for a given value of h than does the regular difference quotient (slope m_2), which is why the symmetric difference quotient is used in the numerical derivative.

Derivatives on a Calculator

Many graphing utilities can approximate derivatives numerically with good accuracy at most points of their domains.

For small values of h, the difference quotient

$$\frac{f(a+h)-f(a)}{h}$$

is often a good numerical approximation of $f'(a)$. However, as suggested by Figure 3.16, the same value of h will usually yield a *better* approximation if we use the **symmetric difference quotient**

$$\frac{f(a+h)-f(a-h)}{2h},$$

which is what our graphing calculator uses to calculate NDER $f(a)$, the **numerical derivative of *f* at a point *a*.** The **numerical derivative of *f*** as a function is denoted by NDER $f(x)$. Sometimes we will use NDER $(f(x), a)$ for NDER $f(a)$ when we want to emphasize both the function *and* the point.

Although the symmetric difference quotient is not the quotient used in the definition of $f'(a)$, it can be proven that

$$\lim_{h\to 0}\frac{f(a+h)-f(a-h)}{2h}$$

equals $f'(a)$ wherever $f'(a)$ exists.

You might think that an extremely small value of h would be required to give an accurate approximation of $f'(a)$, but in most cases $h = 0.001$ is more than adequate. In fact, your calculator probably assumes such a value for h unless you choose to specify otherwise (consult your *Owner's Manual*). The numerical derivatives we compute in this book will use $h = 0.001$; that is,

$$\text{NDER } f(a) = \frac{f(a+0.001)-f(a-0.001)}{0.002}.$$

EXAMPLE 2 Computing a Numerical Derivative

Compute NDER $(x^3, 2)$, the numerical derivative of x^3 at $x = 2$.

SOLUTION

Using $h = 0.001$,

$$\text{NDER }(x^3, 2) = \frac{(2.001)^3-(1.999)^3}{0.002} = 12.000001.$$

Now try Exercise 17.

In Example 1 of Section 3.1, we found the derivative of x^3 to be $3x^2$, whose value at $x = 2$ is $3(2)^2 = 12$. The numerical derivative is accurate to 5 decimal places. Not bad for the push of a button.

Example 2 gives dramatic evidence that NDER is very accurate when $h = 0.001$. Such accuracy is usually the case, although it is also possible for NDER to produce some surprisingly inaccurate results, as in Example 3.

EXAMPLE 3 Fooling the Symmetric Difference Quotient

Compute NDER $(|x|, 0)$, the numerical derivative of $|x|$ at $x = 0$.

continued

SOLUTION

We saw at the start of this section that $|x|$ is not differentiable at $x = 0$ since its right-hand and left-hand derivatives at $x = 0$ are not the same. Nonetheless,

$$\begin{aligned}\text{NDER}\,(|x|, 0) &= \lim_{h\to 0} \frac{|0+h| - |0-h|}{2h} \\ &= \lim_{h\to 0} \frac{|h| - |h|}{2h} \\ &= \lim_{h\to 0} \frac{0}{2h} \\ &= 0.\end{aligned}$$

The symmetric difference quotient, which works symmetrically on either side of 0, never detects the corner! Consequently, most graphing utilities will indicate (wrongly) that $y = |x|$ is differentiable at $x = 0$, with derivative 0.

Now try Exercise 23.

An Alternative to NDER

Graphing

$$y = \frac{f(x + 0.001) - f(x - 0.001)}{0.002}$$

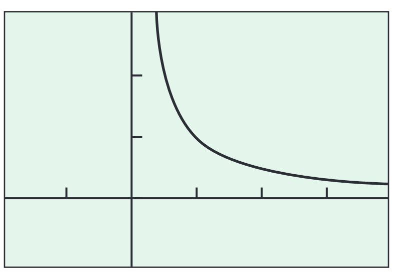
is equivalent to graphing $y = \text{NDER}\ f(x)$ (useful if NDER is not readily available on your calculator).

In light of Example 3, it is worth repeating here that NDER $f(a)$ actually does approach $f'(a)$ *when* $f'(a)$ *exists,* and in fact approximates it quite well (as in Example 2).

EXPLORATION 2 Looking at the Symmetric Difference Quotient Analytically

Let $f(x) = x^2$ and let $h = 0.01$.

1. Find
$$\frac{f(10+h) - f(10)}{h}.$$
How close is it to $f'(10)$?
2. Find
$$\frac{f(10+h) - f(10-h)}{2h}.$$
How close is it to $f'(10)$?
3. Repeat this comparison for $f(x) = x^3$.



[–2, 4] by [–1, 3]

(a)

X	Y1
.1	10
.2	5
.3	3.3333
.4	2.5
.5	2
.6	1.6667
.7	1.4286
X = .1	

(b)

Figure 3.17 (a) The graph of NDER ln (x) and (b) a table of values. What graph could this be? (Example 4)

EXAMPLE 4 Graphing a Derivative Using NDER

Let $f(x) = \ln x$. Use NDER to graph $y = f'(x)$. Can you guess what function $f'(x)$ is by analyzing its graph?

SOLUTION

The graph is shown in Figure 3.17a. The shape of the graph suggests, and the table of values in Figure 3.17b supports, the conjecture that this is the graph of $y = 1/x$. We will prove in Section 3.9 (using analytic methods) that this is indeed the case.

Now try Exercise 27.

Differentiability Implies Continuity

We began this section with a look at the typical ways that a function could fail to have a derivative at a point. As one example, we indicated graphically that a discontinuity in the graph of f would cause one or both of the one-sided derivatives to be nonexistent. It is actually not difficult to give an analytic proof that continuity is an essential condition for the derivative to exist, so we include that as a theorem here.

THEOREM 1 Differentiability Implies Continuity

If f has a derivative at $x = a$, then f is continuous at $x = a$.

Proof Our task is to show that $\lim_{x\to a} f(x) = f(a)$, or, equivalently, that

$$\lim_{x\to a} [f(x) - f(a)] = 0.$$

Using the Limit Product Rule (and noting that $x - a$ is not zero), we can write

$$\begin{aligned}\lim_{x\to a} [f(x) - f(a)] &= \lim_{x\to a} \left[(x - a)\frac{f(x) - f(a)}{x - a}\right] \\ &= \lim_{x\to a} (x - a) \cdot \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \\ &= 0 \cdot f'(a) \\ &= 0.\end{aligned}$$

■

The converse of Theorem 1 is false, as we have already seen. A continuous function might have a corner, a cusp, or a vertical tangent line, and hence not be differentiable at a given point.

Intermediate Value Theorem for Derivatives

Not every function can be a derivative. A derivative must have the intermediate value property, as stated in the following theorem (the proof of which can be found in advanced texts).

THEOREM 2 Intermediate Value Theorem for Derivatives

If a and b are any two points in an interval on which f is differentiable, then f' takes on every value between $f'(a)$ and $f'(b)$.

EXAMPLE 5 Applying Theorem 2

Does any function have the Unit Step Function (see Figure 3.14) as its derivative?

SOLUTION

No. Choose some $a < 0$ and some $b > 0$. Then $U(a) = -1$ and $U(b) = 1$, but U does not take on any value between -1 and 1. ***Now try Exercise 37.***

The question of when a function is a derivative of some function is one of the central questions in all of calculus. The answer, found by Newton and Leibniz, would revolutionize the world of mathematics. We will see what that answer is when we reach Chapter 5.

Quick Review 3.2 *(For help, go to Sections 1.2 and 2.1.)*

In Exercises 1–5, tell whether the limit could be used to define $f'(a)$ (assuming that f is differentiable at a).

1. $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

2. $\lim_{h \to 0} \frac{f(a+h) - f(h)}{h}$

3. $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

4. $\lim_{x \to a} \frac{f(a) - f(x)}{a - x}$

5. $\lim_{h \to 0} \frac{f(a+h) + f(a-h)}{h}$

6. Find the domain of the function $y = x^{4/3}$.

7. Find the domain of the function $y = x^{3/4}$.

8. Find the range of the function $y = |x - 2| + 3$.

9. Find the slope of the line $y - 5 = 3.2(x + \pi)$.

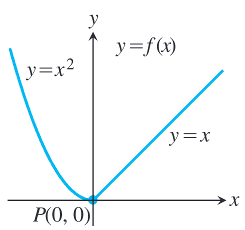
10. If $f(x) = 5x$, find

$$\frac{f(3 + 0.001) - f(3 - 0.001)}{0.002}.$$

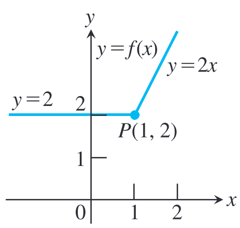
Section 3.2 Exercises

In Exercises 1–4, compare the right-hand and left-hand derivatives to show that the function is not differentiable at the point P. Find all points where f is not differentiable.

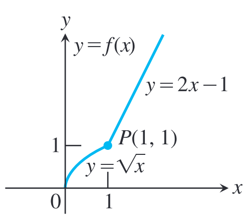
1.



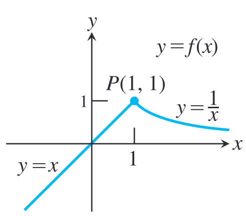
2.



3.



4.

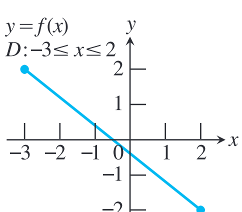


In Exercises 5–10, the graph of a function over a closed interval D is given. At what domain points does the function appear to be

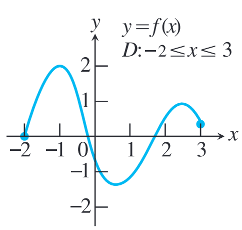
(a) differentiable? **(b)** continuous but not differentiable?

(c) neither continuous nor differentiable?

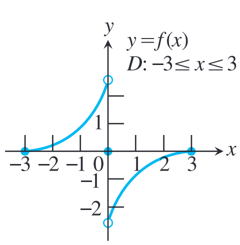
5.



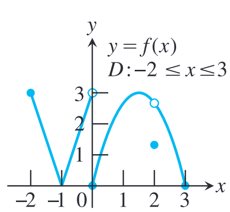
6.



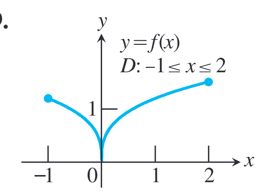
7.



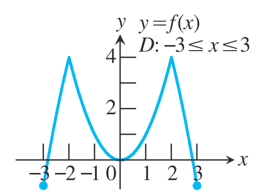
8.



9.



10.



In Exercises 11–16, the function fails to be differentiable at $x = 0$. Tell whether the problem is a corner, a cusp, a vertical tangent, or a discontinuity.

11. $y = \begin{cases} \tan^{-1} x, & x \neq 0 \\ 1, & x = 0 \end{cases}$

12. $y = x^{4/5}$

13. $y = x + \sqrt{x^2} + 2$

14. $y = 3 - \sqrt[3]{x}$

15. $y = 3x - 2|x| - 1$

16. $y = \sqrt[3]{|x|}$

In Exercises 17–26, find the numerical derivative of the given function at the indicated point. Use $h = 0.001$. Is the function differentiable at the indicated point?

17. $f(x) = 4x - x^2,\ x = 0$

18. $f(x) = 4x - x^2,\ x = 3$

19. $f(x) = 4x - x^2,\ x = 1$

20. $f(x) = x^3 - 4x,\ x = 0$

21. $f(x) = x^3 - 4x,\ x = -2$

22. $f(x) = x^3 - 4x,\ x = 2$

23. $f(x) = x^{2/3},\ x = 0$

24. $f(x) = |x - 3|,\ x = 3$

25. $f(x) = x^{2/5},\ x = 0$

26. $f(x) = x^{4/5},\ x = 0$

Group Activity In Exercises 27–30, use NDER to graph the derivative of the function. If possible, identify the derivative function by looking at the graph.

27. $y = -\cos x$

28. $y = 0.25x^4$

29. $y = \frac{x|x|}{2}$

30. $y = -\ln|\cos x|$

In Exercises 31–36, find all values of x for which the function is differentiable.

31. $f(x) = \frac{x^3 - 8}{x^2 - 4x - 5}$

32. $h(x) = \sqrt[3]{3x - 6} + 5$

33. $P(x) = \sin(|x|) - 1$

34. $Q(x) = 3\cos(|x|)$

35. $g(x) = \begin{cases} (x+1)^2, & x \le 0 \\ 2x + 1, & 0 < x < 3 \\ (4 - x)^2, & x \ge 3 \end{cases}$

36. $C(x) = x|x|$

37. Show that the function

$$f(x) = \begin{cases} 0, & -1 \le x < 0 \\ 1, & 0 \le x \le 1 \end{cases}$$

is not the derivative of any function on the interval $-1 \le x \le 1$.

38. Writing to Learn Recall that the numerical derivative (NDER) can give meaningless values at points where a function is not differentiable. In this exercise, we consider the numerical derivatives of the functions $1/x$ and $1/x^2$ at $x = 0$.

(a) Explain why neither function is differentiable at $x = 0$.

(b) Find NDER at $x = 0$ for each function.

(c) By analyzing the definition of the symmetric difference quotient, explain why NDER returns wrong responses that are so different from each other for these two functions.

39. Let f be the function defined as

$$f(x) = \begin{cases} 3 - x, & x < 1 \\ ax^2 + bx, & x \ge 1 \end{cases}$$

where a and b are constants.

(a) If the function is continuous for all x, what is the relationship between a and b?

(b) Find the unique values for a and b that will make f both continuous and differentiable.

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

40. True or False If f has a derivative at $x = a$, then f is continuous at $x = a$. Justify your answer.

41. True or False If f is continuous at $x = a$, then f has a derivative at $x = a$. Justify your answer.

42. Multiple Choice Which of the following is true about the graph of $f(x) = x^{4/5}$ at $x = 0$?

(A) It has a corner.

(B) It has a cusp.

(C) It has a vertical tangent.

(D) It has a discontinuity.

(E) $f(0)$ does not exist.

43. Multiple Choice Let $f(x) = \sqrt[3]{x - 1}$. At which of the following points is $f'(a) \ne \text{NDER}(f, x, a)$?

(A) $a = 1$ **(B)** $a = -1$ **(C)** $a = 2$ **(D)** $a = -2$ **(E)** $a = 0$

In Exercises 44 and 45, let

$$f(x) = \begin{cases} 2x + 1, & x \le 0 \\ x^2 + 1, & x > 0. \end{cases}$$

44. Multiple Choice Which of the following is equal to the left-hand derivative of f at $x = 0$?

(A) $2x$ **(B)** 2 **(C)** 0 **(D)** $-\infty$ **(E)** ∞

45. Multiple Choice Which of the following is equal to the right-hand derivative of f at $x = 0$?

(A) $2x$ **(B)** 2 **(C)** 0 **(D)** $-\infty$ (E) ∞

Explorations

46. (a) Enter the expression "$x < 0$" into Y1 of your calculator using "<" from the TEST menu. Graph Y1 in DOT MODE in the window $[-4.7, 4.7]$ by $[-3.1, 3.1]$.

(b) Describe the graph in part (a).

(c) Enter the expression "$x \ge 0$" into Y1 of your calculator using "≥" from the TEST menu. Graph Y1 in DOT MODE in the window $[-4.7, 4.7]$ by $[-3.1, 3.1]$.

(d) Describe the graph in part (c).

47. *Graphing Piecewise Functions on a Calculator* Let

$$f(x) = \begin{cases} x^2, & x \le 0 \\ 2x, & x > 0. \end{cases}$$

(a) Enter the expression "(X²)(X≤0)+(2X)(X>0)" into Y1 of your calculator and draw its graph in the window $[-4.7, 4.7]$ by $[-3, 5]$.

(b) Explain why the values of Y1 and $f(x)$ are the same.

(c) Enter the numerical derivative of Y1 into Y2 of your calculator and draw its graph in the same window. Turn off the graph of Y1.

(d) Use TRACE to calculate NDER(Y1, x, -0.1), NDER(Y1, x, 0), and NDER(Y1, x, 0.1). Compare with Section 3.1, Example 6.

Extending the Ideas

48. *Oscillation* There is another way that a function might fail to be differentiable, and that is by *oscillation.* Let

$$f(x) = \begin{cases} x \sin \dfrac{1}{x}, & x \ne 0 \\ 0, & x = 0. \end{cases}$$

(a) Show that f is continuous at $x = 0$.

(b) Show that

$$\frac{f(0 + h) - f(0)}{h} = \sin \frac{1}{h}.$$

(c) Explain why

$$\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h}$$

does not exist.

(d) Does f have either a left-hand or right-hand derivative at $x = 0$?

(e) Now consider the function

$$g(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & x \ne 0 \\ 0, & x = 0. \end{cases}$$

Use the definition of the derivative to show that g is differentiable at $x = 0$ and that $g'(0) = 0$.

3.3 Rules for Differentiation

What you'll learn about

- Positive Integer Powers, Multiples, Sums, and Differences
- Products and Quotients
- Negative Integer Powers of x
- Second and Higher Order Derivatives

. . . and why

These rules help us find derivatives of functions analytically more efficiently.

Positive Integer Powers, Multiples, Sums, and Differences

The first rule of differentiation is that the derivative of every constant function is the zero function.

RULE 1 Derivative of a Constant Function

If f is the function with the constant value c, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

Proof of Rule 1 If $f(x) = c$ is a function with a constant value c, then

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0. \quad \blacksquare$$

The next rule is a first step toward a rule for differentiating any polynomial.

RULE 2 Power Rule for Positive Integer Powers of x

If n is a positive integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Proof of Rule 2 If $f(x) = x^n$, then $f(x+h) = (x+h)^n$ and the difference quotient for f is

$$\frac{(x+h)^n - x^n}{h}.$$

We can readily find the limit of this quotient as $h \to 0$ if we apply the algebraic identity

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$$

with $a = x + h$ and $b = x$. For then $(a - b) = h$ and the h's in the numerator and denominator of the quotient cancel, giving

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{(x+h)^n - x^n}{h} \\ &= \frac{h[(x+h)^{n-1} + (x+h)^{n-2}x + \cdots + (x+h)x^{n-2} + x^{n-1}]}{h} \\ &= \underbrace{(x+h)^{n-1} + (x+h)^{n-2}x + \cdots + (x+h)x^{n-2} + x^{n-1}}.\end{aligned}$$

Hence,

$$\frac{d}{dx}(x^n) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = nx^{n-1}. \quad \blacksquare$$

The Power Rule says: To differentiate x^n, multiply by n and subtract 1 from the exponent. For example, the derivatives of x^2, x^3, and x^4 are $2x^1$, $3x^2$, and $4x^3$, respectively.

RULE 3 The Constant Multiple Rule

If u is a differentiable function of x and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

Proof of Rule 3

$$\begin{aligned}\frac{d}{dx}(cu) &= \lim_{h\to 0}\frac{cu(x+h)-cu(x)}{h}\\ &= c\lim_{h\to 0}\frac{u(x+h)-u(x)}{h}\\ &= c\frac{du}{dx}\end{aligned}$$

■

Rule 3 says that if a differentiable function is multiplied by a constant, then its derivative is multiplied by the same constant. Combined with Rule 2, it enables us to find the derivative of any monomial quickly; for example, the derivative of $7x^4$ is $7(4x^3) = 28x^3$.

To find the derivatives of polynomials, we need to be able to differentiate sums and differences of monomials. We can accomplish this by applying the Sum and Difference Rule.

Denoting Functions by *u* and *v*

The functions we work with when we need a differentiation formula are likely to be denoted by letters like f and g. When we apply the formula, we do not want to find the formula using these same letters in some other way. To guard against this, we denote the functions in differentiation rules by letters like u and v that are not likely to be already in use.

RULE 4 The Sum and Difference Rule

If u and v are differentiable functions of x, then their sum and difference are differentiable at every point where u and v are differentiable. At such points,

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}.$$

Proof of Rule 4

We use the difference quotient for $f(x) = u(x) + v(x)$.

$$\begin{aligned}\frac{d}{dx}[u(x)+v(x)] &= \lim_{h\to 0}\frac{[u(x+h)+v(x+h)]-[u(x)+v(x)]}{h}\\ &= \lim_{h\to 0}\left[\frac{u(x+h)-u(x)}{h}+\frac{v(x+h)-v(x)}{h}\right]\\ &= \lim_{h\to 0}\frac{u(x+h)-u(x)}{h}+\lim_{h\to 0}\frac{v(x+h)-v(x)}{h}\\ &= \frac{du}{dx}+\frac{dv}{dx}\end{aligned}$$

The proof of the rule for the difference of two functions is similar. ■

EXAMPLE 1 Differentiating a Polynomial

Find $\dfrac{dp}{dt}$ if $p = t^3 + 6t^2 - \dfrac{5}{3}t + 16.$

SOLUTION

By Rule 4 we can differentiate the polynomial term-by-term, applying Rules 1 through 3 as we go.

$$\begin{aligned}\frac{dp}{dt} &= \frac{d}{dt}(t^3) + \frac{d}{dt}(6t^2) - \frac{d}{dt}\left(\frac{5}{3}t\right) + \frac{d}{dt}(16)\\ &= 3t^2 + 6 \cdot 2t - \frac{5}{3} + 0\\ &= 3t^2 + 12t - \frac{5}{3}\end{aligned}$$

Now try Exercise 5.

EXAMPLE 2 Finding Horizontal Tangents

Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so, where?

SOLUTION

The horizontal tangents, if any, occur where the slope dy/dx is zero. To find these points, we

(a) calculate dy/dx:

$$\frac{dy}{dx} = \frac{d}{dx}(x^4 - 2x^2 + 2) = 4x^3 - 4x.$$

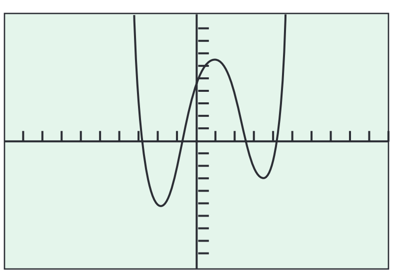
(b) solve the equation $dy/dx = 0$ for x:

$$\begin{aligned}4x^3 - 4x &= 0\\ 4x(x^2 - 1) &= 0\\ x &= 0, 1, -1.\end{aligned}$$

The curve has horizontal tangents at $x = 0, 1,$ and $-1.$ The corresponding points on the curve (found from the equation $y = x^4 - 2x^2 + 2$) are $(0, 2),$ $(1, 1),$ and $(-1, 1).$ You might wish to graph the curve to see where the horizontal tangents go.

Now try Exercise 7.

The derivative in Example 2 was easily factored, making an algebraic solution of the equation $dy/dx = 0$ correspondingly simple. When a simple algebraic solution is not possible, the solutions to $dy/dx = 0$ can still be found to a high degree of accuracy by using the SOLVE capability of your calculator.



[–10, 10] by [–10, 10]

Figure 3.18 The graph of

$y = 0.2x^4 - 0.7x^3 - 2x^2 + 5x + 4$

has three horizontal tangents. (Example 3)

EXAMPLE 3 Using Calculus and Calculator

As can be seen in the viewing window $[-10, 10]$ by $[-10, 10]$, the graph of $y = 0.2x^4 - 0.7x^3 - 2x^2 + 5x + 4$ has three horizontal tangents (Figure 3.18). At what points do these horizontal tangents occur?

continued

SOLUTION

First we find the derivative

$$\frac{dy}{dx} = 0.8x^3 - 2.1x^2 - 4x + 5.$$

Using the calculator solver, we find that $0.8x^3 - 2.1x^2 - 4x + 5 = 0$ when $x \approx -1.862$, 0.9484, and 3.539. We use the calculator again to evaluate the original function at these x-values and find the corresponding points to be approximately $(-1.862, -5.321)$, $(0.9484, 6.508)$, and $(3.539, -3.008)$.

Now try Exercise 11.

On Rounding Calculator Values

Notice in Example 3 that we rounded the x-values to four significant digits when we presented the answers. The calculator actually presented many more digits, but there was no practical reason for writing all of them. When we used the calculator to compute the corresponding y-values, however, we *used the x-values stored in the calculator,* not the rounded values. We then rounded the y-values to four significant digits when we presented the ordered pairs. Significant "round-off errors" can accumulate in a problem if you use rounded intermediate values for doing additional computations, so avoid rounding until the final answer.

Products and Quotients

While the derivative of the sum of two functions is the sum of their derivatives and the derivative of the difference of two functions is the difference of their derivatives, the derivative of the product of two functions is *not* the product of their derivatives.

For instance,

$$\frac{d}{dx}(x \cdot x) = \frac{d}{dx}(x^2) = 2x, \quad \text{while} \quad \frac{d}{dx}(x) \cdot \frac{d}{dx}(x) = 1 \cdot 1 = 1.$$

The derivative of a product is actually the sum of *two* products, as we now explain.

You can remember the Product Rule with the phrase "the first times the derivative of the second plus the second times the derivative of the first."

RULE 5 The Product Rule

The product of two differentiable functions u and v is differentiable, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Proof of Rule 5 We begin, as usual, by applying the definition.

$$\frac{d}{dx}(uv) = \lim_{h\to 0} \frac{u(x+h)v(x+h) - u(x)v(x)}{h}$$

To change the fraction into an equivalent one that contains difference quotients for the derivatives of u and v, we subtract and add $u(x+h)v(x)$ in the numerator. Then,

$$\frac{d}{dx}(uv) = \lim_{h\to 0} \frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)}{h}$$

$$= \lim_{h\to 0} \left[u(x+h)\frac{v(x+h) - v(x)}{h} + v(x)\frac{u(x+h) - u(x)}{h} \right]$$

$$= \lim_{h\to 0} u(x+h) \cdot \lim_{h\to 0} \frac{v(x+h) - v(x)}{h} + v(x) \cdot \lim_{h\to 0} \frac{u(x+h) - u(x)}{h}.$$

As h approaches 0, $u(x+h)$ approaches $u(x)$ because u, being differentiable at x, is continuous at x. The two fractions approach the values of dv/dx and du/dx, respectively, at x. Therefore

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$ ■

Gottfried Wilhelm Leibniz (1646–1716)

The method of limits used in this book was not discovered until nearly a century after Newton and Leibniz, the discoverers of calculus, had died.

To Leibniz, the key idea was the *differential,* an infinitely small quantity that was almost like zero, but which—unlike zero—could be used in the denominator of a fraction. Thus, Leibniz thought of the derivative dy/dx as the quotient of two differentials, dy and dx.

The problem was explaining why these differentials sometimes became zero and sometimes did not! See Exercise 59.

Some 17th-century mathematicians were confident that the calculus of Newton and Leibniz would eventually be found to be fatally flawed because of these mysterious quantities. It was only after later generations of mathematicians had found better ways to prove their results that the calculus of Newton and Leibniz was accepted by the entire scientific community.

EXAMPLE 4 Differentiating a Product

Find $f'(x)$ if $f(x) = (x^2 + 1)(x^3 + 3)$.

SOLUTION

From the Product Rule with $u = x^2 + 1$ and $v = x^3 + 3$, we find

$$\begin{aligned} f'(x) = \frac{d}{dx}[(x^2+1)(x^3+3)] &= (x^2+1)(3x^2) + (x^3+3)(2x) \\ &= 3x^4 + 3x^2 + 2x^4 + 6x \\ &= 5x^4 + 3x^2 + 6x. \end{aligned}$$

Now try Exercise 13.

We could also have done Example 4 by multiplying out the original expression and then differentiating the resulting polynomial. That alternate strategy will not work, however, on a product like $x^2 \sin x$.

Just as the derivative of the product of two differentiable functions is not the product of their derivatives, the derivative of a quotient of two functions is not the quotient of their derivatives. What happens instead is this:

Using the Quotient Rule

Since order is important in subtraction, be sure to set up the numerator of the Quotient Rule correctly:

v times the derivative of *u*

minus

u times the derivative of *v*.

You can remember the Quotient Rule with the phrase "bottom times the derivative of the top minus the top times the derivative of the bottom, all over the bottom squared."

RULE 6 The Quotient Rule

At a point where $v \neq 0$, the quotient $y = u/v$ of two differentiable functions is differentiable, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

Proof of Rule 6

$$\begin{aligned} \frac{d}{dx}\left(\frac{u}{v}\right) &= \lim_{h\to 0} \frac{\dfrac{u(x+h)}{v(x+h)} - \dfrac{u(x)}{v(x)}}{h} \\ &= \lim_{h\to 0} \frac{v(x)u(x+h) - u(x)v(x+h)}{hv(x+h)v(x)} \end{aligned}$$

To change the last fraction into an equivalent one that contains the difference quotients for the derivatives of u and v, we subtract and add $v(x)u(x)$ in the numerator. This allows us to continue with

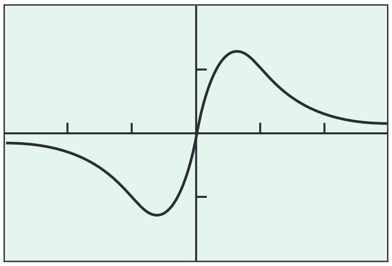
$$\begin{aligned} \frac{d}{dx}\left(\frac{u}{v}\right) &= \lim_{h\to 0} \frac{v(x)u(x+h) - v(x)u(x) + v(x)u(x) - u(x)v(x+h)}{hv(x+h)v(x)} \\ &= \lim_{h\to 0} \frac{v(x)\dfrac{u(x+h)-u(x)}{h} - u(x)\dfrac{v(x+h)-v(x)}{h}}{v(x+h)v(x)}. \end{aligned}$$

Taking the limits in both the numerator and denominator now gives us the Quotient Rule. ■

EXAMPLE 5 Supporting Computations Graphically

Differentiate $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$. Support graphically.

continued



[–3, 3] by [–2, 2]

Figure 3.19 The graph of

$$y = \frac{4x}{(x^2 + 1)^2}$$

and the graph of

$$y = \text{NDER}\left(\frac{x^2 - 1}{x^2 + 1}\right)$$

appear to be the same. (Example 5)

SOLUTION

We apply the Quotient Rule with $u = x^2 - 1$ and $v = x^2 + 1$:

$$\begin{aligned} f'(x) &= \frac{(x^2 + 1) \cdot 2x - (x^2 - 1) \cdot 2x}{(x^2 + 1)^2} \\ &= \frac{2x^3 + 2x - 2x^3 + 2x}{(x^2 + 1)^2} \\ &= \frac{4x}{(x^2 + 1)^2}. \end{aligned}$$

The graphs of $y_1 = f'(x)$ calculated above and of $y_2 = \text{NDER}\, f(x)$ are shown in Figure 3.19. The fact that they appear to be identical provides strong graphical support that our calculations are indeed correct. *Now try Exercise 19.*

EXAMPLE 6 Working with Numerical Values

Let $y = uv$ be the product of the functions u and v. Find $y'(2)$ if

$$u(2) = 3, \quad u'(2) = -4, \quad v(2) = 1, \quad \text{and} \quad v'(2) = 2.$$

SOLUTION

From the Product Rule, $y' = (uv)' = uv' + vu'$. In particular,

$$\begin{aligned} y'(2) &= u(2)v'(2) + v(2)u'(2) \\ &= (3)(2) + (1)(-4) \\ &= 2. \end{aligned}$$

Now try Exercise 23.

Negative Integer Powers of *x*

The rule for differentiating negative powers of x is the same as Rule 2 for differentiating positive powers of x, although our proof of Rule 2 does not work for negative values of n. We can now extend the Power Rule to negative integer powers by a clever use of the Quotient Rule.

RULE 7 Power Rule for Negative Integer Powers of *x*

If n is a negative integer and $x \neq 0$, then

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Proof of Rule 7 If n is a negative integer, then $n = -m$, where m is a positive integer. Hence, $x^n = x^{-m} = 1/x^m$, and

$$\begin{aligned} \frac{d}{dx}(x^n) = \frac{d}{dx}\left(\frac{1}{x^m}\right) &= \frac{x^m \cdot \frac{d}{dx}(1) - 1 \cdot \frac{d}{dx}(x^m)}{(x^m)^2} \\ &= \frac{0 - mx^{m-1}}{x^{2m}} \\ &= -mx^{-m-1} \\ &= nx^{n-1}. \end{aligned}$$

■

EXAMPLE 7 Using the Power Rule

Find an equation for the line tangent to the curve

$$y = \frac{x^2 + 3}{2x}$$

at the point (1, 2). Support your answer graphically.

SOLUTION

We could find the derivative by the Quotient Rule, but it is easier to first simplify the function as a sum of two powers of x.

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2}{2x} + \frac{3}{2x}\right)$$

$$= \frac{d}{dx}\left(\frac{1}{2}x + \frac{3}{2}x^{-1}\right)$$

$$= \frac{1}{2} - \frac{3}{2}x^{-2}$$

The slope at $x = 1$ is

$$\left.\frac{dy}{dx}\right|_{x=1} = \left[\frac{1}{2} - \frac{3}{2}x^{-2}\right]_{x=1} = \frac{1}{2} - \frac{3}{2} = -1.$$

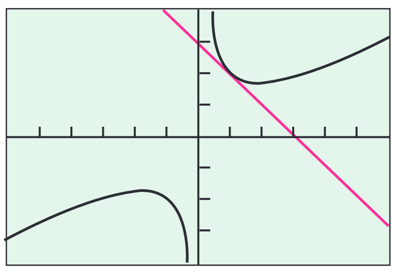
The line through (1, 2) with slope $m = -1$ is

$$y - 2 = (-1)(x - 1)$$

$$y = -x + 1 + 2$$

$$y = -x + 3.$$

We graph $y = (x^2 + 3)/2x$ and $y = -x + 3$ (Figure 3.20), observing that the line appears to be tangent to the curve at (1, 2). Thus, we have graphical support that our computations are correct. *Now try Exercise 27.*



[–6, 6] by [–4, 4]

Figure 3.20 The line $y = -x + 3$ appears to be tangent to the graph of

$$y = \frac{x^2 + 3}{2x}$$

at the point (1, 2). (Example 7)

Second and Higher Order Derivatives

The derivative $y' = dy/dx$ is called the *first derivative* of y with respect to x. The first derivative may itself be a differentiable function of x. If so, its derivative,

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2},$$

is called the *second derivative* of y with respect to x. If y'' ("y double-prime") is differentiable, its derivative,

$$y''' = \frac{dy''}{dx} = \frac{d^3y}{dx^3},$$

is called the *third derivative* of y with respect to x. The names continue as you might expect they would, except that the multiple-prime notation begins to lose its usefulness after about three primes. We use

$$y^{(n)} = \frac{d}{dx}y^{(n-1)}$$

to denote the ***n*th derivative** of y with respect to x. (We also use d^ny/dx^n.) Do not confuse $y^{(n)}$ with the nth power of y, which is y^n.

Technology Tip

HIGHER ORDER DERIVATIVES WITH NDER

Some graphers will allow the *nesting* of the NDER function,

NDER2 f = NDER(NDER f),

but such nesting, in general, is safe only to the second derivative. Beyond that, the error buildup in the algorithm makes the results unreliable.

EXAMPLE 8 Finding Higher Order Derivatives

Find the first four derivatives of $y = x^3 - 5x^2 + 2$.

SOLUTION

The first four derivatives are:

First derivative: $y' = 3x^2 - 10x$;

Second derivative: $y'' = 6x - 10$;

Third derivative: $y''' = 6$;

Fourth derivative: $y^{(4)} = 0$.

This function has derivatives of all orders, the fourth and higher order derivatives all being zero. ***Now try Exercise 33.***

EXAMPLE 9 Finding Instantaneous Rate of Change

An orange farmer currently has 200 trees yielding an average of 15 bushels of oranges per tree. She is expanding her farm at the rate of 15 trees per year, while improved husbandry is improving her average annual yield by 1.2 bushels per tree. What is the current (instantaneous) rate of increase of her total annual production of oranges?

SOLUTION

Let the functions t and y be defined as follows.

$t(x)$ = the number of trees x years from now.

$y(x)$ = yield per tree x years from now.

Then $p(x) = t(x)y(x)$ is the total production of oranges in year x. We know the following values.

$$t(0) = 200, \qquad y(0) = 15$$

$$t'(0) = 15, \qquad y'(0) = 1.2$$

We need to find $p'(0)$, where $p = ty$.

$$\begin{aligned} p'(0) &= t(0)y'(0) + y(0)t'(0) \\ &= (200)(1.2) + (15)(15) \\ &= 465 \end{aligned}$$

The rate we seek is 465 bushels per year. ***Now try Exercise 51.***

Quick Review 3.3 *(For help, go to Sections 1.2 and 3.1.)*

In Exercises 1–6, write the expression as a sum of powers of x.

1. $(x^2 - 2)(x^{-1} + 1)$

2. $\left(\dfrac{x}{x^2 + 1}\right)^{-1}$

3. $3x^2 - \dfrac{2}{x} + \dfrac{5}{x^2}$

4. $\dfrac{3x^4 - 2x^3 + 4}{2x^2}$

5. $(x^{-1} + 2)(x^{-2} + 1)$

6. $\dfrac{x^{-1} + x^{-2}}{x^{-3}}$

7. Find the positive roots of the equation

$$2x^3 - 5x^2 - 2x + 6 = 0$$

and evaluate the function $y = 500x^6$ at each root. Round your answers to the nearest integer, but only in the final step.

8. If $f(x) = 7$ for all real numbers x, find

(a) $f(10)$. **(b)** $f(0)$.

(c) $f(x + h)$. **(d)** $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$.

9. Find the derivatives of these functions with respect to x.

(a) $f(x) = \pi$ **(b)** $f(x) = \pi^2$ **(c)** $f(x) = \pi^{15}$

10. Find the derivatives of these functions with respect to x using the definition of the derivative.

(a) $f(x) = \frac{x}{\pi}$ **(b)** $f(x) = \frac{\pi}{x}$

Section 3.3 Exercises

In Exercises 1–6, find dy/dx.

1. $y = -x^2 + 3$ **2.** $y = \frac{x^3}{3} - x$

3. $y = 2x + 1$ **4.** $y = x^2 + x + 1$

5. $y = \frac{x^3}{3} + \frac{x^2}{2} + x$ **6.** $y = 1 - x + x^2 - x^3$

In Exercises 7–12, find the horizontal tangents of the curve.

7. $y = x^3 - 2x^2 + x + 1$ **8.** $y = x^3 - 4x^2 + x + 2$

9. $y = x^4 - 4x^2 + 1$ **10.** $y = 4x^3 - 6x^2 - 1$

11. $y = 5x^3 - 3x^5$ **12.** $y = x^4 - 7x^3 + 2x^2 + 15$

13. Let $y = (x + 1)(x^2 + 1)$. Find dy/dx **(a)** by applying the Product Rule, and **(b)** by multiplying the factors first and then differentiating.

14. Let $y = (x^2 + 3)/x$. Find dy/dx **(a)** by using the Quotient Rule, and **(b)** by first dividing the terms in the numerator by the denominator and then differentiating.

In Exercises 15–22, find dy/dx. Support your answer graphically.

15. $(x^3 + x + 1)(x^4 + x^2 + 1)$ **16.** $(x^2 + 1)(x^3 + 1)$

17. $y = \frac{2x + 5}{3x - 2}$ **18.** $y = \frac{x^2 + 5x - 1}{x^2}$

19. $y = \frac{(x - 1)(x^2 + x + 1)}{x^3}$ **20.** $y = (1 - x)(1 + x^2)^{-1}$

21. $y = \frac{x^2}{1 - x^3}$ **22.** $y = \frac{(x + 1)(x + 2)}{(x - 1)(x - 2)}$

23. Suppose u and v are functions of x that are differentiable at $x = 0$, and that $u(0) = 5$, $u'(0) = -3$, $v(0) = -1$, $v'(0) = 2$. Find the values of the following derivatives at $x = 0$.

(a) $\frac{d}{dx}(uv)$ **(b)** $\frac{d}{dx}\left(\frac{u}{v}\right)$

(c) $\frac{d}{dx}\left(\frac{v}{u}\right)$ **(d)** $\frac{d}{dx}(7v - 2u)$

24. Suppose u and v are functions of x that are differentiable at $x = 2$ and that $u(2) = 3$, $u'(2) = -4$, $v(2) = 1$, and $v'(2) = 2$. Find the values of the following derivatives at $x = 2$.

(a) $\frac{d}{dx}(uv)$ **(b)** $\frac{d}{dx}\left(\frac{u}{v}\right)$

(c) $\frac{d}{dx}\left(\frac{v}{u}\right)$ **(d)** $\frac{d}{dx}(3u - 2v + 2uv)$

25. Which of the following numbers is the slope of the line tangent to the curve $y = x^2 + 5x$ at $x = 3$?

i. 24 **ii.** $-5/2$ **iii.** 11 **iv.** 8

26. Which of the following numbers is the slope of the line $3x - 2y + 12 = 0$?

i. 6 **ii.** 3 **iii.** 3/2 **iv.** 2/3

In Exercises 27 and 28, find an equation for the line tangent to the curve at the given point.

27. $y = \frac{x^3 + 1}{2x}$, $x = 1$ **28.** $y = \frac{x^4 + 2}{x^2}$, $x = -1$

In Exercises 29–32, find dy/dx.

29. $y = 4x^{-2} - 8x + 1$

30. $y = \frac{x^{-4}}{4} - \frac{x^{-3}}{3} + \frac{x^{-2}}{2} - x^{-1} + 3$

31. $y = \frac{\sqrt{x} - 1}{\sqrt{x} + 1}$ **32.** $y = 2\sqrt{x} - \frac{1}{\sqrt{x}}$

In Exercises 33–36, find the first four derivatives of the function.

33. $y = x^4 + x^3 - 2x^2 + x - 5$ **34.** $y = x^2 + x + 3$

35. $y = x^{-1} + x^2$ **36.** $y = \frac{x + 1}{x}$

In Exercises 37–42, support your answer graphically.

37. Find an equation of the line perpendicular to the tangent to the curve $y = x^3 - 3x + 1$ at the point (2, 3).

38. Find the tangents to the curve $y = x^3 + x$ at the points where the slope is 4. What is the smallest slope of the curve? At what value of x does the curve have this slope?

39. Find the points on the curve $y = 2x^3 - 3x^2 - 12x + 20$ where the tangent is parallel to the x-axis.

40. Find the x- and y-intercepts of the line that is tangent to the curve $y = x^3$ at the point $(-2, -8)$.

41. Find the tangents to *Newton's serpentine,*

$$y = \frac{4x}{x^2 + 1},$$

at the origin and the point (1, 2).

42. Find the tangent to the *witch of Agnesi,*

$$y = \frac{8}{4 + x^2},$$

at the point (2, 1).

43. Use the definition of derivative (given in Section 3.1, Equation 1) to show that

(a) $\frac{d}{dx}(x) = 1.$

(b) $\frac{d}{dx}(-u) = -\frac{du}{dx}.$

44. Use the Product Rule to show that

$$\frac{d}{dx}(c \cdot f(x)) = c \cdot \frac{d}{dx} f(x)$$

for any constant c.

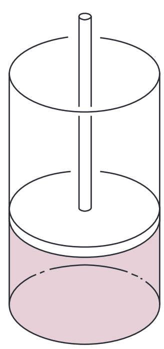
45. Devise a rule for $\frac{d}{dx}\left(\frac{1}{f(x)}\right)$.

When we work with functions of a single variable in mathematics, we often call the independent variable x and the dependent variable y. Applied fields use many different letters, however. Here are some examples.

46. ***Cylinder Pressure*** If gas in a cylinder is maintained at a constant temperature T, the pressure P is related to the volume V by a formula of the form

$$P = \frac{nRT}{V - nb} - \frac{an^2}{V^2},$$

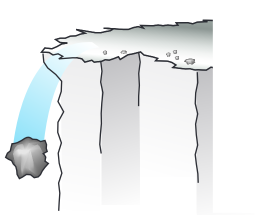
in which a, b, n, and R are constants. Find dP/dV.



47. ***Free Fall*** When a rock falls from rest near the surface of the earth, the distance it covers during the first few seconds is given by the equation

$$s = 4.9t^2.$$

In this equation, s is the distance in meters and t is the elapsed time in seconds. Find ds/dt and d^2s/dt^2.



Group Activity In Exercises 48–52, *work in groups of two or three* to solve the problems.

48. ***The Body's Reaction to Medicine*** The reaction of the body to a dose of medicine can often be represented by an equation of the form

$$R = M^2\left(\frac{C}{2} - \frac{M}{3}\right),$$

where C is a positive constant and M is the amount of medicine absorbed in the blood. If the reaction is a change in blood pressure, R is measured in millimeters of mercury. If the reaction is a change in temperature, R is measured in degrees, and so on.

Find dR/dM. This derivative, as a function of M, is called the sensitivity of the body to medicine. In Chapter 4, we shall see how to find the amount of medicine to which the body is most sensitive. *Source: Some Mathematical Models in Biology,* Revised Edition, December 1967, PB-202 364, p. 221; distributed by N.T.I.S., U.S. Department of Commerce.

49. **Writing to Learn** Recall that the area A of a circle with radius r is πr^2 and that the circumference C is $2\pi r$. Notice that $dA/dr = C$. Explain in terms of geometry why the instantaneous rate of change of the area with respect to the radius should equal the circumference.

50. **Writing to Learn** Recall that the volume V of a sphere of radius r is $(4/3)\pi r^3$ and that the surface area A is $4\pi r^2$. Notice that $dV/dr = A$. Explain in terms of geometry why the instantaneous rate of change of the volume with respect to the radius should equal the surface area.

51. ***Orchard Farming*** An apple farmer currently has 156 trees yielding an average of 12 bushels of apples per tree. He is expanding his farm at a rate of 13 trees per year, while improved husbandry is improving his average annual yield by 1.5 bushels per tree. What is the current (instantaneous) rate of increase of his total annual production of apples? Answer in appropriate units of measure.

52. ***Picnic Pavilion Rental*** The members of the Blue Boar society always divide the pavilion rental fee for their picnics equally among the members. Currently there are 65 members and the pavilion rents for $250. The pavilion cost is increasing at a rate of $10 per year, while the Blue Boar membership is increasing at a rate of 6 members per year. What is the current (instantaneous) rate of change in each member's share of the pavilion rental fee? Answer in appropriate units of measure.

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

53. **True or False** $\frac{d}{dx}(\pi^3) = 3\pi^2$. Justify your answer.

54. **True or False** The graph of $f(x) = 1/x$ has no horizontal tangents. Justify your answer.

55. Multiple Choice Let $y = uv$ be the product of the functions u and v. Find $y'(1)$ if $u(1) = 2$, $u'(1) = 3$, $v(1) = -1$, and $v'(1) = 1$.

(A) -4 **(B)** -1 **(C)** 1 **(D)** 4 **(E)** 7

56. Multiple Choice Let $f(x) = x - \frac{1}{x}$. Find $f''(x)$.

(A) $1 + \frac{1}{x^2}$ **(B)** $1 - \frac{1}{x^2}$ **(C)** $\frac{2}{x^3}$

(D) $-\frac{2}{x^3}$ **(E)** does not exist

57. Multiple Choice Which of the following is $\frac{d}{dx}\left(\frac{x+1}{x-1}\right)$?

(A) $\frac{2}{(x-1)^2}$ **(B)** 0 **(C)** $-\frac{x^2+1}{x^2}$

(D) $2x - \frac{1}{x^2} - 1$ **(E)** $-\frac{2}{(x-1)^2}$

58. Multiple Choice Assume $f(x) = (x^2 - 1)(x^2 + 1)$. Which of the following gives the number of horizontal tangents of f?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Extending the Ideas

59. *Leibniz's Proof of the Product Rule* Here's how Leibniz explained the Product Rule in a letter to his colleague John Wallis:

It is useful to consider quantities infinitely small such that when their ratio is sought, they may not be considered zero, but which are rejected as often as they occur with quantities incomparably greater. Thus if we have $x + dx$, dx is rejected. Similarly we cannot have xdx and $dxdx$ standing together, as xdx is incomparably greater than $dxdx$. Hence if we are to differentiate uv, we write

$$\begin{aligned} d(uv) &= (u + du)(v + dv) - uv \\ &= uv + vdu + udv + dudv - uv \\ &= vdu + udv. \end{aligned}$$

Answer the following questions about Leibniz's proof.

(a) What does Leibniz mean by a quantity being "rejected"?

(b) What happened to $dudv$ in the last step of Leibniz's proof?

(c) Divide both sides of Leibniz's formula

$$d(uv) = vdu + udv$$

by the differential dx. What formula results?

(d) Why would the critics of Leibniz's time have objected to dividing both sides of the equation by dx?

(e) Leibniz had a similar simple (but not-so-clean) proof of the Quotient Rule. Can you reconstruct it?

Quick Quiz for AP* Preparation: Sections 3.1–3.3

You may use a graphing calculator to solve the following problems.

1. Multiple Choice Let $f(x) = |x + 1|$. Which of the following statements about f are true?

I. f is continuous at $x = -1$.

II. f is differentiable at $x = -1$.

III. f has a corner at $x = -1$.

(A) I only **(B)** II only **(C)** III only

(D) I and III only **(E)** I and II only

2. Multiple Choice If the line normal to the graph of f at the point (1, 2) passes through the point $(-1, 1)$, then which of the following gives the value of $f'(1) =$?

(A) -2 **(B)** 2 **(C)** $-1/2$ **(D)** 1/2 **(E)** 3

3. Multiple Choice Find dy/dx if $y = \frac{4x-3}{2x+1}$.

(A) $\frac{10}{(4x-3)^2}$ **(B)** $-\frac{10}{(4x-3)^2}$ **(C)** $\frac{10}{(2x+1)^2}$

(D) $-\frac{10}{(2x+1)^2}$ **(E)** 2

4. Free Response Let $f(x) = x^4 - 4x^2$.

(a) Find all the points where f has horizontal tangents.

(b) Find an equation of the tangent line at $x = 1$.

(c) Find an equation of the normal line at $x = 1$.

3.4 Velocity and Other Rates of Change

What you'll learn about

- Instantaneous Rates of Change
- Motion along a Line
- Sensitivity to Change
- Derivatives in Economics

. . . and why

Derivatives give the rates at which things change in the world.

Instantaneous Rates of Change

In this section we examine some applications in which derivatives as functions are used to represent the rates at which things change in the world around us. It is natural to think of change as change with respect to time, but other variables can be treated in the same way. For example, a physician may want to know how change in dosage affects the body's response to a drug. An economist may want to study how the cost of producing steel varies with the number of tons produced.

If we interpret the difference quotient

$$\frac{f(x+h)-f(x)}{h}$$

as the average rate of change of the function f over the interval from x to $x+h$, we can interpret its limit as h approaches 0 to be the rate at which f is changing at the point x.

DEFINITION Instantaneous Rate of Change

The **(instantaneous) rate of change** of f with respect to x at a is the derivative

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h},$$

provided the limit exists.

It is conventional to use the word *instantaneous* even when x does not represent time. The word, however, is frequently omitted in practice. When we say *rate of change,* we mean *instantaneous rate of change.*

EXAMPLE 1 Enlarging Circles

(a) Find the rate of change of the area A of a circle with respect to its radius r.

(b) Evaluate the rate of change of A at $r = 5$ and at $r = 10$.

(c) If r is measured in inches and A is measured in square inches, what units would be appropriate for dA/dr?

SOLUTION

The area of a circle is related to its radius by the equation $A = \pi r^2$.

(a) The (instantaneous) rate of change of A with respect to r is

$$\frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = \pi \cdot 2r = 2\pi r.$$

(b) At $r = 5$, the rate is 10π (about 31.4). At $r = 10$, the rate is 20π (about 62.8).

Notice that the rate of change gets bigger as r gets bigger. As can be seen in Figure 3.21, the same change in radius brings about a bigger change in area as the circles grow radially away from the center.

(c) The appropriate units for dA/dr are square inches (of area) per inch (of radius).

Now try Exercise 1.

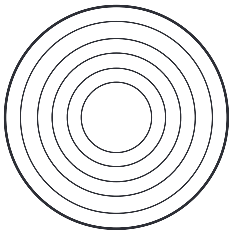


Figure 3.21 The same change in radius brings about a larger change in area as the circles grow radially away from the center. (Example 1, Exploration 1)

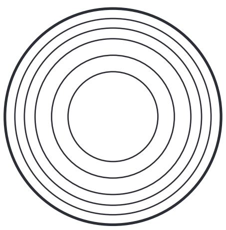


Figure 3.22 Which is the more appropriate model for the growth of rings in a tree—the circles here or those in Figure 3.21? (Exploration 1)

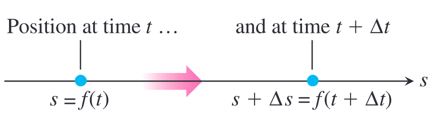


Figure 3.23 The positions of an object moving along a coordinate line at time t and shortly later at time $t + \Delta t$.

EXPLORATION 1 Growth Rings on a Tree

The phenomenon observed in Example 1, that the rate of change in area of a circle with respect to its radius gets larger as the radius gets larger, is reflected in nature in many ways. When trees grow, they add layers of wood directly under the inner bark during the growing season, then form a darker, protective layer for protection during the winter. This results in concentric rings that can be seen in a cross-sectional slice of the trunk. The age of the tree can be determined by counting the rings.

1. Look at the concentric rings in Figure 3.21 and Figure 3.22. Which is a better model for the pattern of growth rings in a tree? Is it likely that a tree could find the nutrients and light necessary to increase its amount of growth every year?
2. Considering how trees grow, explain why the change in *area* of the rings remains relatively constant from year to year.
3. If the change in area is constant, and if

$$\frac{dA}{dr} = \frac{\text{change in area}}{\text{change in radius}} = 2\pi r,$$

explain why the change in radius must get smaller as r gets bigger.

Motion along a Line

Suppose that an object is moving along a coordinate line (say an s-axis) so that we know its position s on that line as a function of time t:

$$s = f(t).$$

The **displacement** of the object over the time interval from t to $t + \Delta t$ is

$$\Delta s = f(t + \Delta t) - f(t)$$

(Figure 3.23) and the **average velocity** of the object over that time interval is

$$v_{\text{av}} = \frac{\text{displacement}}{\text{travel time}} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

To find the object's velocity at the exact instant t, we take the limit of the average velocity over the interval from t to $t + \Delta t$ as Δt shrinks to zero. The limit is the derivative of f with respect to t.

DEFINITION Instantaneous Velocity

The **(instantaneous) velocity** is the derivative of the position function $s = f(t)$ with respect to time. At time t the velocity is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

EXAMPLE 2 Finding the Velocity of a Race Car

Figure 3.24 shows the time-to-distance graph of a 1996 Riley & Scott Mk III-Olds WSC race car. The slope of the secant PQ is the average velocity for the 3-second interval from $t = 2$ to $t = 5$ sec, in this case, about 100 ft/sec or 68 mph. The slope of the tangent at P is the speedometer reading at $t = 2$ sec, about 57 ft/sec or 39 mph. The acceleration for the period shown is a nearly constant 28.5 ft/sec during each second, which is about $0.89g$ where g is the acceleration due to gravity. The race car's top speed is an estimated 190 mph. *Source: Road and Track,* March 1997.

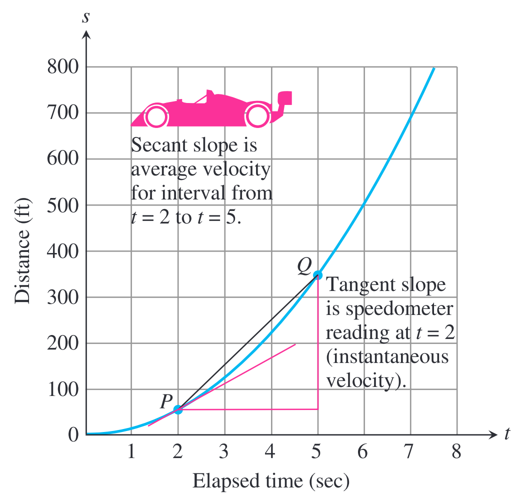


Figure 3.24 The time-to-distance graph for Example 2. ***Now try Exercise 7.***

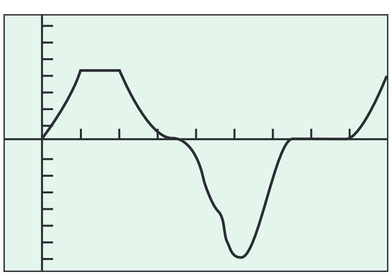
Besides telling how fast an object is moving, velocity tells the direction of motion. When the object is moving forward (when s is increasing), the velocity is positive; when the object is moving backward (when s is decreasing), the velocity is negative.

If we drive to a friend's house and back at 30 mph, the speedometer will show 30 on the way over but will not show -30 on the way back, even though our distance from home is decreasing. The speedometer always shows *speed,* which is the absolute value of velocity. Speed measures the rate of motion regardless of direction.

DEFINITION Speed

Speed is the absolute value of velocity.

$$\text{Speed} = |v(t)| = \left|\frac{ds}{dt}\right|$$



[–4, 36] by [–7.5, 7.5]

Figure 3.25 A student's velocity graph from data recorded by a motion detector. (Example 3)

EXAMPLE 3 Reading a Velocity Graph

A student walks around in front of a motion detector that records her velocity at 1-second intervals for 36 seconds. She stores the data in her graphing calculator and uses it to generate the time-velocity graph shown in Figure 3.25. Describe her motion as a function of time by reading the velocity graph. When is her *speed* a maximum?

SOLUTION

The student moves forward for the first 14 seconds, moves backward for the next 12 seconds, stands still for 6 seconds, and then moves forward again. She achieves her maximum speed at $t \approx 20$, while moving backward. ***Now try Exercise 9.***

The rate at which a body's velocity changes is called the body's *acceleration.* The acceleration measures how quickly the body picks up or loses speed.

DEFINITION **Acceleration**

Acceleration is the derivative of velocity with respect to time. If a body's velocity at time t is $v(t) = ds/dt$, then the body's acceleration at time t is

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

The earliest questions that motivated the discovery of calculus were concerned with velocity and acceleration, particularly the motion of freely falling bodies under the force of gravity. (See Examples 1 and 2 in Section 2.1.) The mathematical description of this type of motion captured the imagination of many great scientists, including Aristotle, Galileo, and Newton. Experimental and theoretical investigations revealed that the distance a body released from rest falls freely is proportional to the square of the amount of time it has fallen. We express this by saying that

$$s = \frac{1}{2}gt^2,$$

where s is distance, g is the acceleration due to Earth's gravity, and t is time. The value of g in the equation depends on the units used to measure s and t. With t in seconds (the usual unit), we have the following values:

Free-fall Constants (Earth)

English units: $g = 32 \dfrac{\text{ft}}{\text{sec}^2}$, $s = \dfrac{1}{2}(32)t^2 = 16t^2$ (s in feet)

Metric units: $g = 9.8 \dfrac{\text{m}}{\text{sec}^2}$, $s = \dfrac{1}{2}(9.8)t^2 = 4.9t^2$ (s in meters)

The abbreviation ft/sec^2 is read "feet per second squared" or "feet per second per second," and m/sec^2 is read "meters per second squared."

EXAMPLE 4 Modeling Vertical Motion

A dynamite blast propels a heavy rock straight up with a launch velocity of 160 ft/sec (about 109 mph) (Figure 3.26a). It reaches a height of $s = 160t - 16t^2$ ft after t seconds.

(a) How high does the rock go?

(b) What is the velocity and speed of the rock when it is 256 ft above the ground on the way up? on the way down?

(c) What is the acceleration of the rock at any time t during its flight (after the blast)?

(d) When does the rock hit the ground?

SOLUTION

In the coordinate system we have chosen, s measures height from the ground up, so velocity is positive on the way up and negative on the way down.

continued

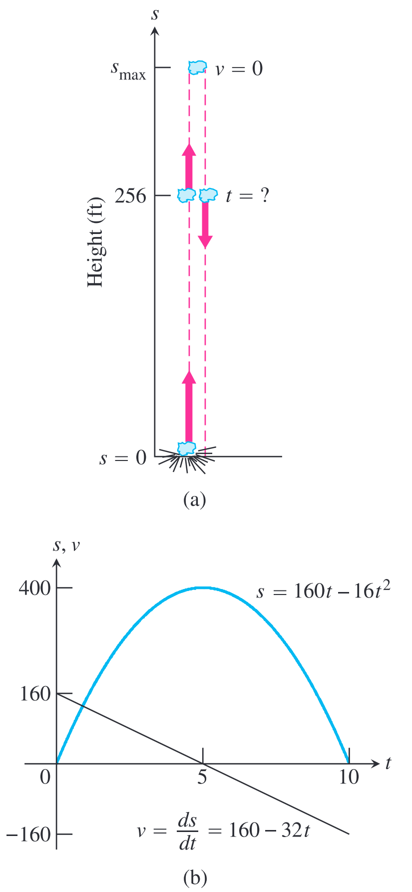


Figure 3.26 (a) The rock in Example 4. (b) The graphs of s and v as functions of time t, showing that s is largest when $v = ds/dt = 0$. (The graph of s is *not* the path of the rock; it is a plot of height as a function of time.) (Example 4)

(a) The instant when the rock is at its highest point is the one instant during the flight when the velocity is 0. At any time t, the velocity is

$$v = \frac{ds}{dt} = \frac{d}{dt}(160t - 16t^2) = 160 - 32t \text{ ft/sec.}$$

The velocity is zero when $160 - 32t = 0$, or at $t = 5$ sec.

The maximum height is the height of the rock at $t = 5$ sec. That is,

$$s_{max} = s(5) = 160(5) - 16(5)^2 = 400 \text{ ft.}$$

See Figure 3.26b.

(b) To find the velocity when the height is 256 ft, we determine the two values of t for which $s(t) = 256$ ft.

$$s(t) = 160t - 16t^2 = 256$$
$$16t^2 - 160t + 256 = 0$$
$$16(t^2 - 10t + 16) = 0$$
$$(t - 2)(t - 8) = 0$$
$$t = 2 \text{ sec} \quad \text{or} \quad t = 8 \text{ sec}$$

The velocity of the rock at each of these times is

$$v(2) = 160 - 32(2) = 96 \text{ ft/sec,}$$
$$v(8) = 160 - 32(8) = -96 \text{ ft/sec.}$$

At both instants, the speed of the rock is 96 ft/sec.

(c) At any time during its flight after the explosion, the rock's acceleration is

$$a = \frac{dv}{dt} = \frac{d}{dt}(160 - 32t) = -32 \text{ ft/sec}^2.$$

The acceleration is always downward. When the rock is rising, it is slowing down; when it is falling, it is speeding up.

(d) The rock hits the ground at the positive time for which $s = 0$. The equation $160t - 16t^2 = 0$ has two solutions: $t = 0$ and $t = 10$. The blast initiated the flight of the rock from ground level at $t = 0$. The rock returned to the ground 10 seconds later.

Now try Exercise 13.

EXAMPLE 5 Studying Particle Motion

A particle moves along a line so that its position at any time $t \geq 0$ is given by the function $s(t) = t^2 - 4t + 3$, where s is measured in meters and t is measured in seconds.

(a) Find the displacement of the particle during the first 2 seconds.

(b) Find the average velocity of the particle during the first 4 seconds.

(c) Find the instantaneous velocity of the particle when $t = 4$.

(d) Find the acceleration of the particle when $t = 4$.

(e) Describe the motion of the particle. At what values of t does the particle change directions?

(f) Use parametric graphing to view the motion of the particle on the horizontal line $y = 2$.

continued

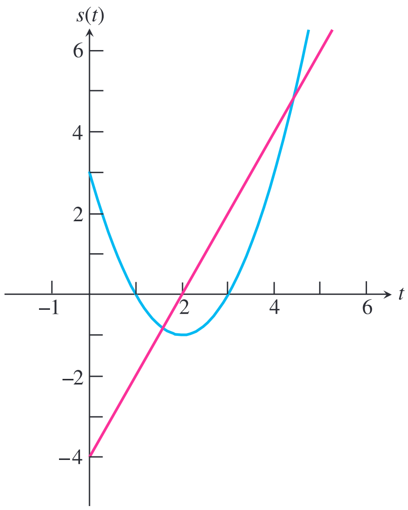
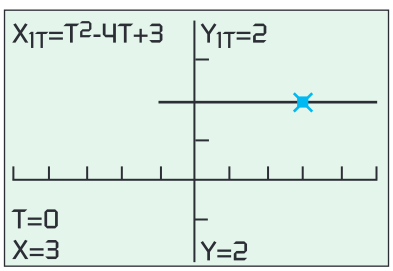


Figure 3.27 The graphs of $s(t) = t^2 - 4t + 3,\ t \geq 0$ (blue) and its derivative $v(t) = 2t - 4,\ t \geq 0$ (red). (Example 5)



[−5, 5] by [−2, 4]

Figure 3.28 The graph of $X_{1T} = T^2 - 4T + 3,\ Y_{1T} = 2$ in parametric mode. (Example 5)

SOLUTION

(a) The displacement is given by $s(2) - s(0) = (-1) - 3 = -4$. This value means that the particle is 4 units left of where it started.

(b) The average velocity we seek is

$$\frac{s(4) - s(0)}{4 - 0} = \frac{3 - 3}{4} = 0 \text{ m/sec.}$$

(c) The velocity $v(t)$ at any time t is $v(t) = ds/dt = 2t - 4$. So $v(4) = 4$ m/sec

(d) The acceleration $a(t)$ at any time t is $a(t) = dv/dt = 2$ m/sec^2. So $a(4) = 2$.

(e) The graphs of $s(t) = t^2 - 4t + 3$ for $t \geq 0$ and its derivative $v(t) = 2t - 4$ shown in Figure 3.27 will help us analyze the motion.

For $0 \leq t < 2$, $v(t) < 0$, so the particle is moving to the left. Notice that $s(t)$ is decreasing. The particle starts $(t = 0)$ at $s = 3$ and moves left, arriving at the origin $t = 1$ when $s = 0$. The particle continues moving to the left until it reaches the point $s = -1$ at $t = 2$.

At $t = 2$, $v = 0$, so the particle is at rest.

For $t > 2$, $v(t) > 0$, so the particle is moving to the right. Notice that $s(t)$ is increasing. In this interval, the particle starts at $s = -1$, moving to the right through the origin and continuing to the right for the rest of time.

The particle changes direction at $t = 2$ when $v = 0$.

(f) Enter $X_{1T} = T^2 - 4T + 3$, $Y_{1T} = 2$ in parametric mode and graph in the window [−5, 5] by [−2, 4] with Tmin = 0, Tmax = 10 (it really should be ∞), and Xscl=Yscl=1. (Figure 3.28) By using TRACE you can follow the path of the particle. You will learn more ways to visualize motion in Explorations 2 and 3.

Now try Exercise 19.

EXPLORATION 2 Modeling Horizontal Motion

The position (x-coordinate) of a particle moving on the horizontal line $y = 2$ is given by $x(t) = 4t^3 - 16t^2 + 15t$ for $t \geq 0$.

1. Graph the parametric equations $x_1(t) = 4t^3 - 16t^2 + 15t,\ y_1(t) = 2$ in [−4, 6] by [−3, 5]. Use TRACE to support that the particle starts at the point (0, 2), moves to the right, then to the left, and finally to the right. At what times does the particle reverse direction?
2. Graph the parametric equations $x_2(t) = x_1(t),\ y_2(t) = t$ in the same viewing window. Explain how this graph shows the back and forth motion of the particle. Use this graph to find when the particle reverses direction.
3. Graph the parametric equations $x_3(t) = t,\ y_3(t) = x_1(t)$ in the same viewing window. Explain how this graph shows the back and forth motion of the particle. Use this graph to find when the particle reverses direction.
4. Use the methods in parts 1, 2, and 3 to represent and describe the *velocity* of the particle.

EXPLORATION 3 **Seeing Motion on a Graphing Calculator**

The graphs in Figure 3.26b give us plenty of information about the flight of the rock in Example 4, but neither graph shows the path of the rock in flight. We can simulate the moving rock by graphing the parametric equations

$$x_1(t) = 3(t < 5) + 3.1(t \geq 5), \qquad y_1(t) = 160t - 16t^2$$

in dot mode.

This will show the upward flight of the rock along the vertical line $x = 3$, and the downward flight of the rock along the line $x = 3.1$.

1. To see the flight of the rock from beginning to end, what should we use for tMin and tMax in our graphing window?
2. Set $x\text{Min} = 0$, $x\text{Max} = 6$, and $y\text{Min} = -10$. Use the results from Example 4 to determine an appropriate value for yMax. (You will want the entire flight of the rock to fit within the vertical range of the screen.)
3. Set tStep initially at 0.1. (A higher number will make the simulation move faster. A lower number will slow it down.)
4. Can you explain why the grapher actually slows down when the rock would slow down, and speeds up when the rock would speed up?

Sensitivity to Change

When a small change in x produces a large change in the value of a function $f(x)$, we say that the function is relatively **sensitive** to changes in x. The derivative $f'(x)$ is a measure of this sensitivity.

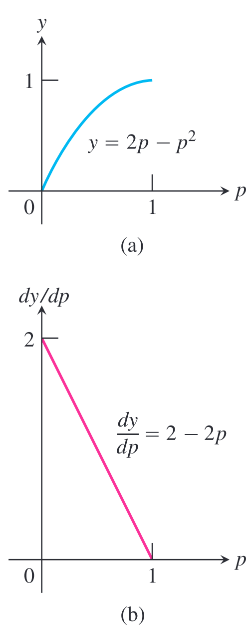


Figure 3.29 (a) The graph of $y = 2p - p^2$ describing the proportion of smooth-skinned peas. (b) The graph of dy/dp. (Example 6)

EXAMPLE 6 Sensitivity to Change

The Austrian monk Gregor Johann Mendel (1822–1884), working with garden peas and other plants, provided the first scientific explanation of hybridization. His careful records showed that if p (a number between 0 and 1) is the relative frequency of the gene for smooth skin in peas (dominant) and $(1 - p)$ is the relative frequency of the gene for wrinkled skin in peas (recessive), then the proportion of smooth-skinned peas in the next generation will be

$$y = 2p(1 - p) + p^2 = 2p - p^2.$$

Compare the graphs of y and dy/dp to determine what values of y are more sensitive to a change in p. The graph of y versus p in Figure 3.29a suggests that the value of y is more sensitive to a change in p when p is small than it is to a change in p when p is large. Indeed, this is borne out by the derivative graph in Figure 3.29b, which shows that dy/dp is close to 2 when p is near 0 and close to 0 when p is near 1.

Now try Exercise 25.

Derivatives in Economics

Engineers use the terms *velocity* and *acceleration* to refer to the derivatives of functions describing motion. Economists, too, have a specialized vocabulary for rates of change and derivatives. They call them *marginals.*

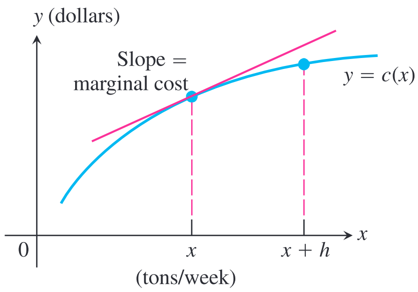


Figure 3.30 Weekly steel production: $c(x)$ is the cost of producing x tons per week. The cost of producing an additional h tons per week is $c(x+h) - c(x)$.

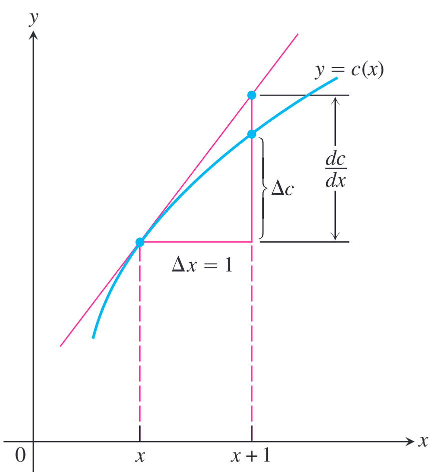


Figure 3.31 Because dc/dx is the slope of the tangent at x, the marginal cost dc/dx approximates the extra cost Δc of producing $\Delta x = 1$ more unit.

In a manufacturing operation, *the cost of production* $c(x)$ is a function of x, the number of units produced. The *marginal cost of production* is the rate of change of cost with respect to the level of production, so it is dc/dx.

Suppose $c(x)$ represents the dollars needed to produce x tons of steel in one week. It costs more to produce $x + h$ tons per week, and the cost difference divided by h is the average cost of producing each additional ton.

$$\frac{c(x+h) - c(x)}{h} = \begin{cases} \text{the average cost of each of the} \\ \text{additional } h \text{ tons produced} \end{cases}$$

The limit of this ratio as $h \to 0$ is the **marginal cost** of producing more steel per week when the current production is x tons (Figure 3.30).

$$\frac{dc}{dx} = \lim_{h \to 0} \frac{c(x+h) - c(x)}{h} = \text{marginal cost of production}$$

Sometimes the marginal cost of production is loosely defined to be the extra cost of producing one more unit,

$$\frac{\Delta c}{\Delta x} = \frac{c(x+1) - c(x)}{1},$$

which is approximated by the value of dc/dx at x. This approximation is acceptable if the slope of c does not change quickly near x, for then the difference quotient is close to its limit dc/dx even if $\Delta x = 1$ (Figure 3.31). The approximation works best for large values of x.

EXAMPLE 7 Marginal Cost and Marginal Revenue

Suppose it costs

$$c(x) = x^3 - 6x^2 + 15x$$

dollars to produce x radiators when 8 to 10 radiators are produced, and that

$$r(x) = x^3 - 3x^2 + 12x$$

gives the dollar revenue from selling x radiators. Your shop currently produces 10 radiators a day. Find the marginal cost and **marginal revenue.**

SOLUTION

The marginal cost of producing one more radiator a day when 10 are being produced is $c'(10)$.

$$c'(x) = \frac{d}{dx}(x^3 - 6x^2 + 15x) = 3x^2 - 12x + 15$$

$$c'(10) = 3(100) - 12(10) + 15 = 195 \text{ dollars}$$

The marginal revenue is

$$r'(x) = \frac{d}{dx}(x^3 - 3x^2 + 12x) = 3x^2 - 6x + 12,$$

so,

$$r'(10) = 3(100) - 6(10) + 12 = 252 \text{ dollars.}$$

Now try Exercises 27 and 28.

Quick Review 3.4 *(For help, go to Sections 1.2, 3.1, and 3.3.)*

In Exercises 1–10, answer the questions about the graph of the quadratic function $y = f(x) = -16x^2 + 160x - 256$ by analyzing the equation algebraically. Then support your answers graphically.

1. Does the graph open upward or downward?

2. What is the y-intercept?

3. What are the x-intercepts?

4. What is the range of the function?

5. What point is the vertex of the parabola?

6. At what x-values does $f(x) = 80$?

7. For what x-value does $dy/dx = 100$?

8. On what interval is $dy/dx > 0$?

9. Find $\lim_{h \to 0} \dfrac{f(3+h) - f(3)}{h}$.

10. Find d^2y/dx^2 at $x = 7$.

Section 3.4 Exercises

1. (a) Write the volume V of a cube as a function of the side length s.

(b) Find the (instantaneous) rate of change of the volume V with respect to a side s.

(c) Evaluate the rate of change of V at $s = 1$ and $s = 5$.

(d) If s is measured in inches and V is measured in cubic inches, what units would be appropriate for dV/ds?

2. (a) Write the area A of a circle as a function of the circumference C.

(b) Find the (instantaneous) rate of change of the area A with respect to the circumference C.

(c) Evaluate the rate of change of A at $C = \pi$ and $C = 6\pi$.

(d) If C is measured in inches and A is measured in square inches, what units would be appropriate for dA/dC?

3. (a) Write the area A of an equilateral triangle as a function of the side length s.

(b) Find the (instantaneous) rate of change of the area A with respect to a side s.

(c) Evaluate the rate of change of A at $s = 2$ and $s = 10$.

(d) If s is measured in inches and A is measured in square inches, what units would be appropriate for dA/ds?

4. A square of side length s is inscribed in a circle of radius r.

(a) Write the area A of the square as a function of the radius r of the circle.

(b) Find the (instantaneous) rate of change of the area A with respect to the radius r of the circle.

(c) Evaluate the rate of change of A at $r = 1$ and $r = 8$.

(d) If r is measured in inches and A is measured in square inches, what units would be appropriate for dA/dr?

Group Activity In Exercises 5 and 6, the coordinates s of a moving body for various values of t are given. **(a)** Plot s versus t on coordinate paper, and sketch a smooth curve through the given points. **(b)** Assuming that this smooth curve represents the motion of the body, estimate the velocity at $t = 1.0$, $t = 2.5$, and $t = 3.5$.

5.

t (sec)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
s (ft)	12.5	26	36.5	44	48.5	50	48.5	44	36.5

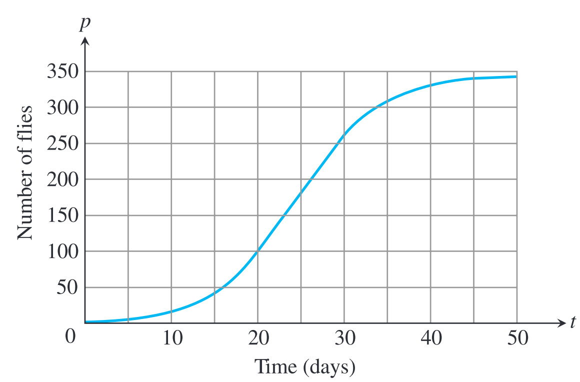
6.

t (sec)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
s (ft)	3.5	−4	−8.5	−10	−8.5	−4	3.5	14	27.5

7. Group Activity ***Fruit Flies*** *(Example 2, Section 2.4 continued)* Populations starting out in closed environments grow slowly at first, when there are relatively few members, then more rapidly as the number of reproducing individuals increases and resources are still abundant, then slowly again as the population reaches the carrying capacity of the environment.

(a) Use the graphical technique of Section 3.1, Example 3, to graph the derivative of the fruit fly population introduced in Section 2.4. The graph of the population is reproduced below. What units should be used on the horizontal and vertical axes for the derivative's graph?

(b) During what days does the population seem to be increasing fastest? slowest?



8. ***Draining a Tank*** The number of gallons of water in a tank t minutes after the tank has started to drain is $Q(t) = 200(30 - t)^2$. How fast is the water running out at the end of 10 min? What is the average rate at which the water flows out during the first 10 min?

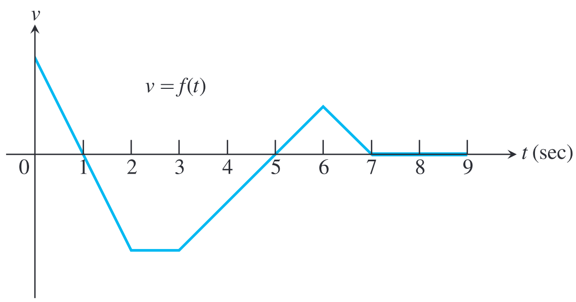
9. ***Particle Motion*** The accompanying figure shows the velocity $v = f(t)$ of a particle moving on a coordinate line.

(a) When does the particle move forward? move backward? speed up? slow down?

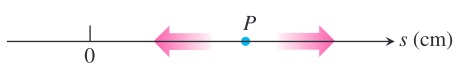
(b) When is the particle's acceleration positive? negative? zero?

(c) When does the particle move at its greatest speed?

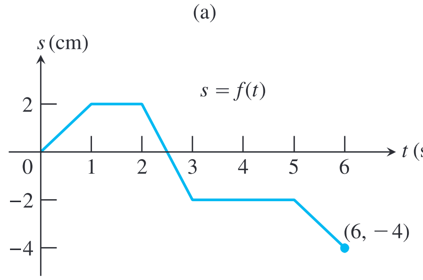
(d) When does the particle stand still for more than an instant?



10. ***Particle Motion*** A particle P moves on the number line shown in part (a) of the accompanying figure. Part (b) shows the position of P as a function of time t.



(a)

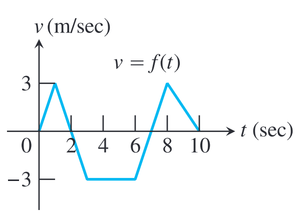


(b)

(a) When is P moving to the left? moving to the right? standing still?

(b) Graph the particle's velocity and speed (where defined).

11. ***Particle Motion*** The accompanying figure shows the velocity $v = ds/dt = f(t)$ (m/sec) of a body moving along a coordinate line.



(a) When does the body reverse direction?

(b) When (approximately) is the body moving at a constant speed?

(c) Graph the body's speed for $0 \le t \le 10$.

(d) Graph the acceleration, where defined.

12. ***Thoroughbred Racing*** A racehorse is running a 10-furlong race. (A furlong is 220 yards, although we will use furlongs and seconds as our units in this exercise.) As the horse passes each furlong marker (F), a steward records the time elapsed (t) since the beginning of the race, as shown in the table below:

F	0	1	2	3	4	5	6	7	8	9	10
t	0	20	33	46	59	73	86	100	112	124	135

(a) How long does it take the horse to finish the race?

(b) What is the average speed of the horse over the first 5 furlongs?

(c) What is the approximate speed of the horse as it passes the 3-furlong marker?

(d) During which portion of the race is the horse running the fastest?

(e) During which portion of the race is the horse accelerating the fastest?

13. ***Lunar Projectile Motion*** A rock thrown vertically upward from the surface of the moon at a velocity of 24 m/sec (about 86 km/h) reaches a height of $s = 24t - 0.8t^2$ meters in t seconds.

(a) Find the rock's velocity and acceleration as functions of time. (The acceleration in this case is the acceleration of gravity on the moon.)

(b) How long did it take the rock to reach its highest point?

(c) How high did the rock go?

(d) When did the rock reach half its maximum height?

(e) How long was the rock aloft?

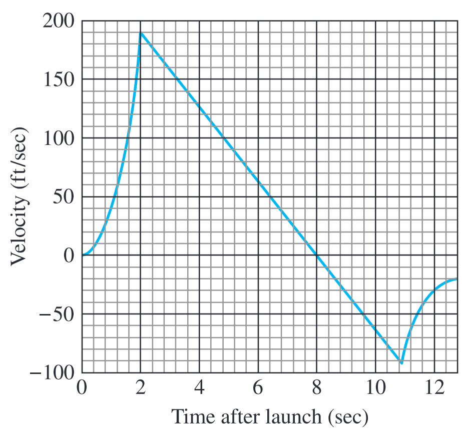
14. ***Free Fall*** The equations for free fall near the surfaces of Mars and Jupiter (s in meters, t in seconds) are: Mars, $s = 1.86t^2$; Jupiter, $s = 11.44t^2$. How long would it take a rock falling from rest to reach a velocity of 16.6 m/sec (about 60 km/h) on each planet?

15. ***Projectile Motion*** On Earth, in the absence of air, the rock in Exercise 13 would reach a height of $s = 24t - 4.9t^2$ meters in t seconds. How high would the rock go?

16. ***Speeding Bullet*** A bullet fired straight up from the moon's surface would reach a height of $s = 832t - 2.6t^2$ ft after t sec. On Earth, in the absence of air, its height would be $s = 832t - 16t^2$ ft after t sec. How long would it take the bullet to get back down in each case?

17. ***Parametric Graphing*** Devise a grapher simulation of the problem situation in Exercise 16. Use it to support the answers obtained analytically.

18. ***Launching a Rocket*** When a model rocket is launched, the propellant burns for a few seconds, accelerating the rocket upward. After burnout, the rocket coasts upward for a while and then begins to fall. A small explosive charge pops out a parachute shortly after the rocket starts downward. The parachute slows the rocket to keep it from breaking when it lands. This graph shows velocity data from the flight.



Use the graph to answer the following.

(a) How fast was the rocket climbing when the engine stopped?

(b) For how many seconds did the engine burn?

(c) When did the rocket reach its highest point? What was its velocity then?

(d) When did the parachute pop out? How fast was the rocket falling then?

(e) How long did the rocket fall before the parachute opened?

(f) When was the rocket's acceleration greatest? When was the acceleration constant?

19. ***Particle Motion*** A particle moves along a line so that its position at any time $t \geq 0$ is given by the function

$$s(t) = t^2 - 3t + 2,$$

where s is measured in meters and t is measured in seconds.

(a) Find the displacement during the first 5 seconds.

(b) Find the average velocity during the first 5 seconds.

(c) Find the instantaneous velocity when $t = 4$.

(d) Find the acceleration of the particle when $t = 4$.

(e) At what values of t does the particle change direction?

(f) Where is the particle when s is a minimum?

20. ***Particle Motion*** A particle moves along a line so that its position at any time $t \geq 0$ is given by the function $s(t) = -t^3 + 7t^2 - 14t + 8$ where s is measured in meters and t is measured in seconds.

(a) Find the instantaneous velocity at any time t.

(b) Find the acceleration of the particle at any time t.

(c) When is the particle at rest?

(d) Describe the motion of the particle. At what values of t does the particle change directions?

21. ***Particle Motion*** A particle moves along a line so that its position at any time $t \geq 0$ is given by the function $s(t) = (t - 2)^2(t - 4)$ where s is measured in meters and t is measured in seconds.

(a) Find the instantaneous velocity at any time t.

(b) Find the acceleration of the particle at any time t.

(c) When is the particle at rest?

(d) Describe the motion of the particle. At what values of t does the particle change directions?

22. ***Particle Motion*** A particle moves along a line so that its position at any time $t \geq 0$ is given by the function $s(t) = t^3 - 6t^2 + 8t + 2$ where s is measured in meters and t is measured in seconds.

(a) Find the instantaneous velocity at any time t.

(b) Find the acceleration of the particle at any time t.

(c) When is the particle at rest?

(d) Describe the motion of the particle. At what values of t does the particle change directions?

23. ***Particle Motion*** The position of a body at time t sec is $s = t^3 - 6t^2 + 9t$ m. Find the body's acceleration each time the velocity is zero.

24. ***Finding Speed*** A body's velocity at time t sec is $v = 2t^3 - 9t^2 + 12t - 5$ m/sec. Find the body's speed each time the acceleration is zero.

25. ***Draining a Tank*** It takes 12 hours to drain a storage tank by opening the valve at the bottom. The depth y of fluid in the tank t hours after the valve is opened is given by the formula

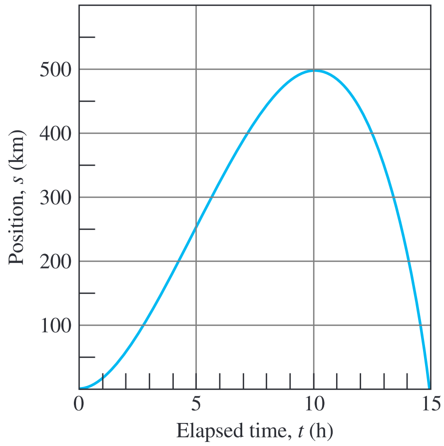
$$y = 6\left(1 - \frac{t}{12}\right)^2 \text{ m.}$$

(a) Find the rate dy/dt (m/h) at which the water level is changing at time t.

(b) When is the fluid level in the tank falling fastest? slowest? What are the values of dy/dt at these times?

(c) Graph y and dy/dt together and discuss the behavior of y in relation to the signs and values of dy/dt.

26. ***Moving Truck*** The graph here shows the position s of a truck traveling on a highway. The truck starts at $t = 0$ and returns 15 hours later at $t = 15$.



(a) Use the technique described in Section 3.1, Example 3, to graph the truck's velocity $v = ds/dt$ for $0 \le t \le 15$. Then repeat the process, with the velocity curve, to graph the truck's acceleration dv/dt.

(b) Suppose $s = 15t^2 - t^3$. Graph ds/dt and d^2s/dt^2, and compare your graphs with those in part (a).

27. ***Marginal Cost*** Suppose that the dollar cost of producing x washing machines is $c(x) = 2000 + 100x - 0.1x^2$.

(a) Find the average cost of producing 100 washing machines.

(b) Find the marginal cost when 100 machines are produced.

(c) Show that the marginal cost when 100 washing machines are produced is approximately the cost of producing one more washing machine after the first 100 have been made, by calculating the latter cost directly.

28. ***Marginal Revenue*** Suppose the weekly revenue in dollars from selling x custom-made office desks is

$$r(x) = 2000\left(1 - \frac{1}{x+1}\right).$$

(a) Draw the graph of r. What values of x make sense in this problem situation?

(b) Find the marginal revenue when x desks are sold.

(c) Use the function $r'(x)$ to estimate the increase in revenue that will result from increasing sales from 5 desks a week to 6 desks a week.

(d) **Writing to Learn** Find the limit of $r'(x)$ as $x \to \infty$. How would you interpret this number?

29. ***Finding Profit*** The monthly profit (in thousands of dollars) of a software company is given by

$$P(x) = \frac{10}{1 + 50 \cdot 2^{5-0.1x}},$$

where x is the number of software packages sold.

(a) Graph $P(x)$.

(b) What values of x make sense in the problem situation?

(c) Use NDER to graph $P'(x)$. For what values of x is P relatively sensitive to changes in x?

(d) What is the profit when the marginal profit is greatest?

(e) What is the marginal profit when 50 units are sold? 100 units, 125 units, 150 units, 175 units, and 300 units?

(f) What is $\lim_{x\to\infty} P(x)$? What is the maximum profit possible?

(g) **Writing to Learn** Is there a practical explanation to the maximum profit answer? Explain your reasoning.

30. In Step 1 of Exploration 2, at what time is the particle at the point (5, 2)?

31. **Group Activity** The graphs in Figure 3.32 show as functions of time t the position s, velocity $v = ds/dt$, and acceleration $a = d^2s/dt^2$ of a body moving along a coordinate line. Which graph is which? Give reasons for your answers.

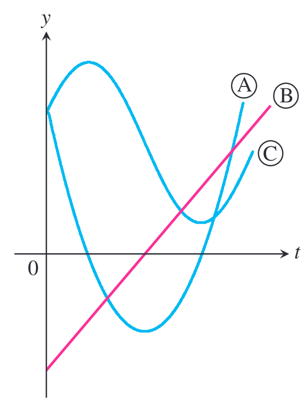


Figure 3.32 The graphs for Exercise 31.

32. **Group Activity** The graphs in Figure 3.33 show as functions of time t the position s, the velocity $v = ds/dt$, and the acceleration $a = d^2s/dt^2$ of a body moving along a coordinate line. Which graph is which? Give reasons for your answers.

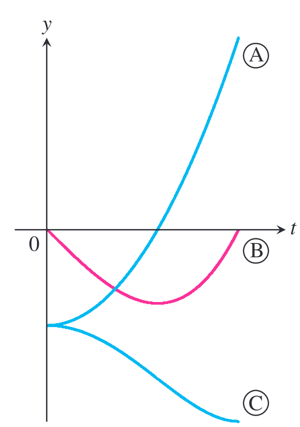


Figure 3.33 The graphs for Exercise 32.

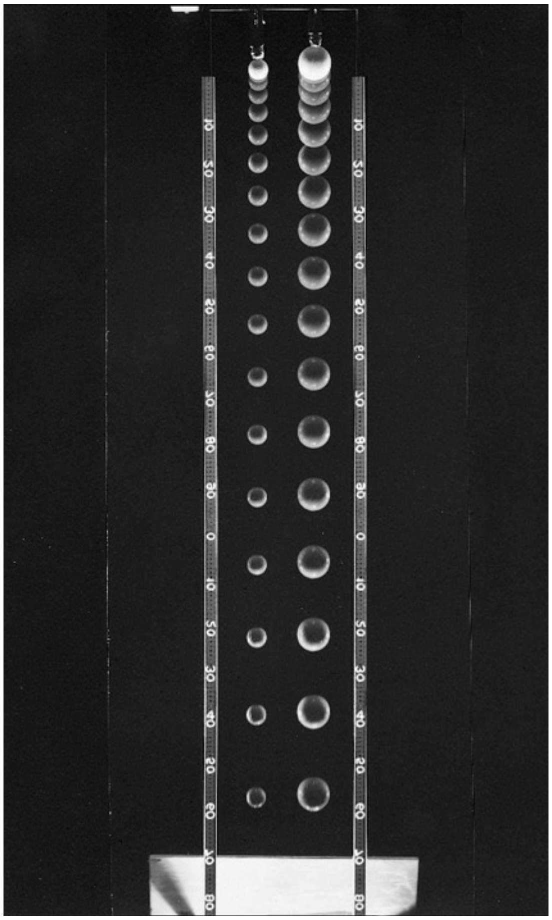


Figure 3.34 Two balls falling from rest. (Exercise 38)

33. ***Pisa by Parachute*** *(continuation of Exercise 18)* A few years ago, Mike McCarthy parachuted 179 ft from the top of the Tower of Pisa. Make a rough sketch to show the shape of the graph of his downward velocity during the jump.

34. ***Inflating a Balloon*** The volume $V = (4/3)\pi r^3$ of a spherical balloon changes with the radius.

(a) At what rate does the volume change with respect to the radius when $r = 2$ ft?

(b) By approximately how much does the volume increase when the radius changes from 2 to 2.2 ft?

35. ***Volcanic Lava Fountains*** Although the November 1959 Kilauea Iki eruption on the island of Hawaii began with a line of fountains along the wall of the crater, activity was later confined to a single vent in the crater's floor, which at one point shot lava 1900 ft straight into the air (a world record). What was the lava's exit velocity in feet per second? in miles per hour? [*Hint:* If v_0 is the exit velocity of a particle of lava, its height t seconds later will be $s = v_0 t - 16t^2$ feet. Begin by finding the time at which $ds/dt = 0$. Neglect air resistance.]

36. **Writing to Learn** Suppose you are looking at a graph of velocity as a function of time. How can you estimate the acceleration at a given point in time?

37. ***Particle Motion*** The position (x-coordinate) of a particle moving on the line $y = 2$ is given by $x(t) = 2t^3 - 13t^2 + 22t - 5$ where t is time in seconds.

(a) Describe the motion of the particle for $t \geq 0$.

(b) When does the particle speed up? slow down?

(c) When does the particle change direction?

(d) When is the particle at rest?

(e) Describe the velocity and speed of the particle.

(f) When is the particle at the point (5, 2)?

38. ***Falling Objects*** The multiflash photograph in Figure 3.34 shows two balls falling from rest. The vertical rulers are marked in centimeters. Use the equation $s = 490t^2$ (the free-fall equation for s in centimeters and t in seconds) to answer the following questions.

(a) How long did it take the balls to fall the first 160 cm? What was their average velocity for the period?

(b) How fast were the balls falling when they reached the 160-cm mark? What was their acceleration then?

(c) About how fast was the light flashing (flashes per second)?

39. **Writing to Learn** Explain how the Sum and Difference Rule (Rule 4 in Section 3.3) can be used to derive a formula for *marginal profit* in terms of marginal revenue and marginal cost.

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

40. **True or False** The speed of a particle at $t = a$ is given by the value of the velocity at $t = a$. Justify your answer.

41. **True or False** The acceleration of a particle is the second derivative of the position function. Justify your answer.

42. **Multiple Choice** Find the instantaneous rate of change of $f(x) = x^2 - 2/x + 4$ at $x = -1$.

(A) -7 **(B)** -4 **(C)** 0 **(D)** 4 **(E)** 7

43. **Multiple Choice** Find the instantaneous rate of change of the volume of a cube with respect to a side length x.

(A) x **(B)** $3x$ **(C)** $6x$ **(D)** $3x^2$ **(E)** x^3

In Exercises 44 and 45, a particle moves along a line so that its position at any time $t \geq 0$ is given by $s(t) = 2 + 7t - t^2$.

44. **Multiple Choice** At which of the following times is the particle moving to the left?

(A) $t = 0$ **(B)** $t = 1$ **(C)** $t = 2$ **(D)** $t = 7/2$ **(E)** $t = 4$

45. **Multiple Choice** When is the particle at rest?

(A) $t = 1$ **(B)** $t = 2$ **(C)** $t = 7/2$ **(D)** $t = 4$ **(E)** $t = 5$

Explorations

46. ***Bacterium Population*** When a bactericide was added to a nutrient broth in which bacteria were growing, the bacterium population continued to grow for a while but then stopped growing and began to decline. The size of the population at time t (hours) was $b(t) = 10^6 + 10^4 t - 10^3 t^2$. Find the growth rates at $t = 0$, $t = 5$, and $t = 10$ hours.

47. *Finding f from f′* Let $f'(x) = 3x^2$.

(a) Compute the derivatives of $g(x) = x^3$, $h(x) = x^3 - 2$, and $t(x) = x^3 + 3$.

(b) Graph the numerical derivatives of g, h, and t.

(c) Describe a *family* of functions, $f(x)$, that have the property that $f'(x) = 3x^2$.

(d) Is there a function f such that $f'(x) = 3x^2$ and $f(0) = 0$? If so, what is it?

(e) Is there a function f such that $f'(x) = 3x^2$ and $f(0) = 3$? If so, what is it?

48. *Airplane Takeoff* Suppose that the distance an aircraft travels along a runway before takeoff is given by $D = (10/9)t^2$, where D is measured in meters from the starting point and t is measured in seconds from the time the brakes are released. If the aircraft will become airborne when its speed reaches 200 km/h, how long will it take to become airborne, and what distance will it have traveled by that time?

Extending the Ideas

49. *Even and Odd Functions*

(a) Show that if f is a differentiable even function, then f' is an odd function.

(b) Show that if f is a differentiable odd function, then f' is an even function.

50. *Extended Product Rule* Derive a formula for the derivative of the product fgh of three differentiable functions.

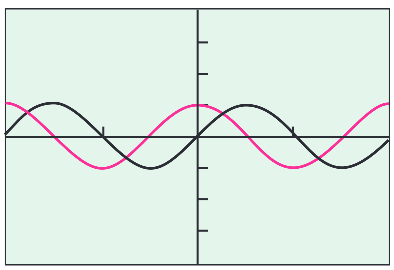
3.5 Derivatives of Trigonometric Functions

What you'll learn about

- Derivative of the Sine Function
- Derivative of the Cosine Function
- Simple Harmonic Motion
- Jerk
- Derivatives of the Other Basic Trigonometric Functions

. . . and why

The derivatives of sines and cosines play a key role in describing periodic change.



$[-2\pi, 2\pi]$ by $[-4, 4]$

Figure 3.35 Sine and its derivative. What is the derivative? (Exploration 1)

Derivative of the Sine Function

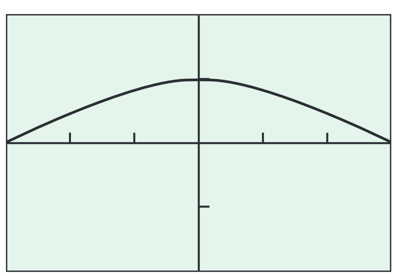
Trigonometric functions are important because so many of the phenomena we want information about are periodic (heart rhythms, earthquakes, tides, weather). It is known that continuous periodic functions can always be expressed in terms of sines and cosines, so the derivatives of sines and cosines play a key role in describing periodic change. This section introduces the derivatives of the six basic trigonometric functions.

EXPLORATION 1 Making a Conjecture with NDER

In the window $[-2\pi, 2\pi]$ by $[-4, 4]$, graph $y_1 = \sin x$ and $y_2 = \text{NDER}(\sin x)$ (Figure 3.35).

1. When the graph of $y_1 = \sin x$ is increasing, what is true about the graph of $y_2 = \text{NDER}(\sin x)$?
2. When the graph of $y_1 = \sin x$ is decreasing, what is true about the graph of $y_2 = \text{NDER}(\sin x)$?
3. When the graph of $y_1 = \sin x$ stops increasing and starts decreasing, what is true about the graph of $y_2 = \text{NDER}(\sin x)$?
4. At the places where $\text{NDER}(\sin x) = \pm 1$, what appears to be the slope of the graph of $y_1 = \sin x$?
5. Make a conjecture about what function the derivative of sine might be. Test your conjecture by graphing your function and $\text{NDER}(\sin x)$ in the same viewing window.
6. Now let $y_1 = \cos x$ and $y_2 = \text{NDER}(\cos x)$. Answer questions (1) through (5) *without* looking at the graph of $\text{NDER}(\cos x)$ until you are ready to test your conjecture about what function the derivative of cosine might be.

If you conjectured that the derivative of the sine function is the cosine function, then you are right. We will confirm this analytically, but first we appeal to technology one more time to evaluate two limits needed in the proof (see Figure 3.36 below and Figure 3.37 on the next page):



$[-3, 3]$ by $[-2, 2]$

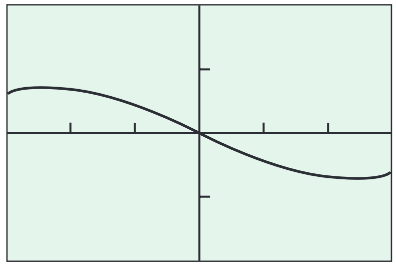
(a)

X	Y1
−.03	.99985
−.02	.99993
−.01	.99998
0	ERROR
.01	.99998
.02	.99993
.03	.99985

Y1 = sin(X)/X

(b)

Figure 3.36 (a) Graphical and (b) tabular support that $\lim_{h\to 0} \frac{\sin(h)}{h} = 1$.



[–3, 3] by [–2, 2]

(a)

X	Y1
–.03	.015
–.02	.01
–.01	.005
0	ERROR
.01	–.005
.02	–.01
.03	–.015

Y1=(cos(X)–1)/X

(b)

Figure 3.37 (a) Graphical and (b) tabular support that $\lim_{h \to 0} \frac{\cos(h) - 1}{h} = 0$.

Confirm Analytically

(Also, see Section 2.1, Exercise 75.) Now, let $y = \sin x$. Then

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h \to 0} \frac{(\sin x)(\cos h - 1) + \cos x \sin h}{h} \\
&= \lim_{h \to 0} \sin x \cdot \lim_{h \to 0} \frac{(\cos h - 1)}{h} + \lim_{h \to 0} \cos x \cdot \lim_{h \to 0} \frac{\sin h}{h} \\
&= \sin x \cdot 0 + \cos x \cdot 1 \\
&= \cos x.
\end{aligned}$$

In short, the derivative of the sine is the cosine.

$$\frac{d}{dx} \sin x = \cos x$$

Now that we know that the sine function is differentiable, we know that sine and its derivative obey all the rules for differentiation. We also know that $\sin x$ is continuous. The same holds for the other trigonometric functions in this section. Each one is differentiable at every point in its domain, so each one is continuous at every point in its domain, and the differentiation rules apply for each one.

Derivative of the Cosine Function

If you conjectured in Exploration 1 that the derivative of the cosine function is the negative of the sine function, you were correct. You can confirm this analytically in Exercise 24.

$$\frac{d}{dx} \cos x = -\sin x$$

Radian Measure in Calculus

In case you have been wondering why calculus uses radian measure when the rest of the world seems to measure angles in degrees, you are now ready to understand the answer. The derivative of sin x is cos x *only* if x is measured in radians! If you look at the analytic confirmation, you will note that the derivative comes down to

$$\cos x \text{ times } \lim_{h\to 0} \frac{\sin h}{h}.$$

We saw that

$$\lim_{h\to 0} \frac{\sin h}{h} = 1$$

in Figure 3.36, but only because the graph in Figure 3.36 is in *radian mode*. If you look at the limit of the same function in *degree* mode you will get a very different limit (and hence a different derivative for sine). See Exercise 50.

EXAMPLE 1 Revisiting the Differentiation Rules

Find the derivatives of **(a)** $y = x^2 \sin x$ and **(b)** $u = \cos x/(1 - \sin x)$.

SOLUTION

(a) $$\frac{dy}{dx} = x^2 \cdot \frac{d}{dx}(\sin x) + \sin x \cdot \frac{d}{dx}(x^2)$$
$$= x^2 \cos x + 2x \sin x$$

(b) $$\frac{du}{dx} = \frac{(1 - \sin x)\cdot \frac{d}{dx}(\cos x) - \cos x \cdot \frac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2}$$
$$= \frac{(1 - \sin x)(-\sin x) - \cos x\,(0 - \cos x)}{(1 - \sin x)^2}$$
$$= \frac{-\sin x + \sin^2 x + \cos^2 x}{(1 - \sin x)^2}$$
$$= \frac{1 - \sin x}{(1 - \sin x)^2}$$
$$= \frac{1}{1 - \sin x}$$

Now try Exercises 5 and 9.

Simple Harmonic Motion

The motion of a weight bobbing up and down on the end of a spring is an example of **simple harmonic motion.** Example 2 describes a case in which there are no opposing forces like friction or buoyancy to slow down the motion.

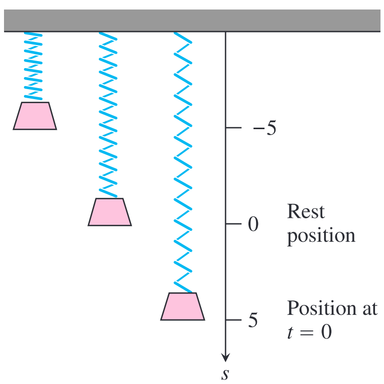


Figure 3.38 The weighted spring in Example 2.

EXAMPLE 2 The Motion of a Weight on a Spring

A weight hanging from a spring (Figure 3.38) is stretched 5 units beyond its rest position ($s = 0$) and released at time $t = 0$ to bob up and down. Its position at any later time t is

$$s = 5 \cos t.$$

What are its velocity and acceleration at time t? Describe its motion.

SOLUTION We have:

Position: $s = 5 \cos t;$

Velocity: $v = \frac{ds}{dt} = \frac{d}{dt}(5 \cos t) = -5 \sin t;$

Acceleration: $a = \frac{dv}{dt} = \frac{d}{dt}(-5 \sin t) = -5 \cos t.$

Notice how much we can learn from these equations:

1. As time passes, the weight moves down and up between $s = -5$ and $s = 5$ on the s-axis. The amplitude of the motion is 5. The period of the motion is 2π.

2. The velocity $v = -5 \sin t$ attains its greatest magnitude, 5, when $\cos t = 0$, as the graphs show in Figure 3.39. Hence the speed of the weight, $|v| = 5\,|\sin t|$, is greatest when $\cos t = 0$, that is, when $s = 0$ (the rest position). The speed of the weight is zero when $\sin t = 0$. This occurs when $s = 5 \cos t = \pm 5$, at the endpoints of the interval of motion.

3. The acceleration value is always the exact opposite of the position value. When the weight is above the rest position, gravity is pulling it back down; when the weight is below the rest position, the spring is pulling it back up.

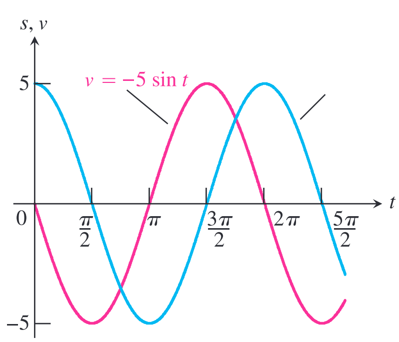


Figure 3.39 Graphs of the position and velocity of the weight in Example 2.

continued

4. The acceleration, $a = -5 \cos t$, is zero only at the rest position where $\cos t = 0$ and the force of gravity and the force from the spring offset each other. When the weight is anywhere else, the two forces are unequal and acceleration is nonzero. The acceleration is greatest in magnitude at the points farthest from the rest position, where $\cos t = \pm 1$. ***Now try Exercise 11.***

Jerk

A sudden change in acceleration is called a "jerk." When a ride in a car or a bus is jerky, it is not that the accelerations involved are necessarily large but that the changes in acceleration are abrupt. Jerk is what spills your soft drink. The derivative responsible for jerk is the *third* derivative of position.

DEFINITION Jerk

Jerk is the derivative of acceleration. If a body's position at time t is $s(t)$, the body's jerk at time t is

$$j(t) = \frac{da}{dt} = \frac{d^3s}{dt^3}.$$

Recent tests have shown that motion sickness comes from accelerations whose changes in magnitude or direction take us by surprise. Keeping an eye on the road helps us to see the changes coming. A driver is less likely to become sick than a passenger who is reading in the back seat.

EXAMPLE 3 A Couple of Jerks

(a) The jerk caused by the constant acceleration of gravity ($g = -32$ ft/sec^2) is zero:

$$j = \frac{d}{dt}(g) = 0.$$

This explains why we don't experience motion sickness while just sitting around.

(b) The jerk of the simple harmonic motion in Example 2 is

$$j = \frac{da}{dt} = \frac{d}{dt}(-5 \cos t)$$
$$= 5 \sin t.$$

It has its greatest magnitude when $\sin t = \pm 1$. This does not occur at the extremes of the displacement, but at the rest position, where the acceleration changes direction and sign.

Now try Exercise 19.

Derivatives of the Other Basic Trigonometric Functions

Because $\sin x$ and $\cos x$ are differentiable functions of x, the related functions

$$\tan x = \frac{\sin x}{\cos x}, \qquad \sec x = \frac{1}{\cos x},$$

$$\cot x = \frac{\cos x}{\sin x}, \qquad \csc x = \frac{1}{\sin x}$$

are differentiable at every value of x for which they are defined. Their derivatives (Exercises 25 and 26) are given by the following formulas.

$$\frac{d}{dx}\tan x = \sec^2 x, \qquad \frac{d}{dx}\sec x = \sec x \tan x$$
$$\frac{d}{dx}\cot x = -\csc^2 x, \qquad \frac{d}{dx}\csc x = -\csc x \cot x$$

EXAMPLE 4 Finding Tangent and Normal Lines

Find equations for the lines that are tangent and normal to the graph of

$$f(x) = \frac{\tan x}{x}$$

at $x = 2$. Support graphically.

SOLUTION

Solve Numerically Since we will be using a calculator approximation for $f(2)$ anyway, this is a good place to use NDER.

We compute $(\tan 2)/2$ on the calculator and store it as k. The slope of the tangent line at $(2, k)$ is

$$\text{NDER}\left(\frac{\tan x}{x}, 2\right),$$

which we compute and store as m. The equation of the tangent line is $y - k = m(x - 2)$, or

$$y = mx + k - 2m.$$

Only after we have found m and $k - 2m$ do we round the coefficients, giving the tangent line as

$$y = 3.43x - 7.96.$$

The equation of the normal line is

$$y - k = -\frac{1}{m}(x - 2), \text{ or}$$

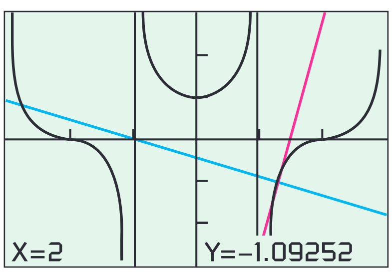
$$y = -\frac{1}{m}x + k + \frac{2}{m}.$$

Again we wait until the end to round the coefficients, giving the normal line as

$$y = -0.291x - 0.51$$

Support Graphically Figure 3.40, showing the original function and the two lines, supports our computations. ***Now try Exercise 23.***

$y_1 = \tan(x)/x$
$y_2 = 3.43x - 7.96$



$[-3\pi/2, 3\pi/2]$ by $[-3, 3]$

Figure 3.40 Graphical support for Example 4.

EXAMPLE 5 A Trigonometric Second Derivative

Find y'' if $y = \sec x$.

SOLUTION

$$\begin{aligned} y &= \sec x \\ y' &= \sec x \tan x \\ y'' &= \frac{d}{dx}(\sec x \tan x) \\ &= \sec x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\sec x) \\ &= \sec x\,(\sec^2 x) + \tan x\,(\sec x \tan x) \\ &= \sec^3 x + \sec x \tan^2 x \end{aligned}$$

Now try Exercise 36.

Quick Review 3.5 *(For help, go to Sections 1.6, 3.1, and 3.4.)*

1. Convert 135 degrees to radians.

2. Convert 1.7 radians to degrees.

3. Find the exact value of $\sin(\pi/3)$ without a calculator.

4. State the domain and the range of the cosine function.

5. State the domain and the range of the tangent function.

6. If $\sin a = -1$, what is $\cos a$?

7. If $\tan a = -1$, what are two possible values of $\sin a$?

8. Verify the identity:
$$\frac{1 - \cos h}{h} = \frac{\sin^2 h}{h(1 + \cos h)}.$$

9. Find an equation of the line tangent to the curve $y = 2x^3 - 7x^2 + 10$ at the point $(3, 1)$.

10. A particle moves along a line with velocity $v = 2t^3 - 7t^2 + 10$ for time $t \geq 0$. Find the acceleration of the particle at $t = 3$.

Section 3.5 Exercises

In Exercises 1–10, find dy/dx. Use your grapher to support your analysis if you are unsure of your answer.

1. $y = 1 + x - \cos x$

2. $y = 2\sin x - \tan x$

3. $y = \dfrac{1}{x} + 5\sin x$

4. $y = x\sec x$

5. $y = 4 - x^2 \sin x$

6. $y = 3x + x\tan x$

7. $y = \dfrac{4}{\cos x}$

8. $y = \dfrac{x}{1 + \cos x}$

9. $y = \dfrac{\cot x}{1 + \cot x}$

10. $y = \dfrac{\cos x}{1 + \sin x}$

In Exercises 11 and 12, a weight hanging from a spring (see Figure 3.38) bobs up and down with position function $s = f(t)$ (s in meters, t in seconds). What are its velocity and acceleration at time t? Describe its motion.

11. $s = 5\sin t$

12. $s = 7\cos t$

In Exercises 13–16, a body is moving in simple harmonic motion with position function $s = f(t)$ (s in meters, t in seconds).

(a) Find the body's velocity, speed, and acceleration at time t.

(b) Find the body's velocity, speed, and acceleration at time $t = \pi/4$.

(c) Describe the motion of the body.

13. $s = 2 + 3\sin t$

14. $s = 1 - 4\cos t$

15. $s = 2\sin t + 3\cos t$

16. $s = \cos t - 3\sin t$

In Exercises 17–20, a body is moving in simple harmonic motion with position function $s = f(t)$ (s in meters, t in seconds). Find the jerk at time t.

17. $s = 2\cos t$

18. $s = 1 + 2\cos t$

19. $s = \sin t - \cos t$

20. $s = 2 + 2\sin t$

21. Find equations for the lines that are tangent and normal to the graph of $y = \sin x + 3$ at $x = \pi$.

22. Find equations for the lines that are tangent and normal to the graph of $y = \sec x$ at $x = \pi/4$.

23. Find equations for the lines that are tangent and normal to the graph of $y = x^2 \sin x$ at $x = 3$.

24. Use the definition of the derivative to prove that $(d/dx)(\cos x) = -\sin x$. (You will need the limits found at the beginning of this section.)

25. Assuming that $(d/dx)(\sin x) = \cos x$ and $(d/dx)(\cos x) = -\sin x$, prove each of the following.

(a) $\dfrac{d}{dx}\tan x = \sec^2 x$ **(b)** $\dfrac{d}{dx}\sec x = \sec x \tan x$

26. Assuming that $(d/dx)(\sin x) = \cos x$ and $(d/dx)(\cos x) = -\sin x$, prove each of the following.

(a) $\dfrac{d}{dx}\cot x = -\csc^2 x$ **(b)** $\dfrac{d}{dx}\csc x = -\csc x \cot x$

27. Show that the graphs of $y = \sec x$ and $y = \cos x$ have horizontal tangents at $x = 0$.

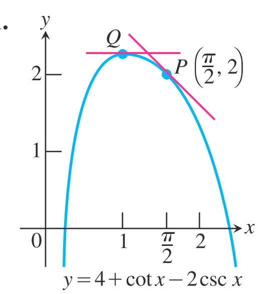
28. Show that the graphs of $y = \tan x$ and $y = \cot x$ have no horizontal tangents.

29. Find equations for the lines that are tangent and normal to the curve $y = \sqrt{2}\cos x$ at the point $(\pi/4, 1)$.

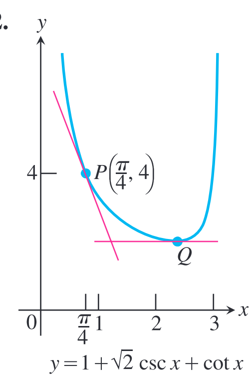
30. Find the points on the curve $y = \tan x$, $-\pi/2 < x < \pi/2$, where the tangent is parallel to the line $y = 2x$.

In Exercises 31 and 32, find an equation for **(a)** the tangent to the curve at P and **(b)** the horizontal tangent to the curve at Q.

31.



32.



Group Activity In Exercises 33 and 34, a body is moving in simple harmonic motion with position $s = f(t)$ (s in meters, t in seconds).

(a) Find the body's velocity, speed, acceleration, and jerk at time t.

(b) Find the body's velocity, speed, acceleration, and jerk at time $t = \pi/4$ sec.

(c) Describe the motion of the body.

33. $s = 2 - 2 \sin t$

34. $s = \sin t + \cos t$

35. Find y'' if $y = \csc x$.

36. Find y'' if $y = \theta \tan \theta$.

37. Writing to Learn Is there a value of b that will make

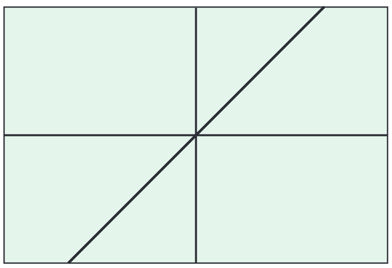
$$g(x) = \begin{cases} x + b, & x < 0 \\ \cos x, & x \geq 0 \end{cases}$$

continuous at $x = 0$? differentiable at $x = 0$? Give reasons for your answers.

38. Find $\dfrac{d^{999}}{dx^{999}}(\cos x)$.

39. Find $\dfrac{d^{725}}{dx^{725}}(\sin x)$.

40. *Local Linearity* This is the graph of the function $y = \sin x$ close to the origin. Since $\sin x$ is differentiable, this graph resembles a line. Find an equation for this line.



41. *(Continuation of Exercise 40)* For values of x close to 0, the linear equation found in Exercise 40 gives a good approximation of $\sin x$.

(a) Use this fact to estimate sin (0.12).

(b) Find sin (0.12) with a calculator. How close is the approximation in part (a)?

42. Use the identity $\sin 2x = 2 \sin x \cos x$ to find the derivative of $\sin 2x$. Then use the identity $\cos 2x = \cos^2 x - \sin^2 x$ to express that derivative in terms of $\cos 2x$.

43. Use the identity $\cos 2x = \cos x \cos x - \sin x \sin x$ to find the derivative of $\cos 2x$. Express the derivative in terms of $\sin 2x$.

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

In Exercises 44 and 45, a spring is bobbing up and down on the end of a spring according to $s(t) = -3 \sin t$.

44. True or False The spring is traveling upward at $t = 3\pi/4$. Justify your answer.

45. True or False The velocity and speed of the particle are the same at $t = \pi/4$. Justify your answer.

46. Multiple Choice Which of the following is an equation of the tangent line to $y = \sin x + \cos x$ at $x = \pi$?

(A) $y = -x + \pi - 1$ **(B)** $y = -x + \pi + 1$

(C) $y = -x - \pi + 1$ **(D)** $y = -x - \pi - 1$

(E) $y = x - \pi + 1$

47. Multiple Choice Which of the following is an equation of the normal line to $y = \sin x + \cos x$ at $x = \pi$?

(A) $y = -x + \pi - 1$ **(B)** $y = x - \pi - 1$ **(C)** $y = x - \pi + 1$

(D) $y = x + \pi + 1$ **(E)** $y = x + \pi - 1$

48. Multiple Choice Find y'' if $y = x \sin x$.

(A) $-x \sin x$ **(B)** $x \cos x + \sin x$ **(C)** $-x \sin x + 2 \cos x$

(D) $x \sin x$ **(E)** $-\sin x + \cos x$

49. Multiple Choice A body is moving in simple harmonic motion with position $s = 3 + \sin t$. At which of the following times is the velocity zero?

(A) $t = 0$ **(B)** $t = \pi/4$ **(C)** $t = \pi/2$

(D) $t = \pi$ **(E)** none of these

Exploration

50. *Radians vs. Degrees* What happens to the derivatives of $\sin x$ and $\cos x$ if x is measured in degrees instead of radians? To find out, take the following steps.

(a) With your grapher in degree mode, graph

$$f(h) = \frac{\sin h}{h}$$

and estimate $\lim_{h \to 0} f(h)$. Compare your estimate with $\pi/180$. Is there any reason to believe the limit should be $\pi/180$?

(b) With your grapher in degree mode, estimate

$$\lim_{h \to 0} \frac{\cos h - 1}{h}.$$

(c) Now go back to the derivation of the formula for the derivative of $\sin x$ in the text and carry out the steps of the derivation using degree-mode limits. What formula do you obtain for the derivative?

(d) Derive the formula for the derivative of $\cos x$ using degree-mode limits.

(e) The disadvantages of the degree-mode formulas become apparent as you start taking derivatives of higher order. What are the second and third degree-mode derivatives of $\sin x$ and $\cos x$?

Extending the Ideas

51. Use analytic methods to show that

$$\lim_{h \to 0} \frac{\cos h - 1}{h} = 0.$$

[*Hint:* Multiply numerator and denominator by $(\cos h + 1)$.]

52. Find A and B in $y = A \sin x + B \cos x$ so that $y'' - y = \sin x$.

3.6 Chain Rule

What you'll learn about

- Derivative of a Composite Function
- "Outside-Inside" Rule
- Repeated Use of the Chain Rule
- Slopes of Parametrized Curves
- Power Chain Rule

. . . and why

The Chain Rule is the most widely used differentiation rule in mathematics.

Derivative of a Composite Function

We now know how to differentiate $\sin x$ and $x^2 - 4$, but how do we differentiate a composite like $\sin(x^2 - 4)$? The answer is with the Chain Rule, which is probably the most widely used differentiation rule in mathematics. This section describes the rule and how to use it.

EXAMPLE 1 Relating Derivatives

The function $y = 6x - 10 = 2(3x - 5)$ is the composite of the functions $y = 2u$ and $u = 3x - 5$. How are the derivatives of these three functions related?

SOLUTION

We have

$$\frac{dy}{dx} = 6, \quad \frac{dy}{du} = 2, \quad \frac{du}{dx} = 3.$$

Since $6 = 2 \cdot 3$,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Now try Exercise 1.

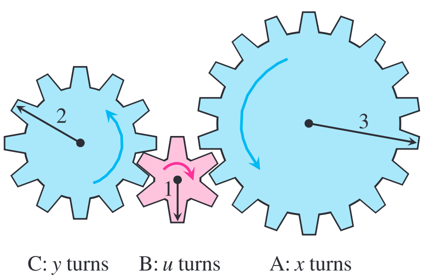


Figure 3.41 When gear A makes x turns, gear B makes u turns, and gear C makes y turns. By comparing circumferences or counting teeth, we see that $y = u/2$ and $u = 3x$, so $y = 3x/2$. Thus $dy/du = 1/2$, $du/dx = 3$, and $dy/dx = 3/2 = (dy/du)(du/dx)$.

Is it an accident that $dy/dx = dy/du \cdot du/dx$?

If we think of the derivative as a rate of change, our intuition allows us to see that this relationship is reasonable. For $y = f(u)$ and $u = g(x)$, if y changes twice as fast as u and u changes three times as fast as x, then we expect y to change six times as fast as x. This is much like the effect of a multiple gear train (Figure 3.41).

Let us try again on another function.

EXAMPLE 2 Relating Derivatives

The polynomial $y = 9x^4 + 6x^2 + 1 = (3x^2 + 1)^2$ is the composite of $y = u^2$ and $u = 3x^2 + 1$. Calculating derivatives, we see that

$$\begin{aligned}\frac{dy}{du} \cdot \frac{du}{dx} &= 2u \cdot 6x \\ &= 2(3x^2 + 1) \cdot 6x \\ &= 36x^3 + 12x.\end{aligned}$$

Also,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(9x^4 + 6x^2 + 1) \\ &= 36x^3 + 12x.\end{aligned}$$

Once again,

$$\frac{dy}{du} \cdot \frac{du}{dx} = \frac{dy}{dx}.$$

Now try Exercise 5.

The derivative of the composite function $f(g(x))$ at x is the derivative of f at $g(x)$ times the derivative of g at x (Figure 3.42). This is known as the Chain Rule.

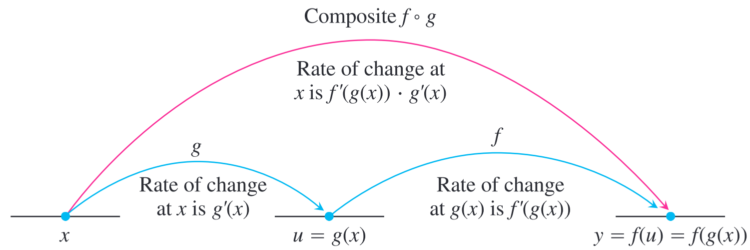


Figure 3.42 Rates of change multiply: the derivative of $f \circ g$ at x is the derivative of f at the point $g(x)$ times the derivative of g at x.

RULE 8 The Chain Rule

If f is differentiable at the point $u = g(x)$, and g is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

It would be tempting to try to prove the Chain Rule by writing

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$$

(a true statement about fractions with nonzero denominators) and taking the limit as $\Delta x \to 0$. This is essentially what is happening, and it would work as a proof if we knew that Δu, the change in u, was nonzero; but we do not know this. A small change in x could conceivably produce no change in u. An air-tight proof of the Chain Rule can be constructed through a different approach, but we will omit it here.

EXAMPLE 3 Applying the Chain Rule

An object moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = \cos(t^2 + 1)$. Find the velocity of the object as a function of t.

SOLUTION

We know that the velocity is dx/dt. In this instance, x is a composite function: $x = \cos(u)$ and $u = t^2 + 1$. We have

$$\frac{dx}{du} = -\sin(u)$$

$$\frac{du}{dt} = 2t.$$

By the Chain Rule,

$$\begin{aligned}\frac{dx}{dt} &= \frac{dx}{du} \cdot \frac{du}{dt} \\ &= -\sin(u) \cdot 2t \\ &= -\sin(t^2 + 1) \cdot 2t \\ &= -2t \sin(t^2 + 1).\end{aligned}$$

Now try Exercise 9.

"Outside-Inside" Rule

It sometimes helps to think about the Chain Rule this way: If $y = f(g(x))$, then

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x).$$

In words, differentiate the "outside" function f and evaluate it at the "inside" function $g(x)$ left alone; then multiply by the derivative of the "inside function."

EXAMPLE 4 Differentiating from the Outside in

Differentiate $\sin(x^2 + x)$ with respect to x.

SOLUTION

$$\frac{d}{dx}\sin\underbrace{(x^2 + x)}_{} = \cos\underbrace{(x^2 + x)}_{} \cdot \underbrace{(2x + 1)}_{}$$

Now try Exercise 13.

Repeated Use of the Chain Rule

We sometimes have to use the Chain Rule two or more times to find a derivative. Here is an example:

EXAMPLE 5 A Three-Link "Chain"

Find the derivative of $g(t) = \tan(5 - \sin 2t)$.

SOLUTION

Notice here that tan is a function of $5 - \sin 2t$, while sin is a function of $2t$, which is itself a function of t. Therefore, by the Chain Rule,

$$\begin{aligned}
g'(t) &= \frac{d}{dt}(\tan(5 - \sin 2t)) \\
&= \sec^2(5 - \sin 2t) \cdot \frac{d}{dt}(5 - \sin 2t) \\
&= \sec^2(5 - \sin 2t) \cdot \left(0 - \cos 2t \cdot \frac{d}{dt}(2t)\right) \\
&= \sec^2(5 - \sin 2t) \cdot (-\cos 2t) \cdot 2 \\
&= -2(\cos 2t)\sec^2(5 - \sin 2t).
\end{aligned}$$

Now try Exercise 23.

Slopes of Parametrized Curves

A parametrized curve $(x(t), y(t))$ is *differentiable at t* if x and y are differentiable at t. At a point on a differentiable parametrized curve where y is also a differentiable function of x, the derivatives dy/dt, dx/dt, and dy/dx are related by the Chain Rule:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

If $dx/dt \neq 0$, we may divide both sides of this equation by dx/dt to solve for dy/dx.

Finding *dy/dx* Parametrically

If all three derivatives exist and $dx/dt \neq 0$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}. \tag{3}$$

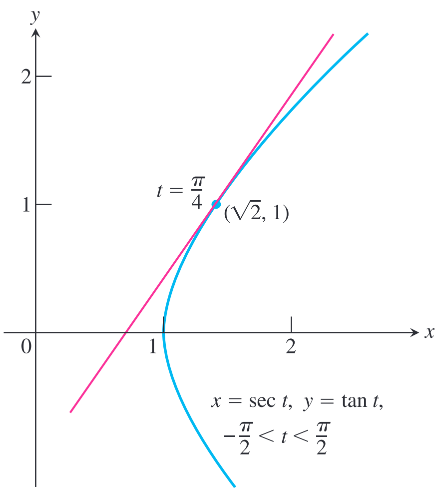


Figure 3.43 The hyperbola branch in Example 6. Equation 3 applies for every point on the graph except (1, 0). Can you state why Equation 3 fails at (1, 0)?

EXAMPLE 6 Differentiating with a Parameter

Find the line tangent to the right-hand hyperbola branch defined parametrically by

$$x = \sec t, \quad y = \tan t, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$$

at the point $(\sqrt{2}, 1)$, where $t = \pi/4$ (Figure 3.43).

SOLUTION

All three of the derivatives in Equation 3 exist and $dx/dt = \sec t \tan t \neq 0$ at the indicated point. Therefore, Equation 3 applies and

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy/dt}{dx/dt} \\ &= \frac{\sec^2 t}{\sec t \tan t} \\ &= \frac{\sec t}{\tan t} \\ &= \csc t.\end{aligned}$$

Setting $t = \pi/4$ gives

$$\left.\frac{dy}{dx}\right|_{t=\pi/4} = \csc(\pi/4) = \sqrt{2}.$$

The equation of the tangent line is

$$\begin{aligned} y - 1 &= \sqrt{2}(x - \sqrt{2}) \\ y &= \sqrt{2}x - 2 + 1 \\ y &= \sqrt{2}x - 1. \end{aligned}$$

Now try Exercise 41.

Power Chain Rule

If f is a differentiable function of u, and u is a differentiable function of x, then substituting $y = f(u)$ into the Chain Rule formula

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

leads to the formula

$$\frac{d}{dx}f(u) = f'(u)\frac{du}{dx}.$$

Here's an example of how it works: If n is an integer and $f(u) = u^n$, the Power Rules (Rules 2 and 7) tell us that $f'(u) = nu^{n-1}$. If u is a differentiable function of x, then we can use the Chain Rule to extend this to the **Power Chain Rule:**

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}.$$

EXAMPLE 7 Finding Slope

(a) Find the slope of the line tangent to the curve $y = \sin^5 x$ at the point where $x = \pi/3$.

(b) Show that the slope of every line tangent to the curve $y = 1/(1 - 2x)^3$ is positive.

SOLUTION

(a)
$$\frac{dy}{dx} = 5 \sin^4 x \cdot \frac{d}{dx} \sin x$$
$$= 5 \sin^4 x \cos x$$

The tangent line has slope

$$\left.\frac{dy}{dx}\right|_{x=\pi/3} = 5\left(\frac{\sqrt{3}}{2}\right)^4\left(\frac{1}{2}\right) = \frac{45}{32}.$$

(b)
$$\frac{dy}{dx} = \frac{d}{dx}(1 - 2x)^{-3}$$
$$= -3(1 - 2x)^{-4} \cdot \frac{d}{dx}(1 - 2x)$$
$$= -3(1 - 2x)^{-4} \cdot (-2)$$
$$= \frac{6}{(1 - 2x)^4}$$

At any point (x, y) on the curve, $x \neq 1/2$ and the slope of the tangent line is

$$\frac{dy}{dx} = \frac{6}{(1 - 2x)^4},$$

the quotient of two positive numbers. *Now try Exercise 53.*

EXAMPLE 8 Radians Versus Degrees

It is important to remember that the formulas for the derivatives of both $\sin x$ and $\cos x$ were obtained under the assumption that x is measured in radians, *not* degrees. The Chain Rule gives us new insight into the difference between the two. Since $180° = \pi$ radians, $x° = \pi x/180$ radians. By the Chain Rule,

$$\frac{d}{dx}\sin(x°) = \frac{d}{dx}\sin\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos(x°).$$

See Figure 3.44.

The factor $\pi/180$, annoying in the first derivative, would compound with repeated differentiation. We see at a glance the compelling reason for the use of radian measure.

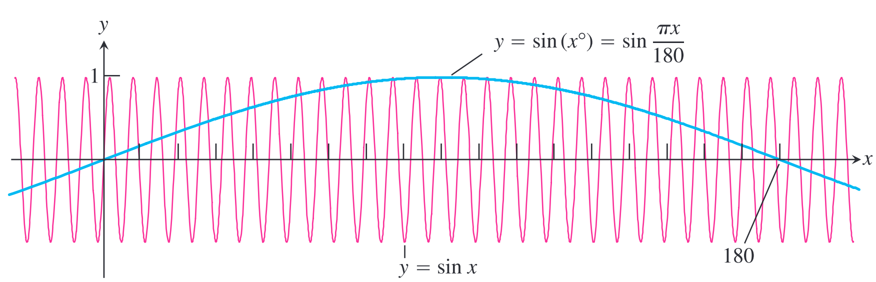


Figure 3.44 $\sin(x°)$ oscillates only $\pi/180$ times as often as $\sin x$ oscillates. Its maximum slope is $\pi/180$. (Example 8)

Quick Review 3.6 *(For help, go to Sections 1.2 and 1.6.)*

In Exercises 1–5, let $f(x) = \sin x,\ g(x) = x^2 + 1,$ and $h(x) = 7x$. Write a simplified expression for the composite function.

1. $f(g(x))$ **2.** $f(g(h(x)))$

3. $(g \circ h)(x)$ **4.** $(h \circ g)(x)$

5. $f\left(\dfrac{g(x)}{h(x)}\right)$

In Exercises 6–10, let $f(x) = \cos x,\ g(x) = \sqrt{x+2},$ and $h(x) = 3x^2$. Write the given function as a composite of two or more of f, g, and h. For example, $\cos 3x^2$ is $f(h(x))$.

6. $\sqrt{\cos x + 2}$ **7.** $\sqrt{3\cos^2 x + 2}$

8. $3\cos x + 6$ **9.** $\cos 27x^4$

10. $\cos\sqrt{2 + 3x^2}$

Section 3.6 Exercises

In Exercises 1–8, use the given substitution and the Chain Rule to find dy/dx.

1. $y = \sin(3x + 1),\ u = 3x + 1$ **2.** $y = \sin(7 - 5x),\ u = 7 - 5x$

3. $y = \cos(\sqrt{3}x),\ u = \sqrt{3}x$ **4.** $y = \tan(2x - x^3),\ u = 2x - x^3$

5. $y = \left(\dfrac{\sin x}{1 + \cos x}\right)^2,\quad u = \dfrac{\sin x}{1 + \cos x}$

6. $y = 5\cot\left(\dfrac{2}{x}\right),\ u = \dfrac{2}{x}$ **7.** $y = \cos(\sin x),\ u = \sin x$

8. $y = \sec(\tan x),\quad u = \tan x$

In Exercises 9–12, an object moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = s(t)$. Find the velocity of the object as a function of t.

9. $s = \cos\left(\dfrac{\pi}{2} - 3t\right)$ **10.** $s = t\cos(\pi - 4t)$

11. $s = \dfrac{4}{3\pi}\sin 3t + \dfrac{4}{5\pi}\cos 5t$

12. $s = \sin\left(\dfrac{3\pi}{2}t\right) + \cos\left(\dfrac{7\pi}{4}t\right)$

In Exercises 13–24, find dy/dx. If you are unsure of your answer, use NDER to support your computation.

13. $y = (x + \sqrt{x})^{-2}$ **14.** $y = (\csc x + \cot x)^{-1}$

15. $y = \sin^{-5} x - \cos^3 x$ **16.** $y = x^3(2x - 5)^4$

17. $y = \sin^3 x \tan 4x$ **18.** $y = 4\sqrt{\sec x + \tan x}$

19. $y = \dfrac{3}{\sqrt{2x + 1}}$ **20.** $y = \dfrac{x}{\sqrt{1 + x^2}}$

21. $y = \sin^2(3x - 2)$ **22.** $y = (1 + \cos 2x)^2$

23. $y = (1 + \cos^2 7x)^3$ **24.** $y = \sqrt{\tan 5x}$

In Exercises 25–28 find $dr/d\theta$.

25. $r = \tan(2 - \theta)$ **26.** $r = \sec 2\theta \tan 2\theta$

27. $r = \sqrt{\theta \sin \theta}$ **28.** $r = 2\theta\sqrt{\sec \theta}$

In Exercises 29–32, find y''.

29. $y = \tan x$ **30.** $y = \cot x$

31. $y = \cot(3x - 1)$ **32.** $y = 9\tan(x/3)$

In Exercises 33–38, find the value of $(f \circ g)'$ at the given value of x.

33. $f(u) = u^5 + 1,\quad u = g(x) = \sqrt{x},\quad x = 1$

34. $f(u) = 1 - \dfrac{1}{u},\quad u = g(x) = \dfrac{1}{1 - x},\quad x = -1$

35. $f(u) = \cot\dfrac{\pi u}{10},\quad u = g(x) = 5\sqrt{x},\quad x = 1$

36. $f(u) = u + \dfrac{1}{\cos^2 u},\quad u = g(x) = \pi x,\quad x = \dfrac{1}{4}$

37. $f(u) = \dfrac{2u}{u^2 + 1},\quad u = g(x) = 10x^2 + x + 1,\quad x = 0$

38. $f(u) = \left(\dfrac{u - 1}{u + 1}\right)^2,\quad u = g(x) = \dfrac{1}{x^2} - 1,\quad x = -1$

What happens if you can write a function as a composite in different ways? Do you get the same derivative each time? The Chain Rule says you should. Try it with the functions in Exercises 39 and 40.

39. Find dy/dx if $y = \cos(6x + 2)$ by writing y as a composite with

(a) $y = \cos u$ and $u = 6x + 2$.

(b) $y = \cos 2u$ and $u = 3x + 1$.

40. Find dy/dx if $y = \sin(x^2 + 1)$ by writing y as a composite with

(a) $y = \sin(u + 1)$ and $u = x^2$.

(b) $y = \sin u$ and $u = x^2 + 1$.

In Exercises 41–48, find the equation of the line tangent to the curve at the point defined by the given value of t.

41. $x = 2\cos t,\quad y = 2\sin t,\quad t = \pi/4$

42. $x = \sin 2\pi t,\quad y = \cos 2\pi t,\quad t = -1/6$

43. $x = \sec^2 t - 1,\quad y = \tan t,\quad t = -\pi/4$

44. $x = \sec t,\quad y = \tan t,\quad t = \pi/6$

45. $x = t,\quad y = \sqrt{t},\quad t = 1/4$

46. $x = 2t^2 + 3,\quad y = t^4,\quad t = -1$

47. $x = t - \sin t,\quad y = 1 - \cos t,\quad t = \pi/3$

48. $x = \cos t,\quad y = 1 + \sin t,\quad t = \pi/2$

49. Let $x = t^2 + t$, and let $y = \sin t$.

(a) Find dy/dx as a function of t.

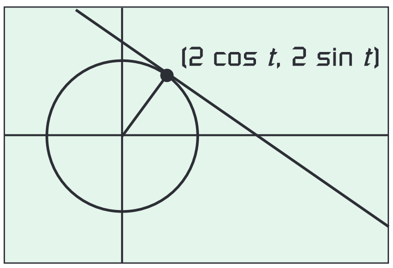
(b) Find $\frac{d}{dt}\left(\frac{dy}{dx}\right)$ as a function of t.

(c) Find $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ as a function of t.

Use the Chain Rule and your answer from part (b).

(d) Which of the expressions in parts (b) and (c) is d^2y/dx^2?

50. A circle of radius 2 and center (0, 0) can be parametrized by the equations $x = 2\cos t$ and $y = 2\sin t$. Show that for any value of t, the line tangent to the circle at $(2\cos t, 2\sin t)$ is perpendicular to the radius.



51. Let $s = \cos\theta$. Evaluate ds/dt when $\theta = 3\pi/2$ and $d\theta/dt = 5$.

52. Let $y = x^2 + 7x - 5$. Evaluate dy/dt when $x = 1$ and $dx/dt = 1/3$.

53. What is the largest value possible for the slope of the curve $y = \sin(x/2)$?

54. Write an equation for the tangent to the curve $y = \sin mx$ at the origin.

55. Find the lines that are tangent and normal to the curve $y = 2\tan(\pi x/4)$ at $x = 1$. Support your answer graphically.

56. ***Working with Numerical Values*** Suppose that functions f and g and their derivatives have the following values at $x = 2$ and $x = 3$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	8	2	1/3	-3
3	3	-4	2π	5

Evaluate the derivatives with respect to x of the following combinations at the given value of x.

(a) $2f(x)$ at $x = 2$ **(b)** $f(x) + g(x)$ at $x = 3$

(c) $f(x) \cdot g(x)$ at $x = 3$ **(d)** $f(x)/g(x)$ at $x = 2$

(e) $f(g(x))$ at $x = 2$ **(f)** $\sqrt{f(x)}$ at $x = 2$

(g) $1/g^2(x)$ at $x = 3$ **(h)** $\sqrt{f^2(x) + g^2(x)}$ at $x = 2$

57. ***Extension of Example 8*** Show that $\frac{d}{dx}\cos(x°)$ is $-\frac{\pi}{180}\sin(x°)$.

58. ***Working with Numerical Values*** Suppose that the functions f and g and their derivatives with respect to x have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1	1	5	1/3
1	3	-4	$-1/3$	$-8/3$

Evaluate the derivatives with respect to x of the following combinations at the given value of x.

(a) $5f(x) - g(x)$, $x = 1$ **(b)** $f(x)g^3(x)$, $x = 0$

(c) $\frac{f(x)}{g(x) + 1}$, $x = 1$ **(d)** $f(g(x))$, $x = 0$

(e) $g(f(x))$, $x = 0$ **(f)** $(g(x) + f(x))^{-2}$, $x = 1$

(g) $f(x + g(x))$, $x = 0$

59. ***Orthogonal Curves*** Two curves are said to cross at right angles if their tangents are perpendicular at the crossing point. The technical word for "crossing at right angles" is **orthogonal.** Show that the curves $y = \sin 2x$ and $y = -\sin(x/2)$ are orthogonal at the origin. Draw both graphs and both tangents in a square viewing window.

60. **Writing to Learn** Explain why the Chain Rule formula

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

is not simply the well-known rule for multiplying fractions.

61. ***Running Machinery Too Fast*** Suppose that a piston is moving straight up and down and that its position at time t seconds is

$$s = A\cos(2\pi bt),$$

with A and b positive. The value of A is the amplitude of the motion, and b is the frequency (number of times the piston moves up and down each second). What effect does doubling the frequency have on the piston's velocity, acceleration, and jerk? (Once you find out, you will know why machinery breaks when you run it too fast.)

Figure 3.45 The internal forces in the engine get so large that they tear the engine apart when the velocity is too great.

62. Group Activity ***Temperatures in Fairbanks, Alaska.*** The graph in Figure 3.46 shows the average Fahrenheit temperature in Fairbanks, Alaska, during a typical 365-day year. The equation that approximates the temperature on day x is

$$y = 37 \sin\left[\frac{2\pi}{365}(x - 101)\right] + 25.$$

(a) On what day is the temperature increasing the fastest?

(b) About how many degrees per day is the temperature increasing when it is increasing at its fastest?

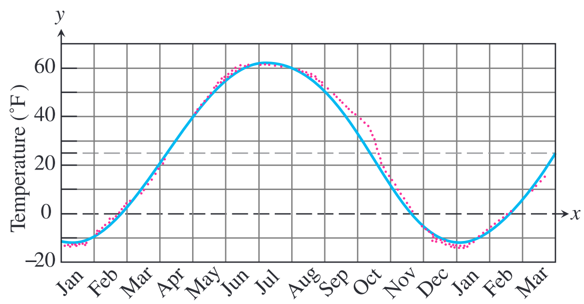


Figure 3.46 Normal mean air temperatures at Fairbanks, Alaska, plotted as data points, and the approximating sine function (Exercise 62).

63. ***Particle Motion*** The position of a particle moving along a coordinate line is $s = \sqrt{1 + 4t}$, with s in meters and t in seconds. Find the particle's velocity and acceleration at $t = 6$ sec.

64. ***Constant Acceleration*** Suppose the velocity of a falling body is $v = k\sqrt{s}$ m/sec (k a constant) at the instant the body has fallen s meters from its starting point. Show that the body's acceleration is constant.

65. ***Falling Meteorite*** The velocity of a heavy meteorite entering the earth's atmosphere is inversely proportional to $\sqrt{s}$ when it is s kilometers from the earth's center. Show that the meteorite's acceleration is inversely proportional to s^2.

66. ***Particle Acceleration*** A particle moves along the x-axis with velocity $dx/dt = f(x)$. Show that the particle's acceleration is $f(x)f'(x)$.

67. ***Temperature and the Period of a Pendulum*** For oscillations of small amplitude (short swings), we may safely model the relationship between the period T and the length L of a simple pendulum with the equation

$$T = 2\pi\sqrt{\frac{L}{g}},$$

where g is the constant acceleration of gravity at the pendulum's location. If we measure g in centimeters per second squared, we measure L in centimeters and T in seconds. If the pendulum is made of metal, its length will vary with temperature, either increasing or decreasing at a rate that is roughly proportional to L. In symbols, with u being temperature and k the proportionality constant,

$$\frac{dL}{du} = kL.$$

Assuming this to be the case, show that the rate at which the period changes with respect to temperature is $kT/2$.

68. Writing to Learn ***Chain Rule*** Suppose that $f(x) = x^2$ and $g(x) = |x|$. Then the composites

$$(f \circ g)(x) = |x|^2 = x^2 \quad \text{and} \quad (g \circ f)(x) = |x^2| = x^2$$

are both differentiable at $x = 0$ even though g itself is not differentiable at $x = 0$. Does this contradict the Chain Rule? Explain.

69. ***Tangents*** Suppose that $u = g(x)$ is differentiable at $x = 1$ and that $y = f(u)$ is differentiable at $u = g(1)$. If the graph of $y = f(g(x))$ has a horizontal tangent at $x = 1$, can we conclude anything about the tangent to the graph of g at $x = 1$ or the tangent to the graph of f at $u = g(1)$? Give reasons for your answer.

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

70. True or False $\frac{d}{dx}(\sin x) = \cos x$, if x is measured in degrees or radians. Justify your answer.

71. True or False The slope of the normal line to the curve $x = 3\cos t$, $y = 3\sin t$ at $t = \pi/4$ is -1. Justify your answer.

72. Multiple Choice Which of the following is dy/dx if $y = \tan(4x)$?

(A) $4\sec(4x)\tan(4x)$ **(B)** $\sec(4x)\tan(4x)$ **(C)** $4\cot(4x)$

(D) $\sec^2(4x)$ **(E)** $4\sec^2(4x)$

73. Multiple Choice Which of the following is dy/dx if $y = \cos^2(x^3 + x^2)$?

(A) $-2(3x^2 + 2x)$

(B) $-(3x^2 + 2x)\cos(x^3 + x^2)\sin(x^3 + x^2)$

(C) $-2(3x^2 + 2x)\cos(x^3 + x^2)\sin(x^3 + x^2)$

(D) $2(3x^2 + 2x)\cos(x^3 + x^2)\sin(x^3 + x^2)$

(E) $2(3x^2 + 2x)$

In Exercises 74 and 75, use the curve defined by the parametric equations $x = t - \cos t$, $y = -1 + \sin t$.

74. Multiple Choice Which of the following is an equation of the tangent line to the curve at $t = 0$?

(A) $y = x$ **(B)** $y = -x$ **(C)** $y = x + 2$

(D) $y = x - 2$ **(E)** $y = -x - 2$

75. Multiple Choice At which of the following values of t is $dy/dx = 0$?

(A) $t = \pi/4$ **(B)** $t = \pi/2$ **(C)** $t = 3\pi/4$

(D) $t = \pi$ **(E)** $t = 2\pi$

Explorations

76. ***The Derivative of sin 2x*** Graph the function $y = 2\cos 2x$ for $-2 \le x \le 3.5$. Then, on the same screen, graph

$$y = \frac{\sin 2(x+h) - \sin 2x}{h}$$

for $h = 1.0$, 0.5, and 0.2. Experiment with other values of h, including negative values. What do you see happening as $h \to 0$? Explain this behavior.

77. ***The Derivative of cos*** (x^2) Graph $y = -2x\sin(x^2)$ for $-2 \le x \le 3$. Then, on screen, graph

$$y = \frac{\cos[(x+h)^2] - \cos(x^2)}{h}$$

for $h = 1.0$, 0.7, and 0.3. Experiment with other values of h. What do you see happening as $h \to 0$? Explain this behavior.

Extending the Ideas

78. ***Absolute Value Functions*** Let u be a differentiable function of x.

(a) Show that $\frac{d}{dx}|u| = u'\frac{u}{|u|}$.

(b) Use part (a) to find the derivatives of $f(x) = |x^2 - 9|$ and $g(x) = |x| \sin x$.

79. ***Geometric and Arithmetic Mean*** The geometric mean of u and v is $G = \sqrt{uv}$ and the arithmetic mean is $A = (u + v)/2$. Show that if $u = x$, $v = x + c$, c a real number, then

$$\frac{dG}{dx} = \frac{A}{G}.$$

Quick Quiz for AP* Preparation: Sections 3.4–3.6

You should solve the following problems without using a graphing calculator.

1. Multiple Choice Which of the following gives dy/dx for $y = \sin^4(3x)$?

(A) $4\sin^3(3x)\cos(3x)$ **(B)** $12\sin^3(3x)\cos(3x)$

(C) $12\sin(3x)\cos(3x)$ **(D)** $12\sin^3(3x)$

(E) $-12\sin^3(3x)\cos(3x)$

2. Multiple Choice Which of the following gives y'' for $y = \cos x + \tan x$?

(A) $-\cos x + 2\sec^2 x \tan x$ **(B)** $\cos x + 2\sec^2 x \tan x$

(C) $-\sin x + \sec^2 x$ **(D)** $-\cos x + \sec^2 x \tan x$

(E) $\cos x + \sec^2 x \tan x$

3. Multiple Choice Which of the following gives dy/dx for the parametric curve $x = 3\sin t$, $y = 2\cos t$?

(A) $-\frac{3}{2}\cot t$ **(B)** $\frac{3}{2}\cot t$ **(C)** $-\frac{2}{3}\tan t$ **(D)** $\frac{2}{3}\tan t$ **(E)** $\tan t$

4. Free Response A particle moves along a line so that its position at any time $t \ge 0$ is given by $s(t) = -t^2 + t + 2$, where s is measured in meters and t is measured in seconds.

(a) What is the initial position of the particle?

(b) Find the velocity of the particle at any time t.

(c) When is the particle moving to the right?

(d) Find the acceleration of the particle at any time t.

(e) Find the speed of the particle at the moment when $s(t) = 0$.

3.7 Implicit Differentiation

What you'll learn about

- Implicitly Defined Functions
- Lenses, Tangents, and Normal Lines
- Derivatives of Higher Order
- Rational Powers of Differentiable Functions

. . . and why

Implicit differentiation allows us to find derivatives of functions that are not defined or written explicitly as a function of a single variable.

Implicitly Defined Functions

The graph of the equation $x^3 + y^3 - 9xy = 0$ (Figure 3.47) has a well-defined slope at nearly every point because it is the union of the graphs of the functions $y = f_1(x)$, $y = f_2(x)$, and $y = f_3(x)$, which are differentiable except at O and A. But how do we find the slope when we cannot conveniently solve the equation to find the functions? The answer is to treat y as a differentiable function of x and differentiate both sides of the equation with respect to x, using the differentiation rules for sums, products, and quotients, and the Chain Rule. Then solve for dy/dx in terms of x and y *together* to obtain a formula that calculates the slope at any point (x, y) on the graph from the values of x and y.

The process by which we find dy/dx is called **implicit differentiation.** The phrase derives from the fact that the equation

$$x^3 + y^3 - 9xy = 0$$

defines the functions f_1, f_2, and f_3 implicitly (i.e., hidden inside the equation), without giving us *explicit* formulas to work with.

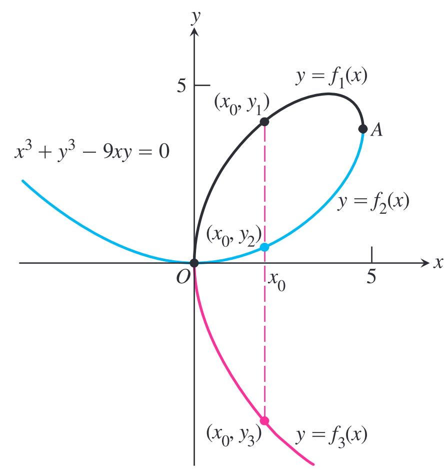


Figure 3.47 The graph of $x^3 + y^3 - 9xy = 0$ (called a *folium*). Although not the graph of a function, it is the union of the graphs of three separate functions. This particular curve dates to Descartes in 1638.

EXAMPLE 1 Differentiating Implicitly

Find dy/dx if $y^2 = x$.

SOLUTION

To find dy/dx, we simply differentiate both sides of the equation $y^2 = x$ with respect to x, treating y as a differentiable function of x and applying the Chain Rule:

$$y^2 = x$$
$$2y\frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{2y}.$$

Now try Exercise 3.

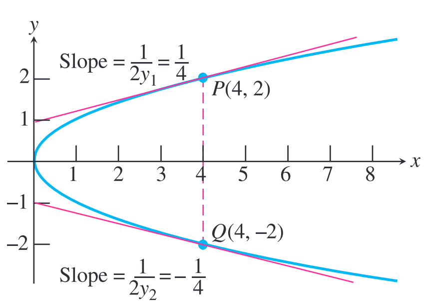


Figure 3.48 The derivative found in Example 1 gives the slope for the tangent lines at both P and Q, because it is a function of y.

In the previous example we differentiated with respect to x, and yet the derivative we obtained appeared as a function of y. Not only is this acceptable, it is actually quite useful. Figure 3.48, for example, shows that the curve has two different tangent lines when $x = 4$: one at the point $(4, 2)$ and the other at the point $(4, -2)$. Since the formula for dy/dx depends on y, our single formula gives the slope in both cases.

Implicit differentiation will frequently yield a derivative that is expressed in terms of both x and y, as in Example 2.

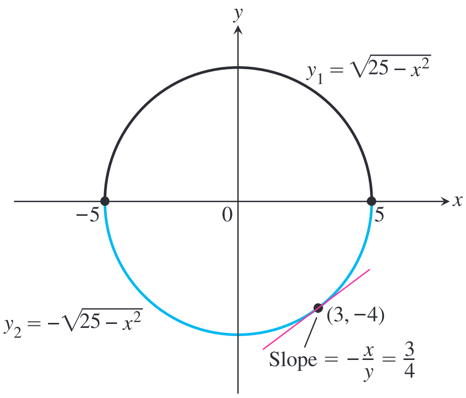


Figure 3.49 The circle combines the graphs of two functions. The graph of y_2 is the lower semicircle and passes through $(3, -4)$. (Example 2)

EXAMPLE 2 Finding Slope on a Circle

Find the slope of the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

SOLUTION

The circle is not the graph of a single function of x, but it is the union of the graphs of two differentiable functions, $y_1 = \sqrt{25 - x^2}$ and $y_2 = -\sqrt{25 - x^2}$ (Figure 3.49). The point $(3, -4)$ lies on the graph of y_2, so it is possible to find the slope by calculating explicitly:

$$\left.\frac{dy_2}{dx}\right|_{x=3} = -\left.\frac{-2x}{2\sqrt{25 - x^2}}\right|_{x=3} = -\frac{-6}{2\sqrt{25 - 9}} = \frac{3}{4}.$$

But we can also find this slope more easily by differentiating both sides of the equation of the circle implicitly with respect to x:

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}.$$

The slope at $(3, -4)$ is

$$-\left.\frac{x}{y}\right|_{(3,\,-4)} = -\frac{3}{-4} = \frac{3}{4}.$$

The implicit solution, besides being computationally easier, yields a formula for dy/dx that applies at any point on the circle (except, of course, $(\pm 5, 0)$, where slope is undefined). The explicit solution derived from the formula for y_2 applies only to the lower half of the circle. ***Now try Exercise 11.***

To calculate the derivatives of other implicitly defined functions, we proceed as in Examples 1 and 2. We treat y as a differentiable function of x and apply the usual rules to differentiate both sides of the defining equation.

EXAMPLE 3 Solving for *dy/dx*

Show that the slope dy/dx is defined at every point on the graph of $2y = x^2 + \sin y$.

SOLUTION

First we need to know dy/dx, which we find by implicit differentiation:

$$2y = x^2 + \sin y$$

$$\frac{d}{dx}(2y) = \frac{d}{dx}(x^2 + \sin y)$$

$$= \frac{d}{dx}(x^2) + \frac{d}{dx}(\sin y)$$

$$2\frac{dy}{dx} = 2x + \cos y\frac{dy}{dx}$$

$$2\frac{dy}{dx} - (\cos y)\frac{dy}{dx} = 2x$$

$$(2 - \cos y)\frac{dy}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{2x}{2 - \cos y}.$$

The formula for dy/dx is defined at every point (x, y), except for those points at which $\cos y = 2$. Since $\cos y$ cannot be greater than 1, this never happens.

Now try Exercise 13.

Ellen Ochoa (1958–)

After earning a doctorate degree in electrical engineering from Stanford University, Ellen Ochoa became a research engineer and, within a few years, received three patents in the field of optics. In 1990, Ochoa joined the NASA astronaut program, and, three years later, became the first Hispanic female to travel in space. Ochoa's message to young people is: "If you stay in school you have the potential to achieve what you want in the future."

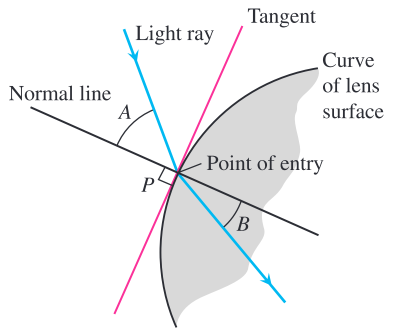


Figure 3.50 The profile of a lens, showing the bending (refraction) of a ray of light as it passes through the lens surface.

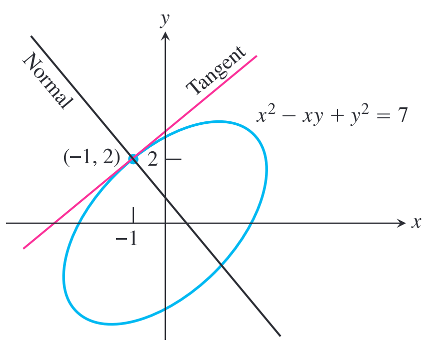


Figure 3.51 Tangent and normal lines to the ellipse $x^2 - xy + y^2 = 7$ at the point $(-1, 2)$. (Example 4)

Implicit Differentiation Process

1. Differentiate both sides of the equation with respect to x.
2. Collect the terms with dy/dx on one side of the equation.
3. Factor out dy/dx.
4. Solve for dy/dx.

Lenses, Tangents, and Normal Lines

In the law that describes how light changes direction as it enters a lens, the important angles are the angles the light makes with the line perpendicular to the surface of the lens at the point of entry (angles A and B in Figure 3.50). This line is called the *normal to the surface* at the point of entry. In a profile view of a lens like the one in Figure 3.50, the normal is a line perpendicular to the tangent to the profile curve at the point of entry.

Profiles of lenses are often described by quadratic curves (see Figure 3.51). When they are, we can use implicit differentiation to find the tangents and normals.

EXAMPLE 4 Tangent and normal to an ellipse

Find the tangent and normal to the ellipse $x^2 - xy + y^2 = 7$ at the point $(-1, 2)$. (See Figure 3.51.)

SOLUTION

We first use implicit differentiation to find dy/dx:

$$x^2 - xy + y^2 = 7$$

$$\frac{d}{dx}(x^2) - \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(7)$$

$$2x - \left(x\frac{dy}{dx} + y\frac{dx}{dx}\right) + 2y\frac{dy}{dx} = 0$$

$$(2y - x)\frac{dy}{dx} = y - 2x$$

$$\frac{dy}{dx} = \frac{y - 2x}{2y - x}.$$

We then evaluate the derivative at $x = -1,\ y = 2$ to obtain

$$\left.\frac{dy}{dx}\right|_{(-1,\,2)} = \left.\frac{y - 2x}{2y - x}\right|_{(-1,\,2)} = \frac{2 - 2(-1)}{2(2) - (-1)} = \frac{4}{5}.$$

The tangent to the curve at $(-1, 2)$ is

$$y - 2 = \frac{4}{5}(x - (-1))$$

$$y = \frac{4}{5}x + \frac{14}{5}.$$

continued

The normal to the curve at $(-1, 2)$ is

$$y - 2 = -\frac{5}{4}(x + 1)$$
$$y = -\frac{5}{4}x + \frac{3}{4}.$$

Now try Exercise 17.

Derivatives of Higher Order

Implicit differentiation can also be used to find derivatives of higher order. Here is an example.

EXAMPLE 5 Finding a Second Derivative Implicitly

Find d^2y/dx^2 if $2x^3 - 3y^2 = 8$.

SOLUTION

To start, we differentiate both sides of the equation with respect to x in order to find $y' = dy/dx$.

$$\frac{d}{dx}(2x^3 - 3y^2) = \frac{d}{dx}(8)$$
$$6x^2 - 6yy' = 0$$
$$x^2 - yy' = 0$$
$$y' = \frac{x^2}{y}, \text{ when } y \neq 0$$

We now apply the Quotient Rule to find y''.

$$y'' = \frac{d}{dx}\left(\frac{x^2}{y}\right) = \frac{2xy - x^2y'}{y^2} = \frac{2x}{y} - \frac{x^2}{y^2} \cdot y'$$

Finally, we substitute $y' = x^2/y$ to express y'' in terms of x and y.

$$y'' = \frac{2x}{y} - \frac{x^2}{y^2}\left(\frac{x^2}{y}\right) = \frac{2x}{y} - \frac{x^4}{y^3}, \text{ when } y \neq 0$$

Now try Exercise 29.

EXPLORATION 1 An Unexpected Derivative

Consider the set of all points (x, y) satisfying the equation $x^2 - 2xy + y^2 = 4$. What does the graph of the equation look like? You can find out in two ways in this Exploration.

1. Use implicit differentiation to find dy/dx. Are you surprised by this derivative?
2. Knowing the derivative, what do you conjecture about the graph?
3. What are the possible values of y when $x = 0$? Does this information enable you to refine your conjecture about the graph?
4. The original equation can be written as $(x - y)^2 - 4 = 0$. By factoring the expression on the left, write two equations whose graphs combine to give the graph of the original equation. Then sketch the graph.
5. Explain why your graph is consistent with the derivative found in part 1.

Rational Powers of Differentiable Functions

We know that the Power Rule

$$\frac{d}{dx}x^n = nx^{n-1}$$

holds for any integer n (Rules 2 and 7 of this chapter). We can now prove that it holds when n is any rational number.

> **RULE 9 Power Rule for Rational Powers of *x***
>
> If n is any rational number, then
> $$\frac{d}{dx}x^n = nx^{n-1}.$$
> If $n < 1$, then the derivative does not exist at $x = 0$.

Proof Let p and q be integers with $q > 0$ and suppose that $y = \sqrt[q]{x^p} = x^{p/q}$. Then

$$y^q = x^p.$$

Since p and q are integers (for which we already have the Power Rule), we can differentiate both sides of the equation with respect to x and obtain

$$qy^{q-1}\frac{dy}{dx} = px^{p-1}.$$

If $y \neq 0$, we can divide both sides of the equation by qy^{q-1} to solve for dy/dx, obtaining

$$\begin{aligned}\frac{dy}{dx} &= \frac{px^{p-1}}{qy^{q-1}} \\ &= \frac{p}{q} \cdot \frac{x^{p-1}}{(x^{p/q})^{q-1}} \\ &= \frac{p}{q} \cdot \frac{x^{p-1}}{x^{p-p/q}} \\ &= \frac{p}{q} \cdot x^{(p-1)-(p-p/q)} \\ &= \frac{p}{q} \cdot x^{(p/q)-1}.\end{aligned}$$

This proves the rule. ■

By combining this result with the Chain Rule, we get an extension of the Power Chain Rule to rational powers of u:

If n is a rational number and u is a differentiable function of x, then u^n is a differentiable function of x and

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx},$$

provided that $u \neq 0$ if $n < 1$.

The restriction that $u \neq 0$ when $n < 1$ is necessary because 0 might be in the domain of u^n but not in the domain of u^{n-1}, as we see in the first two parts of Example 6.

EXAMPLE 6 Using the Rational Power Rule

(a) $\dfrac{d}{dx}(\sqrt{x}) = \dfrac{d}{dx}(x^{1/2}) = \dfrac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}}$

Notice that $\sqrt{x}$ is defined at $x = 0$, but $1/(2\sqrt{x})$ is not.

(b) $\dfrac{d}{dx}(x^{2/3}) = \dfrac{2}{3}(x^{-1/3}) = \dfrac{2}{3x^{1/3}}$

The original function is defined for all real numbers, but the derivative is undefined at $x = 0$. Recall Figure 3.12, which showed that this function's graph has a *cusp* at $x = 0$.

(c)
$$\begin{aligned}\frac{d}{dx}(\cos x)^{-1/5} &= -\frac{1}{5}(\cos x)^{-6/5} \cdot \frac{d}{dx}(\cos x)\\ &= -\frac{1}{5}(\cos x)^{-6/5}(-\sin x)\\ &= \frac{1}{5}\sin x(\cos x)^{-6/5}\end{aligned}$$

Now try Exercise 33.

Quick Review 3.7 *(For help, go to Section 1.2 and Appendix A.5.)*

In Exercises 1–5, sketch the curve defined by the equation and find two functions y_1 and y_2 whose graphs will combine to give the curve.

1. $x - y^2 = 0$ **2.** $4x^2 + 9y^2 = 36$

3. $x^2 - 4y^2 = 0$ **4.** $x^2 + y^2 = 9$

5. $x^2 + y^2 = 2x + 3$

In Exercises 6–8, solve for y' in terms of y and x.

6. $x^2y' - 2xy = 4x - y$

7. $y'\sin x - x\cos x = xy' + y$

8. $x(y^2 - y') = y'(x^2 - y)$

In Exercises 9 and 10, find an expression for the function using rational powers rather than radicals.

9. $\sqrt{x}(x - \sqrt[3]{x})$ **10.** $\dfrac{x + \sqrt[3]{x^2}}{\sqrt{x^3}}$

Section 3.7 Exercises

In Exercises 1–8, find dy/dx.

1. $x^2y + xy^2 = 6$ **2.** $x^3 + y^3 = 18xy$

3. $y^2 = \dfrac{x-1}{x+1}$ **4.** $x^2 = \dfrac{x-y}{x+y}$

5. $x = \tan y$ **6.** $x = \sin y$

7. $x + \tan(xy) = 0$ **8.** $x + \sin y = xy$

In Exercises 9–12, find dy/dx and find the slope of the curve at the indicated point.

9. $x^2 + y^2 = 13$, $(-2, 3)$

10. $x^2 + y^2 = 9$, $(0, 3)$

11. $(x - 1)^2 + (y - 1)^2 = 13$, $(3, 4)$

12. $(x + 2)^2 + (y + 3)^2 = 25$, $(1, -7)$

In Exercises 13–16, find where the slope of the curve is defined.

13. $x^2y - xy^2 = 4$ **14.** $x = \cos y$

15. $x^3 + y^3 = xy$ **16.** $x^2 + 4xy + 4y^2 - 3x = 6$

In Exercises 17–26, find the lines that are **(a)** tangent and **(b)** normal to the curve at the given point.

17. $x^2 + xy - y^2 = 1$, $(2, 3)$

18. $x^2 + y^2 = 25$, $(3, -4)$

19. $x^2y^2 = 9$, $(-1, 3)$

20. $y^2 - 2x - 4y - 1 = 0$, $(-2, 1)$

21. $6x^2 + 3xy + 2y^2 + 17y - 6 = 0$, $(-1, 0)$

22. $x^2 - \sqrt{3}xy + 2y^2 = 5$, $(\sqrt{3}, 2)$

23. $2xy + \pi \sin y = 2\pi$, $(1, \pi/2)$

24. $x \sin 2y = y \cos 2x$, $(\pi/4, \pi/2)$

25. $y = 2\sin(\pi x - y)$, $(1, 0)$

26. $x^2\cos^2 y - \sin y = 0$, $(0, \pi)$

In Exercises 27–30, use implicit differentiation to find dy/dx and then d^2y/dx^2.

27. $x^2 + y^2 = 1$ **28.** $x^{2/3} + y^{2/3} = 1$

29. $y^2 = x^2 + 2x$ **30.** $y^2 + 2y = 2x + 1$

In Exercises 31–42, find dy/dx.

31. $y = x^{9/4}$ **32.** $y = x^{-3/5}$

33. $y = \sqrt[3]{x}$ **34.** $y = \sqrt[4]{x}$

35. $y = (2x + 5)^{-1/2}$ **36.** $y = (1 - 6x)^{2/3}$

37. $y = x\sqrt{x^2 + 1}$ **38.** $y = \dfrac{x}{\sqrt{x^2 + 1}}$

39. $y = \sqrt{1 - \sqrt{x}}$ **40.** $y = 3(2x^{-1/2} + 1)^{-1/3}$

41. $y = 3(\csc x)^{3/2}$ **42.** $y = [\sin(x + 5)]^{5/4}$

43. Which of the following could be true if $f''(x) = x^{-1/3}$?

(a) $f(x) = \frac{3}{2}x^{2/3} - 3$ **(b)** $f(x) = \frac{9}{10}x^{5/3} - 7$

(c) $f'''(x) = -\frac{1}{3}x^{-4/3}$ **(d)** $f'(x) = \frac{3}{2}x^{2/3} + 6$

44. Which of the following could be true if $g''(t) = 1/t^{3/4}$?

(a) $g'(t) = 4\sqrt[4]{t} - 4$ **(b)** $g'''(t) = -4/\sqrt[4]{t}$

(c) $g(t) = t - 7 + (16/5)t^{5/4}$ **(d)** $g'(t) = (1/4)t^{1/4}$

45. ***The Eight Curve*** **(a)** Find the slopes of the figure-eight-shaped curve

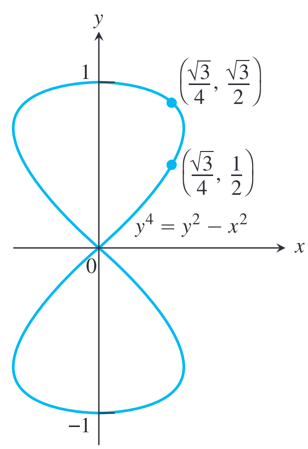
$$y^4 = y^2 - x^2$$

at the two points shown on the graph that follows.

(b) Use parametric mode and the two pairs of parametric equations

$$x_1(t) = \sqrt{t^2 - t^4}, \quad y_1(t) = t,$$
$$x_2(t) = -\sqrt{t^2 - t^4}, \quad y_2(t) = t,$$

to graph the curve. Specify a window and a parameter interval.



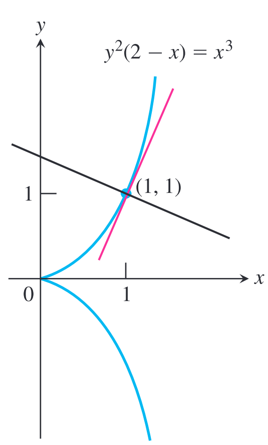
46. ***The Cissoid of Diocles (dates from about 200 B.C.)***

(a) Find equations for the tangent and normal to the cissoid of Diocles,

$$y^2(2 - x) = x^3,$$

at the point (1, 1) as pictured below.

(b) Explain how to reproduce the graph on a grapher.



47. **(a)** Confirm that $(-1, 1)$ is on the curve defined by $x^3y^2 = \cos(\pi y)$.

(b) Use part (a) to find the slope of the line tangent to the curve at $(-1, 1)$.

48. **Grouping Activity**

(a) Show that the relation

$$y^3 - xy = -1$$

cannot be a function of x by showing that there is more than one possible y-value when $x = 2$.

(b) On a small enough square with center (2, 1), the part of the graph of the relation within the square will define a function $y = f(x)$. For this function, find $f'(2)$ and $f''(2)$.

49. Find the two points where the curve $x^2 + xy + y^2 = 7$ crosses the x-axis, and show that the tangents to the curve at these points are parallel. What is the common slope of these tangents?

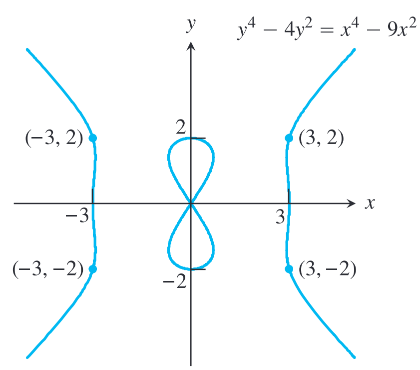
50. Find points on the curve $x^2 + xy + y^2 = 7$ **(a)** where the tangent is parallel to the x-axis and **(b)** where the tangent is parallel to the y-axis. (In the latter case, dy/dx is not defined, but dx/dy is. What value does dx/dy have at these points?)

51. ***Orthogonal Curves*** Two curves are *orthogonal* at a point of intersection if their tangents at that point cross at right angles. Show that the curves $2x^2 + 3y^2 = 5$ and $y^2 = x^3$ are orthogonal at (1, 1) and $(1, -1)$. Use parametric mode to draw the curves and to show the tangent lines.

52. The position of a body moving along a coordinate line at time t is $s = (4 + 6t)^{3/2}$, with s in meters and t in seconds. Find the body's velocity and acceleration when $t = 2$ sec.

53. The velocity of a falling body is $v = 8\sqrt{s - t} + 1$ feet per second at the instant t (sec) the body has fallen s feet from its starting point. Show that the body's acceleration is 32 ft/sec^2.

54. ***The Devil's Curve (Gabriel Cramer [the Cramer of Cramer's Rule], 1750)*** Find the slopes of the devil's curve $y^4 - 4y^2 = x^4 - 9x^2$ at the four indicated points.



55. ***The Folium of Descartes*** (See Figure 3.47 on page 157)

(a) Find the slope of the folium of Descartes, $x^3 + y^3 - 9xy = 0$ at the points (4, 2) and (2, 4).

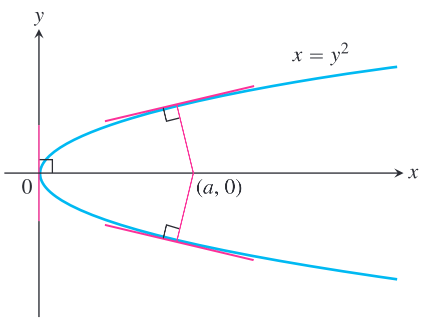
(b) At what point other than the origin does the folium have a horizontal tangent?

(c) Find the coordinates of the point A in Figure 3.47, where the folium has a vertical tangent.

56. The line that is normal to the curve $x^2 + 2xy - 3y^2 = 0$ at $(1, 1)$ intersects the curve at what other point?

57. Find the normals to the curve $xy + 2x - y = 0$ that are parallel to the line $2x + y = 0$.

58. Show that if it is possible to draw these three normals from the point $(a, 0)$ to the parabola $x = y^2$ shown here, then a must be greater than $1/2$. One of the normals is the x-axis. For what value of a are the other two normals perpendicular?



Standardized Test Questions

You should solve the following problems without using a graphing calculator.

59. True or False The slope of $xy^2 + x = 1$ at $(1/2, 1)$ is 2. Justify your answer.

60. True or False The derivative of $y = \sqrt[3]{x}$ is $\frac{1}{3x^{2/3}}$. Justify your answer.

In Exercises 61 and 62, use the curve $x^2 - xy + y^2 = 1$.

61. Multiple Choice Which of the following is equal to dy/dx?

(A) $\frac{y - 2x}{2y - x}$ **(B)** $\frac{y + 2x}{2y - x}$

(C) $\frac{2x}{x - 2y}$ **(D)** $\frac{2x + y}{x - 2y}$

(E) $\frac{y + 2x}{x}$

62. Multiple Choice Which of the following is equal to $\frac{d^2y}{dx^2}$?

(A) $-\frac{6}{(2y - x)^3}$ **(B)** $\frac{10y^2 - 10x^2 - 10xy}{(2y - x)^3}$

(C) $\frac{8x^2 - 4xy + 8y^2}{(x - 2y)^3}$ **(D)** $\frac{10x^2 + 10y^2}{(x - 2y)^3}$

(E) $\frac{2}{x}$

63. Multiple Choice Which of the following is equal to dy/dx if $y = x^{3/4}$?

(A) $\frac{3x^{1/3}}{4}$ **(B)** $\frac{4x^{1/4}}{3}$ **(C)** $\frac{3x^{1/4}}{4}$ **(D)** $\frac{4}{3x^{1/4}}$ **(E)** $\frac{3}{4x^{1/4}}$

64. Multiple Choice Which of the following is equal to the slope of the tangent to $y^2 - x^2 = 1$ at $(1, \sqrt{2})$?

(A) $-\frac{1}{\sqrt{2}}$ **(B)** $-\sqrt{2}$ **(C)** $\frac{1}{\sqrt{2}}$ **(D)** $\sqrt{2}$ **(E)** 0

Extending the Ideas

65. ***Finding Tangents***

(a) Show that the tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point (x_1, y_1) has equation

$$\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1.$$

(b) Find an equation for the tangent to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

at the point (x_1, y_1).

66. ***End Behavior Model*** Consider the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Show that

(a) $y = \pm\frac{b}{a}\sqrt{x^2 - a^2}$.

(b) $g(x) = (b/a)|x|$ is an end behavior model for

$$f(x) = (b/a)\sqrt{x^2 - a^2}.$$

(c) $g(x) = -(b/a)|x|$ is an end behavior model for

$$f(x) = -(b/a)\sqrt{x^2 - a^2}.$$

3.8 Derivatives of Inverse Trigonometric Functions

What you'll learn about

- Derivatives of Inverse Functions
- Derivative of the Arcsine
- Derivative of the Arctangent
- Derivative of the Arcsecant
- Derivatives of the Other Three

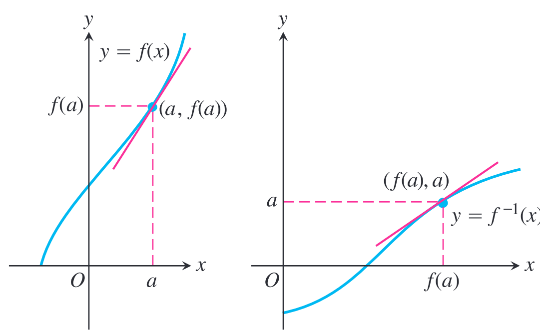
. . . and why

The relationship between the graph of a function and its inverse allows us to see the relationship between their derivatives.

Derivatives of Inverse Functions

In Section 1.5 we learned that the graph of the inverse of a function f can be obtained by reflecting the graph of f across the line $y = x$. If we combine that with our understanding of what makes a function differentiable, we can gain some quick insights into the differentiability of inverse functions.

As Figure 3.52 suggests, the reflection of a continuous curve with no cusps or corners will be another continuous curve with no cusps or corners. Indeed, if there is a tangent line to the graph of f at the point $(a, f(a))$, then that line will reflect across $y = x$ to become a tangent line to the graph of f^{-1} at the point $(f(a), a)$. We can even see geometrically that the *slope* of the reflected tangent line (when it exists and is not zero) will be the *reciprocal* of the slope of the original tangent line, since a change in y becomes a change in x in the reflection, and a change in x becomes a change in y.



The slopes are reciprocal: $\left.\dfrac{df^{-1}}{dx}\right|_{f(a)} = \dfrac{1}{\left.\dfrac{df}{dx}\right|_{a}}$

Figure 3.52 The graphs of a function and its inverse. Notice that the tangent lines have reciprocal slopes.

All of this serves as an introduction to the following theorem, which we will assume as we proceed to find derivatives of inverse functions. Although the essentials of the proof are illustrated in the geometry of Figure 3.52, a careful analytic proof is more appropriate for an advanced calculus text and will be omitted here.

THEOREM 3 Derivatives of Inverse Functions

If f is differentiable at every point of an interval I and df/dx is never zero on I, then f has an inverse and f^{-1} is differentiable at every point of the interval $f(I)$.

EXPLORATION 1 **Finding a Derivative on an Inverse Graph Geometrically**

Let $f(x) = x^5 + 2x - 1$. Since the point (1, 2) is on the graph of f, it follows that the point (2, 1) is on the graph of f^{-1}. Can you find

$$\frac{df^{-1}}{dx}(2),$$

the value of df^{-1}/dx at 2, without knowing a formula for f^{-1}?

1. Graph $f(x) = x^5 + 2x - 1$. A function must be one-to-one to have an inverse function. Is this function one-to-one?
2. Find $f'(x)$. How could this derivative help you to conclude that f has an inverse?
3. Reflect the graph of f across the line $y = x$ to obtain a graph of f^{-1}.
4. Sketch the tangent line to the graph of f^{-1} at the point (2, 1). Call it L.
5. Reflect the line L across the line $y = x$. At what point is the reflection of L tangent to the graph of f?
6. What is the slope of the reflection of L?
7. What is the slope of L?
8. What is $\frac{df^{-1}}{dx}(2)$?

Derivative of the Arcsine

We know that the function $x = \sin y$ is differentiable in the open interval $-\pi/2 < y < \pi/2$ and that its derivative, the cosine, is positive there. Theorem 3 therefore assures us that the inverse function $y = \sin^{-1}(x)$ (the *arcsine* of x) is differentiable throughout the interval $-1 < x < 1$. We cannot expect it to be differentiable at $x = -1$ or $x = 1$, however, because the tangents to the graph are vertical at these points (Figure 3.53).

We find the derivative of $y = \sin^{-1}(x)$ as follows:

$$y = \sin^{-1} x$$

$$\sin y = x$$

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}x$$

$$\cos y\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

The division in the last step is safe because $\cos y \neq 0$ for $-\pi/2 < y < \pi/2$. In fact, $\cos y$ is *positive* for $-\pi/2 < y < \pi/2$, so we can replace $\cos y$ with $\sqrt{1 - (\sin y)^2}$, which is $\sqrt{1 - x^2}$. Thus

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}.$$

If u is a differentiable function of x with $|u| < 1$, we apply the Chain Rule to get

$$\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1 - u^2}}\frac{du}{dx}, \qquad |u| < 1.$$

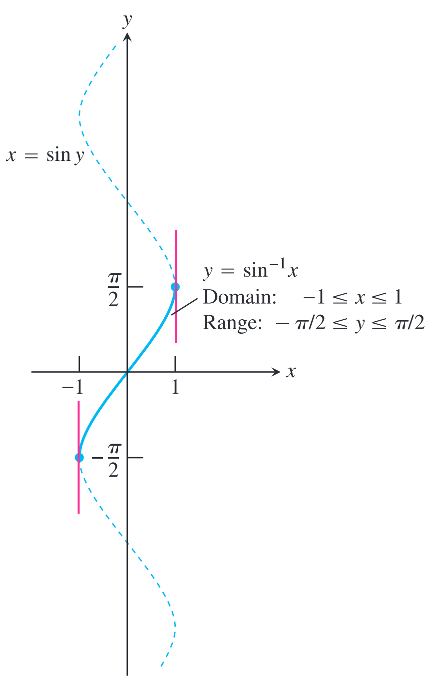


Figure 3.53 The graph of $y = \sin^{-1} x$ has vertical tangents $x = -1$ and $x = 1$.

EXAMPLE 1 Applying the Formula

$$\frac{d}{dx}(\sin^{-1} x^2) = \frac{1}{\sqrt{1-(x^2)^2}} \cdot \frac{d}{dx}(x^2) = \frac{2x}{\sqrt{1-x^4}}$$

Now try Exercise 3.

Derivative of the Arctangent

Although the function $y = \sin^{-1}(x)$ has a rather narrow domain of $[-1, 1]$, the function $y = \tan^{-1} x$ is defined for all real numbers, and is differentiable for all real numbers, as we will now see. The differentiation proceeds exactly as with the arcsine function.

$$\begin{aligned}
y &= \tan^{-1} x \\
\tan y &= x \\
\frac{d}{dx}(\tan y) &= \frac{d}{dx}x \\
\sec^2 y \frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \frac{1}{\sec^2 y} \\
&= \frac{1}{1 + (\tan y)^2} \\
&= \frac{1}{1 + x^2}
\end{aligned}$$

The derivative is defined for all real numbers. If u is a differentiable function of x, we get the Chain Rule form:

$$\frac{d}{dx}\tan^{-1} u = \frac{1}{1+u^2}\frac{du}{dx}.$$

EXAMPLE 2 A Moving Particle

A particle moves along the x-axis so that its position at any time $t \geq 0$ is $x(t) = \tan^{-1}\sqrt{t}$. What is the velocity of the particle when $t = 16$?

SOLUTION

$$v(t) = \frac{d}{dt}\tan^{-1}\sqrt{t} = \frac{1}{1+(\sqrt{t})^2} \cdot \frac{d}{dt}\sqrt{t} = \frac{1}{1+t} \cdot \frac{1}{2\sqrt{t}}$$

When $t = 16$, the velocity is $v(16) = \frac{1}{1+16} \cdot \frac{1}{2\sqrt{16}} = \frac{1}{136}$.

Now try Exercise 11.

Derivative of the Arcsecant

We find the derivative of $y = \sec^{-1} x$, $|x| > 1$, beginning as we did with the other inverse trigonometric functions.

$$\begin{aligned}
y &= \sec^{-1} x \\
\sec y &= x \\
\frac{d}{dx}(\sec y) &= \frac{d}{dx}x \\
\sec y \tan y \frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \frac{1}{\sec y \tan y}
\end{aligned}$$

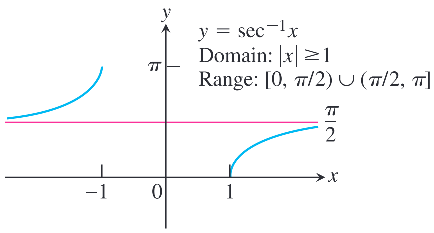


Figure 3.54 The slope of the curve $y = \sec^{-1} x$ is positive for both $x < -1$ and $x > 1$.

To express the result in terms of x, we use the relationships

$$\sec y = x \quad \text{and} \quad \tan y = \pm\sqrt{\sec^2 y - 1} = \pm\sqrt{x^2 - 1}$$

to get

$$\frac{dy}{dx} = \pm\frac{1}{x\sqrt{x^2 - 1}}.$$

Can we do anything about the $\pm$ sign? A glance at Figure 3.54 shows that the slope of the graph $y = \sec^{-1} x$ is always positive. That must mean that

$$\frac{d}{dx}\sec^{-1} x = \begin{cases} +\dfrac{1}{x\sqrt{x^2 - 1}} & \text{if } x > 1 \\ -\dfrac{1}{x\sqrt{x^2 - 1}} & \text{if } x < -1. \end{cases}$$

With the absolute value symbol we can write a single expression that eliminates the "$\pm$" ambiguity:

$$\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2 - 1}}.$$

If u is a differentiable function of x with $|u| > 1$, we have the formula

$$\frac{d}{dx}\sec^{-1} u = \frac{1}{|u|\sqrt{u^2 - 1}}\frac{du}{dx}, \qquad |u| > 1.$$

EXAMPLE 3 Using the Formula

$$\begin{aligned} \frac{d}{dx}\sec^{-1}(5x^4) &= \frac{1}{|5x^4|\sqrt{(5x^4)^2 - 1}}\frac{d}{dx}(5x^4) \\ &= \frac{1}{5x^4\sqrt{25x^8 - 1}}(20x^3) \\ &= \frac{4}{x\sqrt{25x^8 - 1}} \end{aligned}$$

Now try Exercise 17.

Derivatives of the Other Three

We could use the same technique to find the derivatives of the other three inverse trigonometric functions: arccosine, arccotangent, and arccosecant, but there is a much easier way, thanks to the following identities.

Inverse Function–Inverse Cofunction Identities

$$\cos^{-1} x = \pi/2 - \sin^{-1} x$$

$$\cot^{-1} x = \pi/2 - \tan^{-1} x$$

$$\csc^{-1} x = \pi/2 - \sec^{-1} x$$

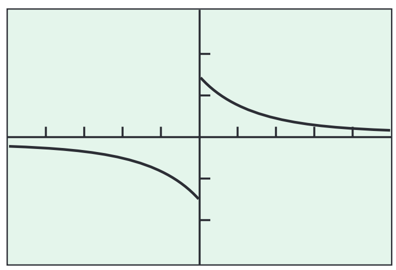
It follows easily that the derivatives of the inverse cofunctions are the negatives of the derivatives of the corresponding inverse functions (see Exercises 32–34).

You have probably noticed by now that most calculators do not have buttons for $\cot^{-1}$, $\sec^{-1}$, or $\csc^{-1}$. They are not needed because of the following identities:

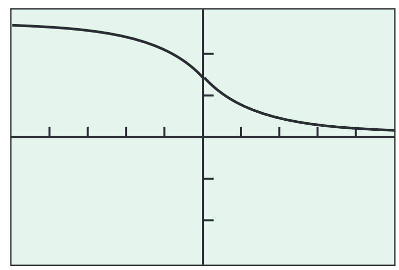
Calculator Conversion Identities

$$\sec^{-1} x = \cos^{-1}(1/x)$$
$$\cot^{-1} x = \pi/2 - \tan^{-1} x$$
$$\csc^{-1} x = \sin^{-1}(1/x)$$

Notice that we do not use $\tan^{-1}(1/x)$ as an identity for $\cot^{-1} x$. A glance at the graphs of $y = \tan^{-1}(1/x)$ and $y = \pi/2 - \tan^{-1} x$ reveals the problem (Figure 3.55).



[–5, 5] by [–3, 3]
(a)



[–5, 5] by [–3, 3]
(b)

Figure 3.55 The graphs of (a) $y = \tan^{-1}(1/x)$ and (b) $y = \pi/2 - \tan^{-1} x$. The graph in (b) is the same as the graph of $y = \cot^{-1} x$.

We cannot replace $\cot^{-1} x$ by the function $y = \tan^{-1}(1/x)$ in the identity for the inverse functions and inverse cofunctions, and so it is not the function we want for $\cot^{-1} x$. The ranges of the inverse trigonometric functions have been chosen in part to make the two sets of identities above hold.

EXAMPLE 4 A Tangent Line to the Arccotangent Curve

Find an equation for the line tangent to the graph of $y = \cot^{-1} x$ at $x = -1$.

SOLUTION

First, we note that

$$\cot^{-1}(-1) = \pi/2 - \tan^{-1}(-1) = \pi/2 - (-\pi/4) = 3\pi/4.$$

The slope of the tangent line is

$$\left.\frac{dy}{dx}\right|_{x=-1} = -\left.\frac{1}{1+x^2}\right|_{x=-1} = -\frac{1}{1+(-1)^2} = -\frac{1}{2}.$$

So the tangent line has equation $y - 3\pi/4 = (-1/2)(x + 1)$.

Now try Exercise 23.

Quick Review 3.8 *(For help, go to Sections 1.2, 1.5, and 1.6.)*

In Exercises 1–5, give the *domain* and *range* of the function, and evaluate the function at $x = 1$.

1. $y = \sin^{-1} x$

2. $y = \cos^{-1} x$

3. $y = \tan^{-1} x$

4. $y = \sec^{-1} x$

5. $y = \tan(\tan^{-1} x)$

In Exercises 6–10, find the inverse of the given function.

6. $y = 3x - 8$

7. $y = \sqrt[3]{x + 5}$

8. $y = \dfrac{8}{x}$

9. $y = \dfrac{3x - 2}{x}$

10. $y = \arctan(x/3)$

Section 3.8 Exercises

In Exercises 1–8, find the derivative of y with respect to the appropriate variable.

1. $y = \cos^{-1}(x^2)$

2. $y = \cos^{-1}(1/x)$

3. $y = \sin^{-1}\sqrt{2}\,t$

4. $y = \sin^{-1}(1 - t)$

5. $y = \sin^{-1}\dfrac{3}{t^2}$

6. $y = s\sqrt{1 - s^2} + \cos^{-1} s$

7. $y = x\sin^{-1} x + \sqrt{1 - x^2}$

8. $y = \dfrac{1}{\sin^{-1}(2x)}$

In Exercises 9–12, a particle moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t)$. Find the velocity at the indicated value of t.

9. $x(t) = \sin^{-1}\left(\dfrac{t}{4}\right), \quad t = 3$

10. $x(t) = \sin^{-1}\left(\dfrac{\sqrt{t}}{4}\right), \quad t = 4$

11. $x(t) = \tan^{-1} t, \quad t = 2$

12. $x(t) = \tan^{-1}(t^2), \quad t = 1$

In Exercises 13–22, find the derivatives of y with respect to the appropriate variable.

13. $y = \sec^{-1}(2s + 1)$

14. $y = \sec^{-1} 5s$

15. $y = \csc^{-1}(x^2 + 1), \ x > 0$

16. $y = \csc^{-1} x/2$

17. $y = \sec^{-1}\dfrac{1}{t}, \quad 0 < t < 1$

18. $y = \cot^{-1}\sqrt{t}$

19. $y = \cot^{-1}\sqrt{t - 1}$

20. $y = \sqrt{s^2 - 1} - \sec^{-1} s$

21. $y = \tan^{-1}\sqrt{x^2 - 1} + \csc^{-1} x, \quad x > 1$

22. $y = \cot^{-1}\dfrac{1}{x} - \tan^{-1} x$

In Exercises 23–26, find an equation for the tangent to the graph of y at the indicated point.

23. $y = \sec^{-1} x, \quad x = 2$

24. $y = \tan^{-1} x, \quad x = 2$

25. $y = \sin^{-1}\left(\dfrac{x}{4}\right), \quad x = 3$

26. $y = \tan^{-1}(x^2), \quad x = 1$

27. **(a)** Find an equation for the line tangent to the graph of $y = \tan x$ at the point $(\pi/4, 1)$.

(b) Find an equation for the line tangent to the graph of $y = \tan^{-1} x$ at the point $(1, \pi/4)$.

28. Let $f(x) = x^5 + 2x^3 + x - 1$.

(a) Find $f(1)$ and $f'(1)$.

(b) Find $f^{-1}(3)$ and $(f^{-1})'(3)$.

29. Let $f(x) = \cos x + 3x$.

(a) Show that f has a differentiable inverse.

(b) Find $f(0)$ and $f'(0)$.

(c) Find $f^{-1}(1)$ and $(f^{-1})'(1)$.

30. **Group Activity** Graph the function $f(x) = \sin^{-1}(\sin x)$ in the viewing window $[-2\pi, 2\pi]$ by $[-4, 4]$. Then answer the following questions:

(a) What is the domain of f?

(b) What is the range of f?

(c) At which points is f not differentiable?

(d) Sketch a graph of $y = f'(x)$ without using NDER or computing the derivative.

(e) Find $f'(x)$ algebraically. Can you reconcile your answer with the graph in part (d)?

31. **Group Activity** A particle moves along the x-axis so that its position at any time $t \geq 0$ is given by $x = \arctan t$.

(a) Prove that the particle is always moving to the right.

(b) Prove that the particle is always decelerating.

(c) What is the limiting position of the particle as t approaches infinity?

In Exercises 32–34, use the inverse function–inverse cofunction identities to derive the formula for the derivative of the function.

32. arccosine

33. arccotangent

34. arccosecant

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

35. True or False The domain of $y = \sin^{-1}x$ is $-1 \le x \le 1$. Justify your answer.

36. True or False The domain of $y = \tan^{-1}x$ is $-1 \le x \le 1$. Justify your answer.

37. Multiple Choice Which of the following is $\frac{d}{dx}\sin^{-1}\left(\frac{x}{2}\right)$?

(A) $-\frac{2}{\sqrt{4-x^2}}$ **(B)** $-\frac{1}{\sqrt{4-x^2}}$ **(C)** $\frac{2}{4+x^2}$

(D) $\frac{2}{\sqrt{4-x^2}}$ **(E)** $\frac{1}{\sqrt{4-x^2}}$

38. Multiple Choice Which of the following is $\frac{d}{dx}\tan^{-1}(3x)$?

(A) $-\frac{3}{1+9x^2}$ **(B)** $-\frac{1}{1+9x^2}$ **(C)** $\frac{1}{1+9x^2}$

(D) $\frac{3}{1+9x^2}$ **(E)** $\frac{3}{\sqrt{1-9x^2}}$

39. Multiple Choice Which of the following is $\frac{d}{dx}\sec^{-1}(x^2)$?

(A) $\frac{2}{x\sqrt{x^4-1}}$ **(B)** $\frac{2}{x\sqrt{x^2-1}}$ **(C)** $\frac{2}{x\sqrt{1-x^4}}$

(D) $\frac{2}{x\sqrt{1-x^2}}$ **(E)** $\frac{2x}{\sqrt{1-x^4}}$

40. Multiple Choice Which of the following is the slope of the tangent line to $y = \tan^{-1}(2x)$ at $x = 1$?

(A) $-2/5$ **(B)** $1/5$ **(C)** $2/5$ **(D)** $5/2$ **(E)** 5

Explorations

In Exercises 41–46, find **(a)** the right end behavior model, **(b)** the left end behavior model, and **(c)** any horizontal tangents for the function if they exist.

41. $y = \tan^{-1} x$

42. $y = \cot^{-1} x$

43. $y = \sec^{-1} x$

44. $y = \csc^{-1} x$

45. $y = \sin^{-1} x$

46. $y = \cos^{-1} x$

Extending the Ideas

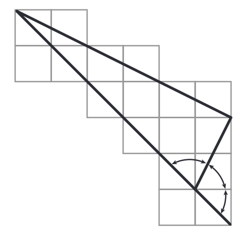
47. *Identities* Confirm the following identities for $x > 0$.

(a) $\cos^{-1} x + \sin^{-1} x = \pi/2$

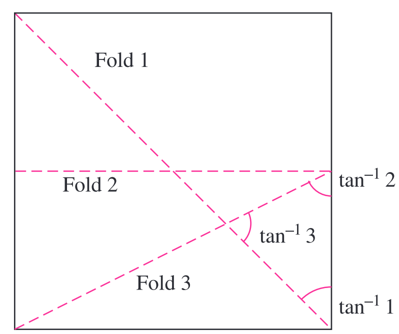
(b) $\tan^{-1} x + \cot^{-1} x = \pi/2$

(c) $\sec^{-1} x + \csc^{-1} x = \pi/2$

48. *Proof Without Words* The figure gives a proof without words that $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$. Explain what is going on.



49. *(Continuation of Exercise 48)* Here is a way to construct $\tan^{-1} 1$, $\tan^{-1} 2$, and $\tan^{-1} 3$ by folding a square of paper. Try it and explain what is going on.



3.9 Derivatives of Exponential and Logarithmic Functions

What you'll learn about

- Derivative of e^x
- Derivative of a^x
- Derivative of $\ln x$
- Derivative of $\log_a x$
- Power Rule for Arbitrary Real Powers

. . . and why

The relationship between exponential and logarithmic functions provides a powerful differentiation tool called logarithmic differentiation.

Derivative of e^x

At the end of the brief review of exponential functions in Section 1.3, we mentioned that the function $y = e^x$ was a particularly important function for modeling exponential growth. The number e was defined in that section to be the limit of $(1 + 1/x)^x$ as $x \to \infty$. This intriguing number shows up in other interesting limits as well, but the one with the most interesting implications for the *calculus* of exponential functions is this one:

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

(The graph and the table in Figure 3.56 provide strong support for this limit being 1. A formal algebraic proof that begins with our limit definition of e would require some rather subtle limit arguments, so we will not include one here.)

The fact that the limit is 1 creates a remarkable relationship between the function e^x and its derivative, as we will now see.

$$\begin{aligned}
\frac{d}{dx}(e^x) &= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} \\
&= \lim_{h \to 0} \frac{e^x \cdot e^h - e^x}{h} \\
&= \lim_{h \to 0} \left(e^x \cdot \frac{e^h - 1}{h} \right) \\
&= e^x \cdot \lim_{h \to 0} \left(\frac{e^h - 1}{h} \right) \\
&= e^x \cdot 1 \\
&= e^x
\end{aligned}$$

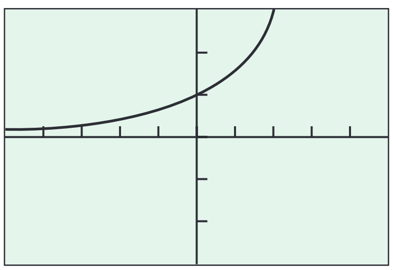
In other words, the derivative of this particular function is itself!

$$\frac{d}{dx}(e^x) = e^x$$

If u is a differentiable function of x, then we have

$$\frac{d}{dx} e^u = e^u \frac{du}{dx}.$$

We will make extensive use of this formula when we study exponential growth and decay in Chapter 6.



[–4.9, 4.9] by [–2.9, 2.9]

(a)

X	Y1	
–.03	.98515	
–.02	.99007	
–.01	.99502	
0	ERROR	
.01	1.005	
.02	1.0101	
.03	1.0152	

X=0

(b)

Figure 3.56 (a) The graph and (b) the table support the conclusion that

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

EXAMPLE 1 Using the Formula

Find dy/dx if $y = e^{(x+x^2)}$.

SOLUTION

Let $u = x + x^2$ then $y = e^u$. Then

$$\frac{dy}{dx} = e^u \frac{du}{dx}, \qquad \text{and} \qquad \frac{du}{dx} = 1 + 2x.$$

Thus, $\dfrac{dy}{dx} = e^u \dfrac{du}{dx} = e^{(x+x^2)}(1 + 2x).$

Now try Exercise 9.

Is any other function its own derivative?

The zero function is also its own derivative, but this hardly seems worth mentioning. (Its value is always 0 and its slope is always 0.) In addition to e^x, however, we can also say that any constant *multiple* of e^x is its own derivative:

$$\frac{d}{dx}(c \cdot e^x) = c \cdot e^x.$$

The next obvious question is whether there are still *other* functions that are their own derivatives, and this time the answer is no. The only functions that satisfy the condition $dy/dx = y$ are functions of the form $y = ke^x$ (and notice that the zero function can be included in this category). We will prove this significant fact in Chapter 6.

Derivative of a^x

What about an exponential function with a base other than e? We will assume that the base is positive and different from 1, since negative numbers to arbitrary real powers are not always real numbers, and $y = 1^x$ is a constant function.

If $a > 0$ and $a \neq 1$, we can use the properties of logarithms to write a^x in terms of e^x. The formula for doing so is

$$a^x = e^{x \ln a}.$$

We can then find the derivative of a^x with the Chain Rule.

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x \ln a} = e^{x \ln a} \cdot \frac{d}{dx}(x \ln a) = e^{x \ln a} \cdot \ln a = a^x \ln a$$

Thus, if u is a differentiable function of x, we get the following rule.

For $a > 0$ and $a \neq 1$,

$$\frac{d}{dx}(a^u) = a^u \ln a \frac{du}{dx}.$$

EXAMPLE 2 Reviewing the Algebra of Logarithms

At what point on the graph of the function $y = 2^t - 3$ does the tangent line have slope 21?

SOLUTION

The slope is the derivative:

$$\frac{d}{dt}(2^t - 3) = 2^t \cdot \ln 2 - 0 = 2^t \ln 2.$$

We want the value of t for which $2^t \ln 2 = 21$. We could use the solver on the calculator, but we will use logarithms for the sake of review.

$$2^t \ln 2 = 21$$

$$2^t = \frac{21}{\ln 2}$$

$$\ln 2^t = \ln\left(\frac{21}{\ln 2}\right)$$

$$t \cdot \ln 2 = \ln 21 - \ln(\ln 2)$$

$$t = \frac{\ln 21 - \ln(\ln 2)}{\ln 2}$$

$$t \approx 4.921$$

$$y = 2^t - 3 \approx 27.297$$

The point is approximately (4.9, 27.3). *Now try Exercise 29.*

EXPLORATION 1 **Leaving Milk on the Counter**

A glass of cold milk from the refrigerator is left on the counter on a warm summer day. Its temperature y (in degrees Fahrenheit) after sitting on the counter t minutes is

$$y = 72 - 30(0.98)^t.$$

Answer the following questions by interpreting y and dy/dt.

1. What is the temperature of the refrigerator? How can you tell?
2. What is the temperature of the room? How can you tell?
3. When is the milk warming up the fastest? How can you tell?
4. Determine algebraically when the temperature of the milk reaches 55°F.
5. At what rate is the milk warming when its temperature is 55°F? Answer with an appropriate unit of measure.

Derivative of ln *x*

Now that we know the derivative of e^x, it is relatively easy to find the derivative of its inverse function, ln x.

$$y = \ln x$$
$$e^y = x$$
$$\frac{d}{dx}(e^y) = \frac{d}{dx}(x)$$
$$e^y \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

This equation answers what was once a perplexing problem: Is there a function with derivative x^{-1}? All of the other power functions follow the Power Rule,

$$\frac{d}{dx}x^n = nx^{n-1}.$$

However, this formula is not much help if one is looking for a function with x^{-1} as its derivative! Now we know why: The function we should be looking for is not a power function at all; it is the natural logarithm function.

If u is a differentiable function of x and $u > 0$,

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}.$$

EXAMPLE 3 **A Tangent through the Origin**

A line with slope m passes through the origin and is tangent to the graph of $y = \ln x$. What is the value of m?

SOLUTION

This problem is a little harder than it looks, since we do not know the point of tangency. However, we do know two important facts about that point:

1. it has coordinates $(a, \ln a)$ for some positive a, and
2. the tangent line there has slope $m = 1/a$ (Figure 3.57).

Since the tangent line passes through the origin, its slope is

$$m = \frac{\ln a - 0}{a - 0} = \frac{\ln a}{a}.$$

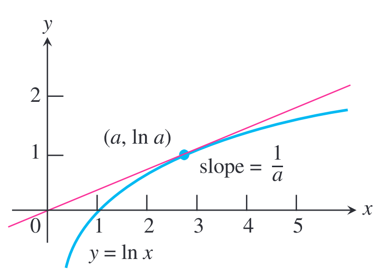


Figure 3.57 The tangent line intersects the curve at some point $(a, \ln a)$, where the slope of the curve is $1/a$. (Example 3)

continued

Setting these two formulas for m equal to each other, we have

$$\frac{\ln a}{a} = \frac{1}{a}$$
$$\ln a = 1$$
$$e^{\ln a} = e^1$$
$$a = e$$
$$m = \frac{1}{e}.$$

Now try Exercise 31.

Derivative of $\log_a x$

To find the derivative of $\log_a x$ for an arbitrary base ($a > 0$, $a \neq 1$), we use the change-of-base formula for logarithms to express $\log_a x$ in terms of natural logarithms, as follows:

$$\log_a x = \frac{\ln x}{\ln a}.$$

The rest is easy:

$$\begin{aligned}\frac{d}{dx}\log_a x &= \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right)\\ &= \frac{1}{\ln a}\cdot\frac{d}{dx}\ln x\\ &= \frac{1}{\ln a}\cdot\frac{1}{x}\\ &= \frac{1}{x\ln a}.\end{aligned}$$

So, if u is a differentiable function of x and $u > 0$, the formula is as follows.

For $a > 0$ and $a \neq 1$,

$$\frac{d}{dx}\log_a u = \frac{1}{u\ln a}\frac{du}{dx}.$$

EXAMPLE 4 Going the Long Way with the Chain Rule

Find dy/dx if $y = \log_a a^{\sin x}$.

SOLUTION

Carefully working from the outside in, we apply the Chain Rule to get:

$$\begin{aligned}\frac{d}{dx}(\log_a a^{\sin x}) &= \frac{1}{a^{\sin x}\ln a}\cdot\frac{d}{dx}(a^{\sin x})\\ &= \frac{1}{a^{\sin x}\ln a}\cdot a^{\sin x}\ln a\cdot\frac{d}{dx}(\sin x)\\ &= \frac{a^{\sin x}\ln a}{a^{\sin x}\ln a}\cdot\cos x\\ &= \cos x.\end{aligned}$$

Now try Exercise 23.

We could have saved ourselves a lot of work in Example 4 if we had noticed at the beginning that $\log_a a^{\sin x}$, being the composite of inverse functions, is equal to $\sin x$. It is always a good idea to simplify functions *before* differentiating, wherever possible. On the other hand, it is comforting to know that all these rules do work if applied correctly.

Power Rule for Arbitrary Real Powers

We are now ready to prove the Power Rule in its final form. As long as $x > 0$, we can write any real power of x as a power of e, specifically

$$x^n = e^{n \ln x}.$$

This enables us to differentiate x^n for any real power n, as follows:

$$\begin{aligned}\frac{d}{dx}(x^n) &= \frac{d}{dx}(e^{n \ln x}) \\ &= e^{n \ln x} \cdot \frac{d}{dx}(n \ln x) \\ &= e^{n \ln x} \cdot \frac{n}{x} \\ &= x^n \cdot \frac{n}{x} \\ &= nx^{n-1}.\end{aligned}$$

The Chain Rule extends this result to the Power Rule's final form.

RULE 10 Power Rule for Arbitrary Real Powers

If u is a positive differentiable function of x and n is any real number, then u^n is a differentiable function of x, and

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}.$$

EXAMPLE 5 Using the Power Rule in all its Power

(a) If $y = x^{\sqrt{2}}$, then

$$\frac{dy}{dx} = \sqrt{2}x^{(\sqrt{2}-1)}.$$

(b) If $y = (2 + \sin 3x)^{\pi}$, then

$$\begin{aligned}\frac{d}{dx}(2 + \sin 3x)^{\pi} &= \pi(2 + \sin 3x)^{\pi-1}(\cos 3x) \cdot 3 \\ &= 3\pi(2 + \sin 3x)^{\pi-1}(\cos 3x).\end{aligned}$$

Now try Exercise 35.

EXAMPLE 6 Finding Domain

If $f(x) = \ln(x - 3)$, find $f'(x)$. State the domain of f'.

SOLUTION

The domain of f is $(3, \infty)$ and

$$f'(x) = \frac{1}{x - 3}.$$

continued

The domain of f' appears to be all $x \neq 3$. However, since f is not defined for $x < 3$, neither is f'. Thus,

$$f'(x) = \frac{1}{x-3}, \qquad x > 3.$$

That is, the domain of f' is $(3, \infty)$. ***Now try Exercise 37.***

Sometimes the properties of logarithms can be used to simplify the differentiation process, even if we must introduce the logarithms ourselves as a step in the process. Example 7 shows a clever way to differentiate $y = x^x$ for $x > 0$.

EXAMPLE 7 Logarithmic Differentiation

Find dy/dx for $y = x^x$, $x > 0$.

SOLUTION

$$y = x^x$$
$$\ln y = \ln x^x$$
$$\ln y = x \ln x$$
$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(x \ln x)$$
$$\frac{1}{y}\frac{dy}{dx} = 1 \cdot \ln x + x \cdot \frac{1}{x}$$
$$\frac{dy}{dx} = y(\ln x + 1)$$
$$\frac{dy}{dx} = x^x(\ln x + 1)$$

Now try Exercise 43.

EXAMPLE 8 How Fast does a Flu Spread?

The spread of a flu in a certain school is modeled by the equation

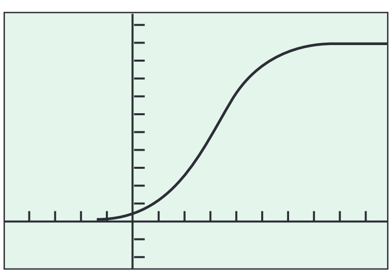
$$P(t) = \frac{100}{1 + e^{3-t}},$$

where $P(t)$ is the total number of students infected t days after the flu was first noticed. Many of them may already be well again at time t.

(a) Estimate the initial number of students infected with the flu.

(b) How fast is the flu spreading after 3 days?

(c) When will the flu spread at its maximum rate? What is this rate?



[–5, 10] by [–25, 120]

Figure 3.58 The graph of

$$P(t) = \frac{100}{1 + e^{3-t}},$$

modeling the spread of a flu. (Example 8)

SOLUTION

The graph of P as a function of t is shown in Figure 3.58.

(a) $P(0) = 100/(1 + e^3) = 5$ students (to the nearest whole number).

continued

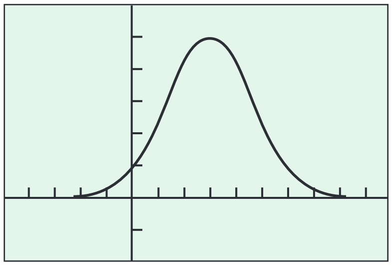


Figure 3.59 The graph of dP/dt, the rate of spread of the flu in Example 8. The graph of P is shown in Figure 3.58.

(b) To find the rate at which the flu spreads, we find dP/dt. To find dP/dt, we need to invoke the Chain Rule twice:

$$\begin{aligned}\frac{dP}{dt} = \frac{d}{dt}(100(1 + e^{3-t})^{-1}) &= 100 \cdot (-1)(1 + e^{3-t})^{-2} \cdot \frac{d}{dt}(1 + e^{3-t}) \\ &= -100(1 + e^{3-t})^{-2} \cdot (0 + e^{3-t} \cdot \frac{d}{dt}(3 - t)) \\ &= -100(1 + e^{3-t})^{-2}(e^{3-t} \cdot (-1)) \\ &= \frac{100e^{3-t}}{(1 + e^{3-t})^2}\end{aligned}$$

At $t = 3$, then, $dP/dt = 100/4 = 25$. The flu is spreading to 25 students per day.

(c) We could estimate when the flu is spreading the fastest by seeing where the graph of $y = P(t)$ has the steepest upward slope, but we can answer both the "when" and the "what" parts of this question most easily by finding the maximum point on the graph of the derivative (Figure 3.59).

We see by tracing on the curve that the maximum rate occurs at about 3 days, when (as we have just calculated) the flu is spreading at a rate of 25 students per day.

Now try Exercise 51.

Quick Review 3.9 *(For help, go to Sections 1.3 and 1.5.)*

1. Write $\log_5 8$ in terms of natural logarithms.

2. Write 7^x as a power of e.

In Exercises 3–7, simplify the expression using properties of exponents and logarithms.

3. $\ln(e^{\tan x})$

4. $\ln(x^2 - 4) - \ln(x + 2)$

5. $\log_2(8^{x-5})$

6. $(\log_4 x^{15})/(\log_4 x^{12})$

7. $3 \ln x - \ln 3x + \ln(12x^2)$

In Exercises 8–10, solve the equation algebraically using logarithms. Give an *exact* answer, such as $(\ln 2)/3$, and also an approximate answer to the nearest hundredth.

8. $3^x = 19$

9. $5^t \ln 5 = 18$

10. $3^{x+1} = 2^x$

Section 3.9 Exercises

In Exercises 1–28, find dy/dx. Remember that you can use NDER to support your computations.

1. $y = 2e^x$

2. $y = e^{2x}$

3. $y = e^{-x}$

4. $y = e^{-5x}$

5. $y = e^{2x/3}$

6. $y = e^{-x/4}$

7. $y = xe^2 - e^x$

8. $y = x^2e^x - xe^x$

9. $y = e^{\sqrt{x}}$

10. $y = e^{(x^2)}$

11. $y = 8^x$

12. $y = 9^{-x}$

13. $y = 3^{\csc x}$

14. $y = 3^{\cot x}$

15. $y = \ln(x^2)$

16. $y = (\ln x)^2$

17. $y = \ln(1/x)$

18. $y = \ln(10/x)$

19. $y = \ln(\ln x)$

20. $y = x \ln x - x$

21. $y = \log_4 x^2$

22. $y = \log_5 \sqrt{x}$

23. $y = \log_2(1/x)$

24. $y = 1/\log_2 x$

25. $y = \ln 2 \cdot \log_2 x$

26. $y = \log_3(1 + x \ln 3)$

27. $y = \log_{10} e^x$

28. $y = \ln 10^x$

29. At what point on the graph of $y = 3^x + 1$ is the tangent line parallel to the line $y = 5x - 1$?

30. At what point on the graph of $y = 2e^x - 1$ is the tangent line perpendicular to the line $y = -3x + 2$?

31. A line with slope m passes through the origin and is tangent to $y = \ln(2x)$. What is the value of m?

32. A line with slope m passes through the origin and is tangent to $y = \ln(x/3)$. What is the value of m?

In Exercises 33–36, find dy/dx.

33. $y = x^{\pi}$

34. $y = x^{1+\sqrt{2}}$

35. $y = x^{-\sqrt{2}}$

36. $y = x^{1-e}$

In Exercises 37–42, find $f'(x)$ and state the domain of f'.

37. $f(x) = \ln(x + 2)$

38. $f(x) = \ln(2x + 2)$

39. $f(x) = \ln(2 - \cos x)$

40. $f(x) = \ln(x^2 + 1)$

41. $f(x) = \log_2(3x + 1)$

42. $f(x) = \log_{10}\sqrt{x + 1}$

Group Activity In Exercises 43–48, use the technique of logarithmic differentiation to find dy/dx.

43. $y = (\sin x)^x, \quad 0 < x < \pi/2$

44. $y = x^{\tan x}, \quad x > 0$

45. $y = \sqrt[5]{\dfrac{(x - 3)^4(x^2 + 1)}{(2x + 5)^3}}$

46. $y = \dfrac{x\sqrt{x^2 + 1}}{(x + 1)^{2/3}}$

47. $y = x^{\ln x}$

48. $y = x^{(1/\ln x)}$

49. Find an equation for a line that is tangent to the graph of $y = e^x$ and goes through the origin.

50. Find an equation for a line that is normal to the graph of $y = xe^x$ and goes through the origin.

51. ***Spread of a Rumor*** The spread of a rumor in a certain school is modeled by the equation

$$P(t) = \frac{300}{1 + 2^{4-t}},$$

where $P(t)$ is the total number of students who have heard the rumor t days after the rumor first started to spread.

(a) Estimate the initial number of students who first heard the rumor.

(b) How fast is the rumor spreading after 4 days?

(c) When will the rumor spread at its maximum rate? What is that rate?

52. ***Spread of Flu*** The spread of flu in a certain school is modeled by the equation

$$P(t) = \frac{200}{1 + e^{5-t}},$$

where $P(t)$ is the total number of students infected t days after the flu first started to spread.

(a) Estimate the initial number of students infected with this flu.

(b) How fast is the flu spreading after 4 days?

(c) When will the flu spread at its maximum rate? What is that rate?

53. ***Radioactive Decay*** The amount A (in grams) of radioactive plutonium remaining in a 20-gram sample after t days is given by the formula

$$A = 20 \cdot (1/2)^{t/140}.$$

At what rate is the plutonium decaying when $t = 2$ days? Answer in appropriate units.

54. For any positive constant k, the derivative of $\ln(kx)$ is $1/x$. Prove this fact

(a) by using the Chain Rule.

(b) by using a property of logarithms and differentiating.

55. Let $f(x) = 2^x$.

(a) Find $f'(0)$.

(b) Use the definition of the derivative to write $f'(0)$ as a limit.

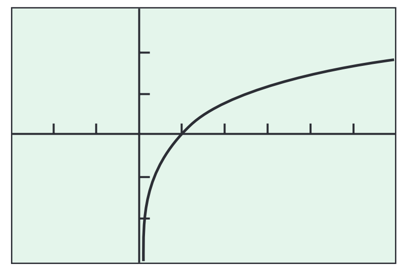
(c) Deduce the exact value of

$$\lim_{h \to 0} \frac{2^h - 1}{h}.$$

(d) What is the exact value of

$$\lim_{h \to 0} \frac{7^h - 1}{h}?$$

56. **Writing to Learn** The graph of $y = \ln x$ looks as though it might be approaching a horizontal asymptote. Write an argument based on the graph of $y = e^x$ to explain why it does not.



[–3, 6] by [–3, 3]

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

57. **True or False** The derivative of $y = 2^x$ is 2^x. Justify your answer.

58. **True or False** The derivative of $y = e^{2x}$ is $2(\ln 2)\,e^{2x}$. Justify your answer.

59. **Multiple Choice** If a flu is spreading at the rate of

$$P(t) = \frac{150}{1 + e^{4-t}},$$

which of the following is the initial number of persons infected?

(A) 1 **(B)** 3 **(C)** 7 **(D)** 8 **(E)** 75

60. **Multiple Choice** Which of the following is the domain of $f'(x)$ if $f(x) = \log_2(x + 3)$?

(A) $x < -3$ **(B)** $x \le 3$ **(C)** $x \ne -3$ **(D)** $x > -3$ **(E)** $x \ge -3$

61. **Multiple Choice** Which of the following gives dy/dx if $y = \log_{10}(2x - 3)$?

(A) $\dfrac{2}{(2x - 3)\ln 10}$ **(B)** $\dfrac{2}{2x - 3}$ **(C)** $\dfrac{1}{(2x - 3)\ln 10}$ **(D)** $\dfrac{1}{2x - 3}$ **(E)** $\dfrac{1}{2x}$

62. **Multiple Choice** Which of the following gives the slope of the tangent line to the graph of $y = 2^{1-x}$ at $x = 2$?

(A) $-\dfrac{1}{2}$ **(B)** $\dfrac{1}{2}$ **(C)** -2 **(D)** 2 **(E)** $-\dfrac{\ln 2}{2}$

Exploration

63. Let $y_1 = a^x$, $y_2 = \text{NDER } y_1$, $y_3 = y_2/y_1$, and $y_4 = e^{y_3}$.

(a) Describe the graph of y_4 for $a = 2, 3, 4, 5$. Generalize your description to an arbitrary $a > 1$.

(b) Describe the graph of y_3 for $a = 2, 3, 4, 5$. Compare a table of values for y_3 for $a = 2, 3, 4, 5$ with $\ln a$. Generalize your description to an arbitrary $a > 1$.

(c) Explain how parts (a) and (b) support the statement

$$\frac{d}{dx} a^x = a^x \quad \text{if and only if} \quad a = e.$$

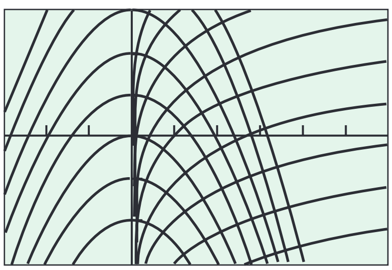
(d) Show algebraically that $y_1 = y_2$ if and only if $a = e$.

Extending the Ideas

64. ***Orthogonal Families of Curves*** Prove that all curves in the family

$$y = -\frac{1}{2}x^2 + k$$

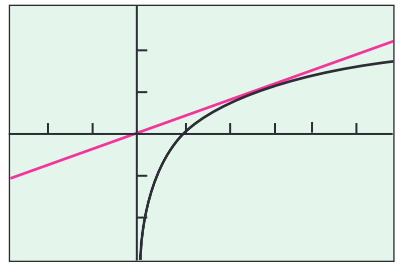
(k any constant) are perpendicular to all curves in the family $y = \ln x + c$ (c any constant) at their points of intersection. (See accompanying figure.)



[–3, 6] by [–3, 3]

65. ***Which is Bigger, π^e or e^π?*** Calculators have taken some of the mystery out of this once-challenging question. (Go ahead and check; you will see that it is a surprisingly close call.) You can answer the question without a calculator, though, by using the result from Example 3 of this section.

Recall from that example that the line through the origin tangent to the graph of $y = \ln x$ has slope $1/e$.



[–3, 6] by [–3, 3]

(a) Find an equation for this tangent line.

(b) Give an argument based on the graphs of $y = \ln x$ and the tangent line to explain why $\ln x < x/e$ for all positive $x \neq e$.

(c) Show that $\ln(x^e) < x$ for all positive $x \neq e$.

(d) Conclude that $x^e < e^x$ for all positive $x \neq e$.

(e) So which is bigger, π^e or e^π?

Quick Quiz for AP* Preparation: Sections 3.7–3.9

You may use a graphing calculator to solve the following problems.

1. Multiple Choice Which of the following gives dy/dx at $x = 1$ if $x^3 + 2xy = 9$?

(A) 11/2 **(B)** 5/2 **(C)** 3/2 **(D)** −5/2 **(E)** −11/2

2. Multiple Choice Which of the following gives dy/dx if $y = \cos^3(3x - 2)$?

(A) $-9\cos^2(3x-2)\sin(3x-2)$

(B) $-3\cos^2(3x-2)\sin(3x-2)$

(C) $9\cos^2(3x-2)\sin(3x-2)$

(D) $-9\cos^2(3x-2)$

(E) $-3\cos^2(3x-2)$

3. Multiple Choice Which of the following gives dy/dx if $y = \sin^{-1}(2x)$?

(A) $-\frac{2}{\sqrt{1-4x^2}}$ **(B)** $-\frac{1}{\sqrt{1-4x^2}}$ **(C)** $\frac{2}{\sqrt{1-4x^2}}$

(D) $\frac{1}{\sqrt{1-4x^2}}$ **(E)** $\frac{2x}{1+4x^2}$

4. Free Response A curve in the xy-plane is defined by $xy^2 - x^3y = 6$.

(a) Find dy/dx.

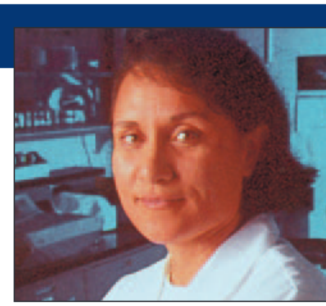
(b) Find an equation for the tangent line at each point on the curve with x-coordinate 1.

(c) Find the x-coordinate of each point on the curve where the tangent line is vertical.

Calculus at Work

I work at Ramsey County Hospital and other community hospitals in the Minneapolis area, both with patients and in a laboratory. I have wanted to be a physician since I was about 12 years old, and I began attending medical school when I was 30 years old. I am now working in the field of internal medicine.

Cardiac patients are common in my field, especially in the diagnostic stages. One of the machines that is sometimes used in the emergency room to diagnose problems is called a Swan-Ganz catheter, named after its inventors Harold James Swan and William Ganz. The catheter is inserted into the pulmonary artery and then is hooked up to a cardiac monitor. A program measures cardiac output by looking at changes of slope in the curve. This information alerts me to left-sided heart failure.



Lupe Bolding, M.D.

Chapter 3 Key Terms

acceleration (p. 130)
average velocity (p. 128)
Chain Rule (p. 149)
Constant Multiple Rule (p. 117)
Derivative of a Constant Function (p. 116)
derivative of f at a (p. 99)
differentiable function (p. 99)
differentiable on a closed interval (p. 104)
displacement (p. 128)
free-fall constants (p. 130)
implicit differentiation (p. 157)
instantaneous rate of change (p. 127)
instantaneous velocity (p. 128)
Intermediate Value Theorem for Derivatives (p. 113)
inverse function–inverse cofunction identities (p. 168)
jerk (p. 144)
left-hand derivative (p. 104)
local linearity (p. 110)
logarithmic differentiation (p. 177)
marginal cost (p. 134)
marginal revenue (p. 134)
nth derivative (p. 122)
normal to the surface (p. 159)
numerical derivative (NDER) (p. 111)
orthogonal curves (p. 154)
orthogonal families (p. 180)
Power Chain Rule (p. 151)
Power Rule for Arbitrary Real Powers (p. 176)
Power Rule for Negative Integer Powers of x (p. 121)
Power Rule for Positive Integer Powers of x (p. 116)
Power Rule for Rational Powers of x (p. 161)
Product Rule (p. 119)
Quotient Rule (p. 120)
right-hand derivative (p. 104)
sensitivity to change (p. 133)
simple harmonic motion (p. 143)
speed (p. 129)
Sum and Difference Rule (p. 117)
symmetric difference quotient (p. 111)
velocity (p. 128)

Chapter 3 Review Exercises

The collection of exercises marked in red could be used as a chapter test.

In Exercises 1–30, find the derivative of the function.

1. $y = x^5 - \frac{1}{8}x^2 + \frac{1}{4}x$

2. $y = 3 - 7x^3 + 3x^7$

3. $y = 2 \sin x \cos x$

4. $y = \frac{2x+1}{2x-1}$

5. $s = \cos(1 - 2t)$

6. $s = \cot \frac{2}{t}$

7. $y = \sqrt{x} + 1 + \frac{1}{\sqrt{x}}$

8. $y = x\sqrt{2x+1}$

9. $r = \sec(1 + 3\theta)$

10. $r = \tan^2(3 - \theta^2)$

11. $y = x^2 \csc 5x$

12. $y = \ln \sqrt{x}$

13. $y = \ln(1 + e^x)$

14. $y = xe^{-x}$

15. $y = e^{(1+\ln x)}$

16. $y = \ln(\sin x)$

17. $r = \ln(\cos^{-1} x)$

18. $r = \log_2(\theta^2)$

19. $s = \log_5(t - 7)$

20. $s = 8^{-t}$

21. $y = x^{\ln x}$

22. $y = \frac{(2x)2^x}{\sqrt{x^2+1}}$

23. $y = e^{\tan^{-1} x}$

24. $y = \sin^{-1}\sqrt{1-u^2}$

25. $y = t \sec^{-1} t - \frac{1}{2}\ln t$

26. $y = (1 + t^2)\cot^{-1} 2t$

27. $y = z\cos^{-1} z - \sqrt{1 - z^2}$

28. $y = 2\sqrt{x-1}\csc^{-1}\sqrt{x}$

29. $y = \csc^{-1}(\sec x),\ 0 \le x \le 2\pi$

30. $r = \left(\dfrac{1 + \sin\theta}{1 - \cos\theta}\right)^2$

In Exercises 31–34, find all values of x for which the function is differentiable.

31. $y = \ln x^2$

32. $y = \sin x - x\cos x$

33. $y = \sqrt{\dfrac{1 - x}{1 + x^2}}$

34. $y = (2x - 7)^{-1}(x + 5)$

In Exercises 35–38, find dy/dx.

35. $xy + 2x + 3y = 1$

36. $5x^{4/5} + 10y^{6/5} = 15$

37. $\sqrt{xy} = 1$

38. $y^2 = \dfrac{x}{x + 1}$

In Exercises 39–42, find d^2y/dx^2 by implicit differentiation.

39. $x^3 + y^3 = 1$

40. $y^2 = 1 - \dfrac{2}{x}$

41. $y^3 + y = 2\cos x$

42. $x^{1/3} + y^{1/3} = 4$

In Exercises 43 and 44, find all derivatives of the function.

43. $y = \dfrac{x^4}{2} - \dfrac{3}{2}x^2 - x$

44. $y = \dfrac{x^5}{120}$

In Exercises 45–48, find an equation for the **(a)** tangent and **(b)** normal to the curve at the indicated point.

45. $y = \sqrt{x^2 - 2x}, \quad x = 3$

46. $y = 4 + \cot x - 2\csc x, \quad x = \pi/2$

47. $x^2 + 2y^2 = 9, \quad (1, 2)$

48. $x + \sqrt{xy} = 6, \quad (4, 1)$

In Exercises 49–52, find an equation for the line tangent to the curve at the point defined by the given value of t.

49. $x = 2\sin t, \quad y = 2\cos t, \quad t = 3\pi/4$

50. $x = 3\cos t, \quad y = 4\sin t, \quad t = 3\pi/4$

51. $x = 3\sec t, \quad y = 5\tan t, \quad t = \pi/6$

52. $x = \cos t, \quad y = t + \sin t, \quad t = -\pi/4$

53. Writing to Learn

(a) Graph the function

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2. \end{cases}$$

(b) Is f continuous at $x = 1$? Explain.

(c) Is f differentiable at $x = 1$? Explain.

54. Writing to Learn For what values of the constant m is

$$f(x) = \begin{cases} \sin 2x, & x \le 0 \\ mx, & x > 0 \end{cases}$$

(a) continuous at $x = 0$? Explain.

(b) differentiable at $x = 0$? Explain.

In Exercises 55–58, determine where the function is **(a)** differentiable, **(b)** continuous but not differentiable, and **(c)** neither continuous nor differentiable.

55. $f(x) = x^{4/5}$

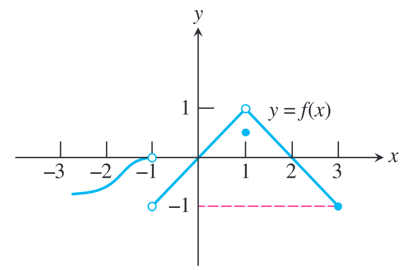
56. $g(x) = \sin(x^2 + 1)$

57. $f(x) = \begin{cases} 2x - 3, & -1 \le x < 0 \\ x - 3, & 0 \le x \le 4 \end{cases}$

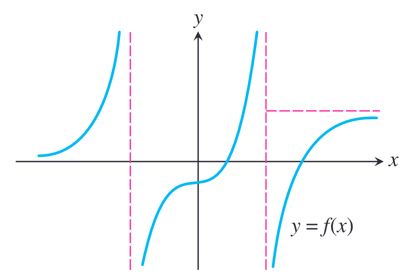
58. $g(x) = \begin{cases} \dfrac{x - 1}{x}, & -2 \le x < 0 \\ \dfrac{x + 1}{x}, & 0 \le x \le 2 \end{cases}$

In Exercises 59 and 60, use the graph of f to sketch the graph of f'.

59. Sketching f' from f

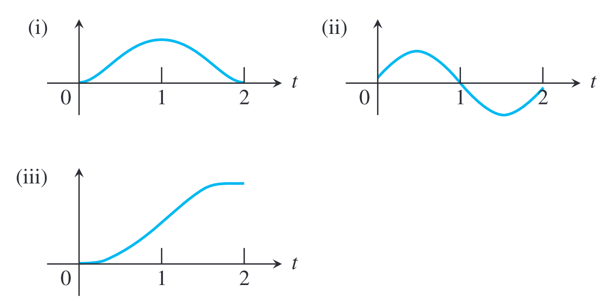


60. Sketching f' from f



61. Recognizing Graphs The following graphs show the distance traveled, velocity, and acceleration for each second of a 2-minute automobile trip. Which graph shows

(a) distance ? **(b)** velocity? **(c)** acceleration?



62. Sketching f from f' Sketch the graph of a continuous function f with $f(0) = 5$ and

$$f'(x) = \begin{cases} -2, & x < 2 \\ -0.5, & x > 2. \end{cases}$$

63. Sketching f from f' Sketch the graph of a continuous function f with $f(-1) = 2$ and

$$f'(x) = \begin{cases} -2, & x < 1 \\ 1, & 1 < x < 4 \\ -1, & 4 < x < 6. \end{cases}$$

64. Which of the following statements could be true if $f''(x) = x^{1/3}$?

i. $f(x) = \dfrac{9}{28}x^{7/3} + 9$ **ii.** $f'(x) = \dfrac{9}{28}x^{7/3} - 2$

iii. $f'(x) = \dfrac{3}{4}x^{4/3} + 6$ **iv.** $f(x) = \dfrac{3}{4}x^{4/3} - 4$

A. i only **B.** iii only

C. ii and iv only **D.** i and iii only

65. ***Derivative from Data*** The following data give the coordinates of a moving body for various values of t.

t (sec)	0	0.5	1	1.5	2	2.5	3	3.5	4
s (ft)	10	38	58	70	74	70	58	38	10

(a) Make a scatter plot of the (t, s) data and sketch a smooth curve through the points.

(b) Compute the average velocity between consecutive points of the table.

(c) Make a scatter plot of the data in part (b) using the midpoints of the t values to represent the data. Then sketch a smooth curve through the points.

(d) **Writing to Learn** Why does the curve in part (c) approximate the graph of ds/dt?

66. ***Working with Numerical Values*** Suppose that a function f and its first derivative have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$f'(x)$
0	9	−2
1	−3	1/5

Find the first derivative of the following combinations at the given value of x.

(a) $\sqrt{x}f(x)$, $x = 1$ **(b)** $\sqrt{f(x)}$, $x = 0$

(c) $f(\sqrt{x})$, $x = 1$ **(d)** $f(1 - 5\tan x)$, $x = 0$

(e) $\dfrac{f(x)}{2 + \cos x}$, $x = 0$ **(f)** $10\sin\left(\dfrac{\pi x}{2}\right)f^2(x)$, $x = 1$

67. ***Working with Numerical Values*** Suppose that functions f and g and their first derivatives have the following values at $x = -1$ and $x = 0$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
−1	0	−1	2	1
0	−1	−3	−2	4

Find the first derivative of the following combinations at the given value of x.

(a) $3f(x) - g(x)$, $x = -1$ **(b)** $f^2(x)g^3(x)$, $x = 0$

(c) $g(f(x))$, $x = -1$ **(d)** $f(g(x))$, $x = -1$

(e) $\dfrac{f(x)}{g(x) + 2}$, $x = 0$ **(f)** $g(x + f(x))$, $x = 0$

68. Find the value of dw/ds at $s = 0$ if $w = \sin(\sqrt{r} - 2)$ and $r = 8\sin(s + \pi/6)$.

69. Find the value of dr/dt at $t = 0$ if $r = (\theta^2 + 7)^{1/3}$ and $\theta^2 t + \theta = 1$.

70. ***Particle Motion*** The position at time $t \geq 0$ of a particle moving along the s-axis is

$$s(t) = 10\cos(t + \pi/4).$$

(a) Give parametric equations that can be used to simulate the motion of the particle.

(b) What is the particle's initial position $(t = 0)$?

(c) What points reached by the particle are farthest to the left and right of the origin?

(d) When does the particle first reach the origin? What are its velocity, speed, and acceleration then?

71. ***Vertical Motion*** On Earth, if you shoot a paper clip 64 ft straight up into the air with a rubber band, the paper clip will be $s(t) = 64t - 16t^2$ feet above your hand at t sec after firing.

(a) Find ds/dt and d^2s/dt^2.

(b) How long does it take the paper clip to reach its maximum height?

(c) With what velocity does it leave your hand?

(d) On the moon, the same force will send the paper clip to a height of $s(t) = 64t - 2.6t^2$ ft in t sec. About how long will it take the paper clip to reach its maximum height, and how high will it go?

72. ***Free Fall*** Suppose two balls are falling from rest at a certain height in centimeters above the ground. Use the equation $s = 490t^2$ to answer the following questions.

(a) How long does it take the balls to fall the first 160 cm? What is their average velocity for the period?

(b) How fast are the balls falling when they reach the 160-cm mark? What is their acceleration then?

73. ***Filling a Bowl*** If a hemispherical bowl of radius 10 in. is filled with water to a depth of x in., the volume of water is given by $V = \pi[10 - (x/3)]x^2$. Find the rate of increase of the volume per inch increase of depth.

74. ***Marginal Revenue*** A bus will hold 60 people. The fare charged (p dollars) is related to the number x of people who use the bus by the formula $p = [3 - (x/40)]^2$.

(a) Write a formula for the total revenue per trip received by the bus company.

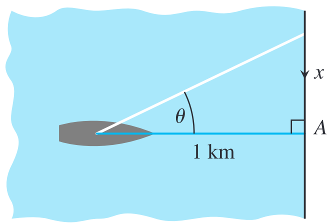
(b) What number of people per trip will make the marginal revenue equal to zero? What is the corresponding fare?

(c) **Writing to Learn** Do you think the bus company's fare policy is good for its business?

75. ***Searchlight*** The figure shows a boat 1 km offshore sweeping the shore with a searchlight. The light turns at a constant rate, $d\theta/dt = -0.6$ rad/sec.

(a) How fast is the light moving along the shore when it reaches point A?

(b) How many revolutions per minute is 0.6 rad/sec?



76. ***Horizontal Tangents*** The graph of $y = \sin(x - \sin x)$ appears to have horizontal tangents at the x-axis. Does it?

77. ***Fundamental Frequency of a Vibrating Piano String*** We measure the frequencies at which wires vibrate in cycles (trips back and forth) per sec. The unit of measure is a *hertz:* 1 cycle per sec. Middle A on a piano has a frequency 440 hertz. For any given wire, the fundamental frequency y is a function of four variables:

r: the radius of the wire;

l: the length;

d: the density of the wire;

T: the tension (force) holding the wire taut.

With r and l in centimeters, d in grams per cubic centimeter, and T in dynes (it takes about 100,000 dynes to lift an apple), the fundamental frequency of the wire is

$$y = \frac{1}{2rl}\sqrt{\frac{T}{\pi d}}.$$

If we keep all the variables fixed except one, then y can be alternatively thought of as four different functions of one variable, $y(r)$, $y(l)$, $y(d)$, and $y(T)$. How would changing each variable affect the string's fundamental frequency? To find out, calculate $y'(r)$, $y'(l)$, $y'(d)$, and $y'(T)$.

78. ***Spread of Measles*** The spread of measles in a certain school is given by

$$P(t) = \frac{200}{1 + e^{5-t}},$$

where t is the number of days since the measles first appeared, and $P(t)$ is the total number of students who have caught the measles to date.

(a) Estimate the initial number of students infected with measles.

(b) About how many students in all will get the measles?

(c) When will the rate of spread of measles be greatest? What is this rate?

79. Graph the function $f(x) = \tan^{-1}(\tan 2x)$ in the window $[-\pi, \pi]$ by $[-4, 4]$. Then answer the following questions.

(a) What is the domain of f?

(b) What is the range of f?

(c) At which points is f not differentiable?

(d) Describe the graph of f'.

80. If $x^2 - y^2 = 1$, find d^2y/dx^2 at the point $(2, \sqrt{3})$.

AP* Examination Preparation

You may use a graphing calculator to solve the following problems.

81. A particle moves along the x-axis so that at any time $t \geq 0$ its position is given by $x(t) = t^3 - 12t + 5$.

(a) Find the velocity of the particle at any time t.

(b) Find the acceleration of the particle at any time t.

(c) Find all values of t for which the particle is at rest.

(d) Find the speed of the particle when its acceleration is zero.

(e) Is the particle moving toward the origin or away from the origin when $t = 3$? Justify your answer.

82. Let $y = \dfrac{e^x + e^{-x}}{2}$.

(a) Find $\dfrac{dy}{dx}$.

(b) Find $\dfrac{d^2y}{dx^2}$.

(c) Find an equation of the line tangent to the curve at $x = 1$.

(d) Find an equation of the line normal to the curve at $x = 1$.

(e) Find any points where the tangent line is horizontal.

83. Let $f(x) = \ln(1 - x^2)$.

(a) State the domain of f.

(b) Find $f'(x)$.

(c) State the domain of f'.

(d) Prove that $f''(x) < 0$ for all x in the domain of f.

Chapter 4 Applications of Derivatives

An automobile's gas mileage is a function of many variables, including road surface, tire type, velocity, fuel octane rating, road angle, and the speed and direction of the wind. If we look only at velocity's effect on gas mileage, the mileage of a certain car can be approximated by:

$$m(v) = 0.00015v^3 - 0.032v^2 + 1.8v + 1.7$$

(where v is velocity)

At what speed should you drive this car to obtain the best gas mileage? The ideas in Section 4.1 will help you find the answer.

Chapter 4 Overview

In the past, when virtually all graphing was done by hand—often laboriously—derivatives were the key tool used to sketch the graph of a function. Now we can graph a function quickly, and usually correctly, using a grapher. However, confirmation of much of what we see and conclude true from a grapher view must still come from calculus.

This chapter shows how to draw conclusions from derivatives about the extreme values of a function and about the general shape of a function's graph. We will also see how a tangent line captures the shape of a curve near the point of tangency, how to deduce rates of change we cannot measure from rates of change we already know, and how to find a function when we know only its first derivative and its value at a single point. The key to recovering functions from derivatives is the Mean Value Theorem, a theorem whose corollaries provide the gateway to *integral calculus,* which we begin in Chapter 5.

4.1 Extreme Values of Functions

What you'll learn about

- Absolute (Global) Extreme Values
- Local (Relative) Extreme Values
- Finding Extreme Values

. . . and why

Finding maximum and minimum values of functions, called optimization, is an important issue in real-world problems.

Absolute (Global) Extreme Values

One of the most useful things we can learn from a function's derivative is whether the function assumes any maximum or minimum values on a given interval and where these values are located if it does. Once we know how to find a function's extreme values, we will be able to answer such questions as "What is the most effective size for a dose of medicine?" and "What is the least expensive way to pipe oil from an offshore well to a refinery down the coast?" We will see how to answer questions like these in Section 4.4.

DEFINITION Absolute Extreme Values

Let f be a function with domain D. Then $f(c)$ is the

(a) **absolute maximum value** on D if and only if $f(x) \le f(c)$ for all x in D.

(b) **absolute minimum value** on D if and only if $f(x) \ge f(c)$ for all x in D.

Absolute (or **global**) maximum and minimum values are also called **absolute extrema** (plural of the Latin *extremum*). We often omit the term "absolute" or "global" and just say maximum and minimum.

Example 1 shows that extreme values can occur at interior points or endpoints of intervals.

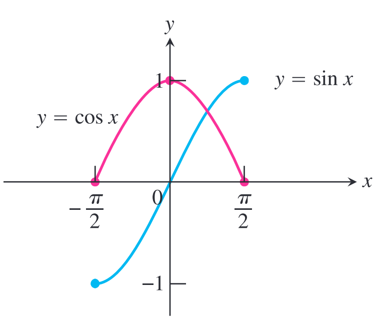


Figure 4.1 (Example 1)

EXAMPLE 1 Exploring Extreme Values

On $[-\pi/2, \pi/2]$, $f(x) = \cos x$ takes on a maximum value of 1 (once) and a minimum value of 0 (twice). The function $g(x) = \sin x$ takes on a maximum value of 1 and a minimum value of -1 (Figure 4.1). ***Now try Exercise 1.***

Functions with the same defining rule can have different extrema, depending on the domain.

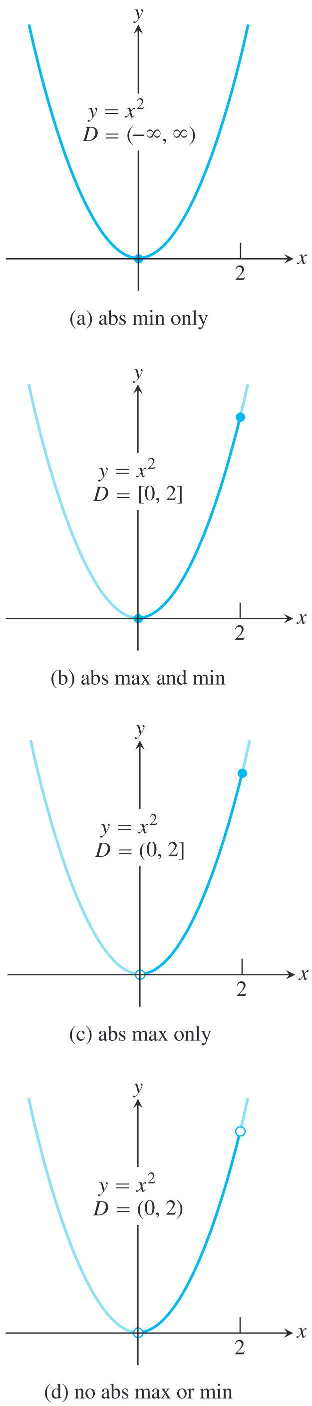


Figure 4.2 (Example 2)

EXAMPLE 2 Exploring Absolute Extrema

The absolute extrema of the following functions on their domains can be seen in Figure 4.2.

	Function Rule	Domain D	Absolute Extrema on D
(a)	$y = x^2$	$(-\infty, \infty)$	No absolute maximum. Absolute minimum of 0 at $x = 0$.
(b)	$y = x^2$	$[0, 2]$	Absolute maximum of 4 at $x = 2$. Absolute minimum of 0 at $x = 0$.
(c)	$y = x^2$	$(0, 2]$	Absolute maximum of 4 at $x = 2$. No absolute minimum.
(d)	$y = x^2$	$(0, 2)$	No absolute extrema.

Now try Exercise 3.

Example 2 shows that a function may fail to have a maximum or minimum value. This cannot happen with a continuous function on a finite closed interval.

THEOREM 1 The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f has both a maximum value and a minimum value on the interval. (Figure 4.3)

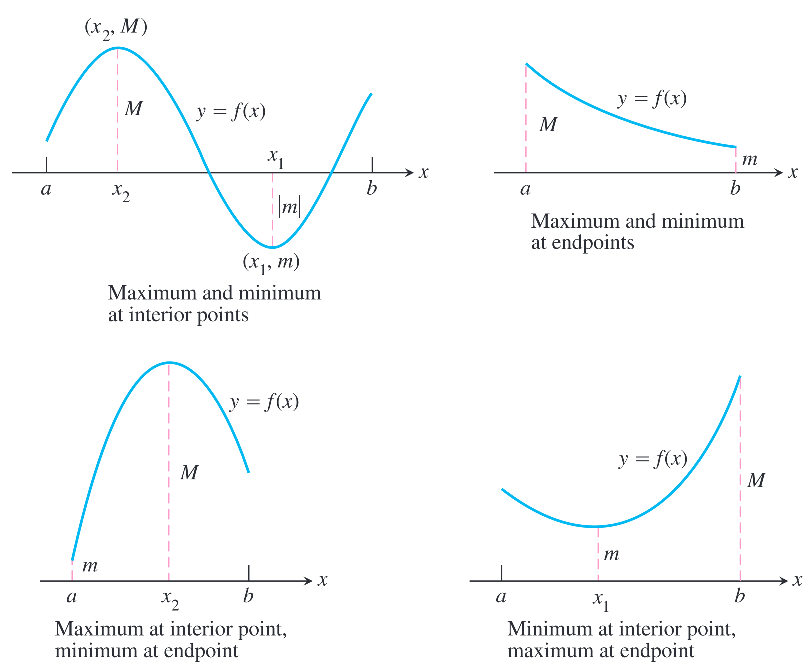


Figure 4.3 Some possibilities for a continuous function's maximum (M) and minimum (m) on a closed interval $[a, b]$.

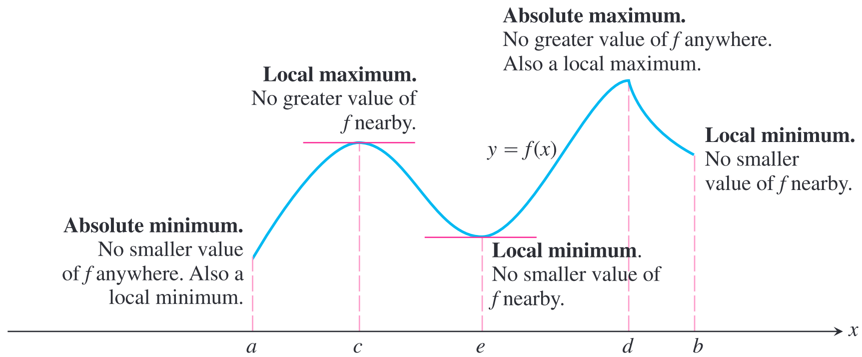


Figure 4.4 Classifying extreme values.

Local (Relative) Extreme Values

Figure 4.4 shows a graph with five points where a function has extreme values on its domain $[a, b]$. The function's absolute minimum occurs at a even though at e the function's value is smaller than at any other point *nearby.* The curve rises to the left and falls to the right around c, making $f(c)$ a maximum locally. The function attains its absolute maximum at d.

DEFINITION **Local Extreme Values**

Let c be an interior point of the domain of the function f. Then $f(c)$ is a

(a) **local maximum value** at c if and only if $f(x) \le f(c)$ for all x in some open interval containing c.

(b) **local minimum value** at c if and only if $f(x) \ge f(c)$ for all x in some open interval containing c.

A function f has a local maximum or local minimum *at an endpoint* c if the appropriate inequality holds for all x in some half-open domain interval containing c.

Local extrema are also called **relative extrema.**

An **absolute extremum** is also a local extremum, because being an extreme value overall makes it an extreme value in its immediate neighborhood. Hence, *a list of local extrema will automatically include absolute extrema if there are any.*

Finding Extreme Values

The interior domain points where the function in Figure 4.4 has local extreme values are points where either f' is zero or f' does not exist. This is generally the case, as we see from the following theorem.

THEOREM 2 **Local Extreme Values**

If a function f has a local maximum value or a local minimum value at an interior point c of its domain, and if f' exists at c, then

$$f'(c) = 0.$$

Because of Theorem 2, we usually need to look at only a few points to find a function's extrema. These consist of the interior domain points where $f' = 0$ or f' does not exist (the domain points covered by the theorem) and the domain endpoints (the domain points not covered by the theorem). At all other domain points, $f' > 0$ or $f' < 0$.

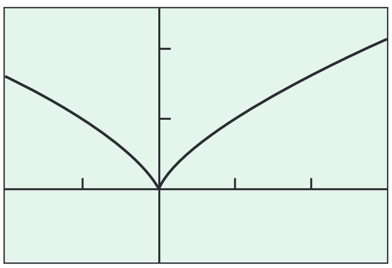
The following definition helps us summarize these findings.

DEFINITION Critical Point

A point in the interior of the domain of a function f at which $f' = 0$ or f' does not exist is a **critical point** of f.

Thus, in summary, extreme values occur only at critical points and endpoints.

$y = x^{2/3}$



[–2, 3] by [–1, 2.5]

Figure 4.5 (Example 3)

EXAMPLE 3 Finding Absolute Extrema

Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on the interval $[-2, 3]$.

SOLUTION

Solve Graphically Figure 4.5 suggests that f has an absolute maximum value of about 2 at $x = 3$ and an absolute minimum value of 0 at $x = 0$.

Confirm Analytically We evaluate the function at the critical points and endpoints and take the largest and smallest of the resulting values.

The first derivative

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

has no zeros but is undefined at $x = 0$. The values of f at this one critical point and at the endpoints are

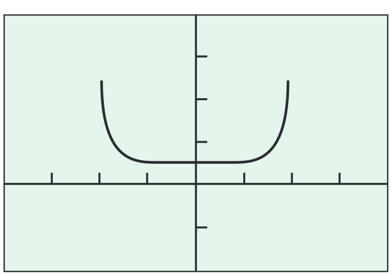
Critical point value: $f(0) = 0$;

Endpoint values: $f(-2) = (-2)^{2/3} = \sqrt[3]{4}$;

$f(3) = (3)^{2/3} = \sqrt[3]{9}$.

We can see from this list that the function's absolute maximum value is $\sqrt[3]{9} \approx 2.08$, and occurs at the right endpoint $x = 3$. The absolute minimum value is 0, and occurs at the interior point $x = 0$. ***Now try Exercise 11.***

In Example 4, we investigate the reciprocal of the function whose graph was drawn in Example 3 of Section 1.2 to illustrate "grapher failure."



[–4, 4] by [–2, 4]

Figure 4.6 The graph of

$$f(x) = \frac{1}{\sqrt{4 - x^2}}.$$

(Example 4)

EXAMPLE 4 Finding Extreme Values

Find the extreme values of $f(x) = \dfrac{1}{\sqrt{4 - x^2}}$.

SOLUTION

Solve Graphically Figure 4.6 suggests that f has an absolute minimum of about 0.5 at $x = 0$. There also appear to be local maxima at $x = -2$ and $x = 2$. However, f is not defined at these points and there do not appear to be maxima anywhere else.

continued

Confirm Analytically The function f is defined only for $4 - x^2 > 0$, so its domain is the open interval $(-2, 2)$. The domain has no endpoints, so all the extreme values must occur at critical points. We rewrite the formula for f to find f':

$$f(x) = \frac{1}{\sqrt{4 - x^2}} = (4 - x^2)^{-1/2}.$$

Thus,

$$f'(x) = -\frac{1}{2}(4 - x^2)^{-3/2}(-2x) = \frac{x}{(4 - x^2)^{3/2}}.$$

The only critical point in the domain $(-2, 2)$ is $x = 0$. The value

$$f(0) = \frac{1}{\sqrt{4 - 0^2}} = \frac{1}{2}$$

is therefore the sole candidate for an extreme value.

To determine whether $1/2$ is an extreme value of f, we examine the formula

$$f(x) = \frac{1}{\sqrt{4 - x^2}}.$$

As x moves away from 0 on either side, the denominator gets smaller, the values of f increase, and the graph rises. We have a minimum value at $x = 0$, and the minimum is absolute.

The function has no maxima, either local or absolute. This does not violate Theorem 1 (The Extreme Value Theorem) because here f is defined on an *open* interval. To invoke Theorem 1's guarantee of extreme points, the interval must be closed.

Now try Exercise 25.

While a function's extrema can occur only at critical points and endpoints, not every critical point or endpoint signals the presence of an extreme value. Figure 4.7 illustrates this for interior points. Exercise 55 describes a function that fails to assume an extreme value at an endpoint of its domain.

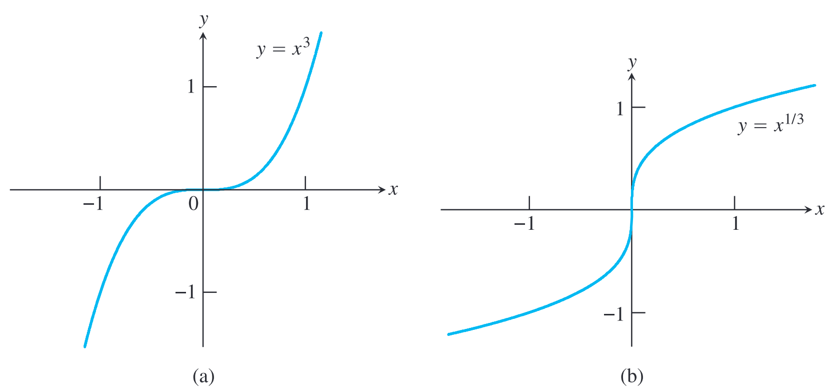


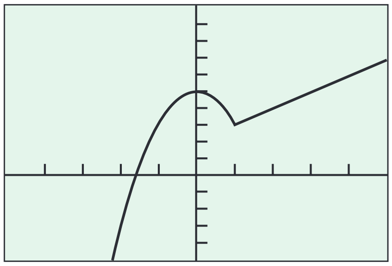
Figure 4.7 Critical points without extreme values. (a) $y' = 3x^2$ is 0 at $x = 0$, but $y = x^3$ has no extremum there. (b) $y' = (1/3)x^{-2/3}$ is undefined at $x = 0$, but $y = x^{1/3}$ has no extremum there.

EXAMPLE 5 Finding Extreme Values

Find the extreme values of

$$f(x) = \begin{cases} 5 - 2x^2, & x \le 1 \\ x + 2, & x > 1. \end{cases}$$

continued



[–5, 5] by [–5, 10]

Figure 4.8 The function in Example 5.

SOLUTION

Solve Graphically The graph in Figure 4.8 suggests that $f'(0) = 0$ and that $f'(1)$ does not exist. There appears to be a local maximum value of 5 at $x = 0$ and a local minimum value of 3 at $x = 1$.

Confirm Analytically For $x \neq 1$, the derivative is

$$f'(x) = \begin{cases} \dfrac{d}{dx}(5 - 2x^2) = -4x, & x < 1 \\ \dfrac{d}{dx}(x + 2) = 1, & x > 1. \end{cases}$$

The only point where $f' = 0$ is $x = 0$. What happens at $x = 1$?

At $x = 1$, the right- and left-hand derivatives are respectively

$$\lim_{h \to 0^+} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0^+} \frac{(1 + h) + 2 - 3}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1,$$

$$\lim_{h \to 0^-} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0^-} \frac{5 - 2(1 + h)^2 - 3}{h}$$

$$= \lim_{h \to 0^-} \frac{-2h(2 + h)}{h} = -4.$$

Since these one-sided derivatives differ, f has no derivative at $x = 1$, and 1 is a second critical point of f.

The domain $(-\infty, \infty)$ has no endpoints, so the only values of f that might be local extrema are those at the critical points:

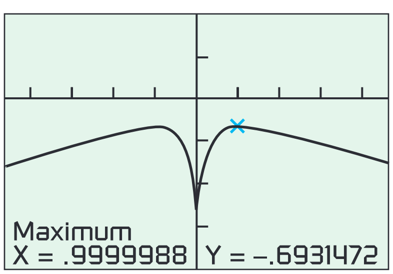
$$f(0) = 5 \quad \text{and} \quad f(1) = 3.$$

From the formula for f, we see that the values of f immediately to either side of $x = 0$ are less than 5, so 5 is a local maximum. Similarly, the values of f immediately to either side of $x = 1$ are greater than 3, so 3 is a local minimum. ***Now try Exercise 41.***

Most graphing calculators have built-in methods to find the coordinates of points where extreme values occur. We must, of course, be sure that we use correct graphs to find these values. The calculus that you learn in this chapter should make you feel more confident about working with graphs.

EXAMPLE 6 Using Graphical Methods

Find the extreme values of $f(x) = \ln \left| \dfrac{x}{1 + x^2} \right|$.



[–4.5, 4.5] by [–4, 2]

Figure 4.9 The function in Example 6.

SOLUTION

Solve Graphically The domain of f is the set of all nonzero real numbers. Figure 4.9 suggests that f is an even function with a maximum value at two points. The coordinates found in this window suggest an extreme value of about -0.69 at approximately $x = 1$. Because f is even, there is another extreme of the same value at approximately $x = -1$. The figure also suggests a minimum value at $x = 0$, but f is not defined there.

Confirm Analytically The derivative

$$f'(x) = \frac{1 - x^2}{x(1 + x^2)}$$

is defined at every point of the function's domain. The critical points where $f'(x) = 0$ are $x = 1$ and $x = -1$. The corresponding values of f are both $\ln(1/2) = -\ln 2 \approx -0.69$.

Now try Exercise 37.

EXPLORATION 1 **Finding Extreme Values**

Let $f(x) = \left|\dfrac{x}{x^2 + 1}\right|, \ -2 \le x \le 2.$

1. Determine graphically the extreme values of f and where they occur. Find f' at these values of x.
2. Graph f and f' (or NDER $(f(x), x, x)$) in the same viewing window. Comment on the relationship between the graphs.
3. Find a formula for $f'(x)$.

Quick Review 4.1 *(For help, go to Sections 1.2, 2.1, 3.5, and 3.6.)*

In Exercises 1–4, find the first derivative of the function.

1. $f(x) = \sqrt{4 - x}$

2. $f(x) = \dfrac{2}{\sqrt{9 - x^2}}$

3. $g(x) = \cos(\ln x)$

4. $h(x) = e^{2x}$

In Exercises 5–8, match the table with a graph of $f(x)$.

5.

x	$f'(x)$
a	0
b	0
c	5

6.

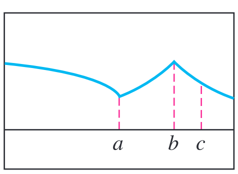
x	$f'(x)$
a	0
b	0
c	-5

7.

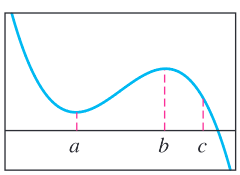
x	$f'(x)$
a	does not exist
b	0
c	-2

8.

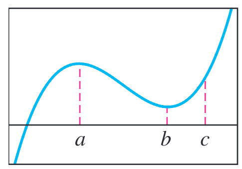
x	$f'(x)$
a	does not exist
b	does not exist
c	-1.7



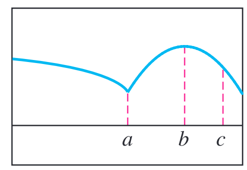
(a)



(b)



(c)



(d)

In Exercises 9 and 10, find the limit for

$$f(x) = \frac{2}{\sqrt{9 - x^2}}.$$

9. $\lim_{x\to 3^-} f(x)$

10. $\lim_{x\to -3^+} f(x)$

In Exercises 11 and 12, let

$$f(x) = \begin{cases} x^3 - 2x, & x \le 2 \\ x + 2, & x > 2. \end{cases}$$

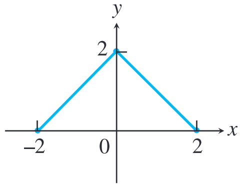
11. Find **(a)** $f'(1)$, **(b)** $f'(3)$, **(c)** $f'(2)$.

12. **(a)** Find the domain of f'.
(b) Write a formula for $f'(x)$.

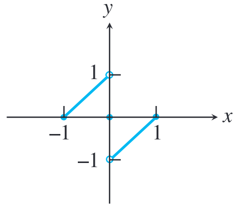
Section 4.1 Exercises

In Exercises 1–4, find the extreme values and where they occur.

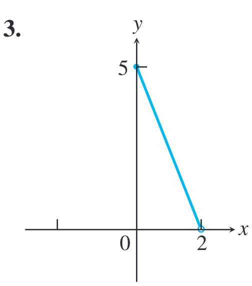
1.



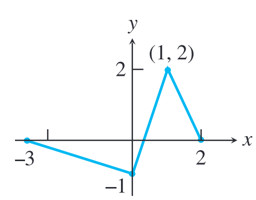
2.



3.

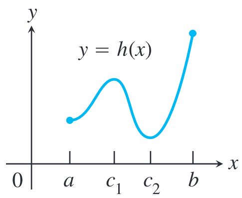


4.

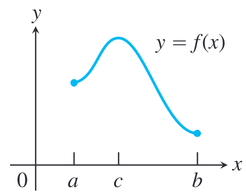


In Exercises 5–10, identify each x-value at which any absolute extreme value occurs. Explain how your answer is consistent with the Extreme Value Theorem.

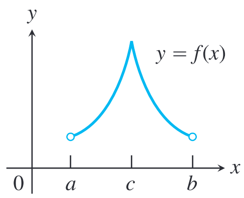
5.



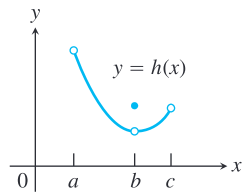
6.



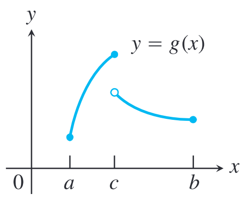
7.



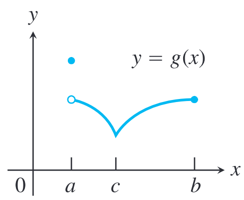
8.



9.



10.



In Exercises 11–18, use analytic methods to find the extreme values of the function on the interval and where they occur.

11. $f(x) = \dfrac{1}{x} + \ln x, \quad 0.5 \le x \le 4$

12. $g(x) = e^{-x}, \quad -1 \le x \le 1$

13. $h(x) = \ln(x + 1), \quad 0 \le x \le 3$

14. $k(x) = e^{-x^2}, \quad -\infty < x < \infty$

15. $f(x) = \sin\left(x + \dfrac{\pi}{4}\right), \quad 0 \le x \le \dfrac{7\pi}{4}$

16. $g(x) = \sec x, \quad -\dfrac{\pi}{2} < x < \dfrac{3\pi}{2}$

17. $f(x) = x^{2/5}, \quad -3 \le x < 1$

18. $f(x) = x^{3/5}, \quad -2 < x \le 3$

In Exercises 19–30, find the extreme values of the function and where they occur.

19. $y = 2x^2 - 8x + 9$

20. $y = x^3 - 2x + 4$

21. $y = x^3 + x^2 - 8x + 5$

22. $y = x^3 - 3x^2 + 3x - 2$

23. $y = \sqrt{x^2 - 1}$

24. $y = \dfrac{1}{x^2 - 1}$

25. $y = \dfrac{1}{\sqrt{1 - x^2}}$

26. $y = \dfrac{1}{\sqrt[3]{1 - x^2}}$

27. $y = \sqrt{3 + 2x - x^2}$

28. $y = \dfrac{3}{2}x^4 + 4x^3 - 9x^2 + 10$

29. $y = \dfrac{x}{x^2 + 1}$

30. $y = \dfrac{x + 1}{x^2 + 2x + 2}$

Group Activity In Exercises 31–34, find the extreme values of the function on the interval and where they occur.

31. $f(x) = |x - 2| + |x + 3|, \quad -5 \le x \le 5$

32. $g(x) = |x - 1| - |x - 5|, \quad -2 \le x \le 7$

33. $h(x) = |x + 2| - |x - 3|, \quad -\infty < x < \infty$

34. $k(x) = |x + 1| + |x - 3|, \quad -\infty < x < \infty$

In Exercises 35–42, identify the critical point and determine the local extreme values.

35. $y = x^{2/3}(x + 2)$

36. $y = x^{2/3}(x^2 - 4)$

37. $y = x\sqrt{4 - x^2}$

38. $y = x^2\sqrt{3 - x}$

39. $y = \begin{cases} 4 - 2x, & x \le 1 \\ x + 1, & x > 1 \end{cases}$

40. $y = \begin{cases} 3 - x, & x < 0 \\ 3 + 2x - x^2, & x \ge 0 \end{cases}$

41. $y = \begin{cases} -x^2 - 2x + 4, & x \le 1 \\ -x^2 + 6x - 4, & x > 1 \end{cases}$

42. $y = \begin{cases} -\dfrac{1}{4}x^2 - \dfrac{1}{2}x + \dfrac{15}{4}, & x \le 1 \\ x^3 - 6x^2 + 8x, & x > 1 \end{cases}$

43. Writing to Learn The function

$$V(x) = x(10 - 2x)(16 - 2x), \quad 0 < x < 5,$$

models the volume of a box.

(a) Find the extreme values of V.

(b) Interpret any values found in (a) in terms of volume of the box.

44. Writing to Learn The function

$$P(x) = 2x + \frac{200}{x}, \quad 0 < x < \infty,$$

models the perimeter of a rectangle of dimensions x by $100/x$.

(a) Find any extreme values of P.

(b) Give an interpretation in terms of perimeter of the rectangle for any values found in (a).

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

45. True or False If $f(c)$ is a local maximum of a continuous function f on an open interval (a, b), then $f'(c) = 0$. Justify your answer.

46. True or False If m is a local minimum and M is a local maximum of a continuous function f on (a, b), then $m < M$. Justify your answer.

47. Multiple Choice Which of the following values is the absolute maximum of the function $f(x) = 4x - x^2 + 6$ on the interval $[0, 4]$?

(A) 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** 10

48. **Multiple Choice** If f is a continuous, decreasing function on [0, 10] with a critical point at (4, 2), which of the following statements *must be false*?

(A) $f(10)$ is an absolute minimum of f on [0, 10].

(B) $f(4)$ is neither a relative maximum nor a relative minimum.

(C) $f'(4)$ does not exist.

(D) $f'(4) = 0$

(E) $f'(4) < 0$

49. **Multiple Choice** Which of the following functions has exactly two local extrema on its domain?

(A) $f(x) = |x - 2|$

(B) $f(x) = x^3 - 6x + 5$

(C) $f(x) = x^3 + 6x - 5$

(D) $f(x) = \tan x$

(E) $f(x) = x + \ln x$

50. **Multiple Choice** If an even function f with domain all real numbers has a local maximum at $x = a$, then $f(-a)$

(A) is a local minimum.

(B) is a local maximum.

(C) is both a local minimum and a local maximum.

(D) could be either a local minimum or a local maximum.

(E) is neither a local minimum nor a local maximum.

Explorations

In Exercises 51 and 52, give reasons for your answers.

51. **Writing to Learn** Let $f(x) = (x - 2)^{2/3}$.

(a) Does $f'(2)$ exist?

(b) Show that the only local extreme value of f occurs at $x = 2$.

(c) Does the result in (b) contradict the Extreme Value Theorem?

(d) Repeat parts (a) and (b) for $f(x) = (x - a)^{2/3}$, replacing 2 by a.

52. **Writing to Learn** Let $f(x) = |x^3 - 9x|$.

(a) Does $f'(0)$ exist?

(b) Does $f'(3)$ exist?

(c) Does $f'(-3)$ exist?

(d) Determine all extrema of f.

Extending the Ideas

53. ***Cubic Functions*** Consider the cubic function

$$f(x) = ax^3 + bx^2 + cx + d.$$

(a) Show that f can have 0, 1, or 2 critical points. Give examples and graphs to support your argument.

(b) How many local extreme values can f have?

54. ***Proving Theorem 2*** Assume that the function f has a local maximum value at the interior point c of its domain and that $f'(c)$ exists.

(a) Show that there is an open interval containing c such that $f(x) - f(c) \le 0$ for all x in the open interval.

(b) **Writing to Learn** Now explain why we may say

$$\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \le 0.$$

(c) **Writing to Learn** Now explain why we may say

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \ge 0.$$

(d) **Writing to Learn** Explain how parts (b) and (c) allow us to conclude $f'(c) = 0$.

(e) **Writing to Learn** Give a similar argument if f has a local minimum value at an interior point.

55. ***Functions with No Extreme Values at Endpoints***

(a) Graph the function

$$f(x) = \begin{cases} \sin \dfrac{1}{x}, & x > 0 \\ 0, & x = 0. \end{cases}$$

Explain why $f(0) = 0$ is not a local extreme value of f.

(b) **Group Activity** Construct a function of your own that fails to have an extreme value at a domain endpoint.

4.2 Mean Value Theorem

What you'll learn about

- Mean Value Theorem
- Physical Interpretation
- Increasing and Decreasing Functions
- Other Consequences

. . . and why

The Mean Value Theorem is an important theoretical tool to connect the average and instantaneous rates of change.

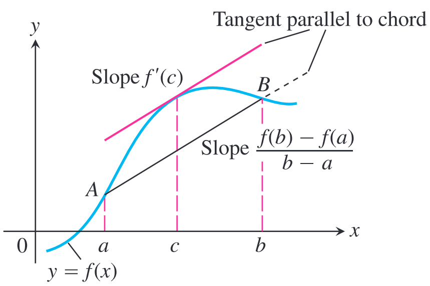


Figure 4.10 Figure for the Mean Value Theorem.

Mean Value Theorem

The Mean Value Theorem connects the average rate of change of a function over an interval with the instantaneous rate of change of the function at a point within the interval. Its powerful corollaries lie at the heart of some of the most important applications of the calculus.

The theorem says that somewhere between points A and B on a differentiable curve, there is at least one tangent line parallel to chord AB (Figure 4.10).

THEOREM 3 Mean Value Theorem for Derivatives

If $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b), then there is at least one point c in (a, b) at which

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The hypotheses of Theorem 3 cannot be relaxed. If they fail at even one point, the graph may fail to have a tangent parallel to the chord. For instance, the function $f(x) = |x|$ is continuous on $[-1, 1]$ and differentiable at every point of the interior $(-1, 1)$ except $x = 0$. The graph has no tangent parallel to chord AB (Figure 4.11a). The function $g(x) = \text{int}\,(x)$ is differentiable at every point of $(1, 2)$ and continuous at every point of $[1, 2]$ except $x = 2$. Again, the graph has no tangent parallel to chord AB (Figure 4.11b).

The Mean Value Theorem is an *existence theorem.* It tells us the number c exists without telling how to find it. We can sometimes satisfy our curiosity about the value of c but the real importance of the theorem lies in the surprising conclusions we can draw from it.

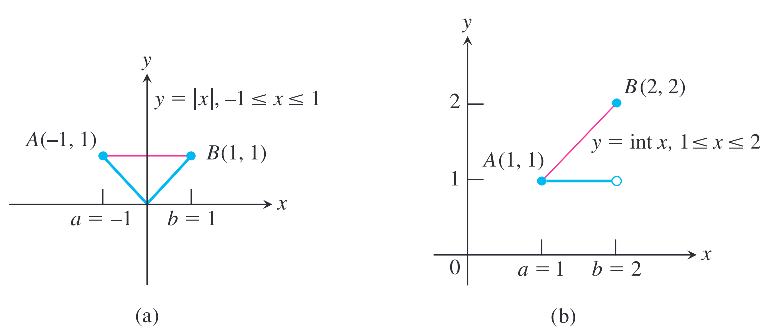
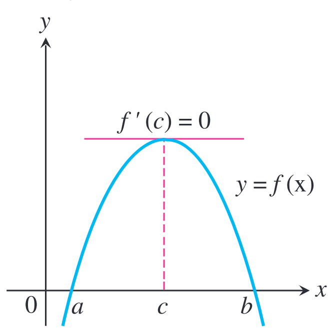


Figure 4.11 No tangent parallel to chord AB.

Rolle's Theorem

The first version of the Mean Value Theorem was proved by French mathematician Michel Rolle (1652–1719). His version had $f(a) = f(b) = 0$ and was proved only for polynomials, using algebra and geometry.



Rolle distrusted calculus and spent most of his life denouncing it. It is ironic that he is known today only for an unintended contribution to a field he tried to suppress.

EXAMPLE 1 Exploring the Mean Value Theorem

Show that the function $f(x) = x^2$ satisfies the hypotheses of the Mean Value Theorem on the interval $[0, 2]$. Then find a solution c to the equation

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

on this interval.

continued

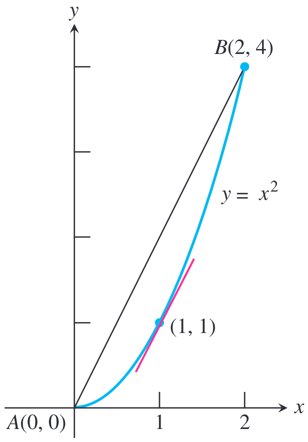


Figure 4.12 (Example 1)

SOLUTION

The function $f(x) = x^2$ is continuous on $[0, 2]$ and differentiable on $(0, 2)$. Since $f(0) = 0$ and $f(2) = 4$, the Mean Value Theorem guarantees a point c in the interval $(0, 2)$ for which

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$2c = \frac{f(2) - f(0)}{2 - 0} = 2$$

$$c = 1.$$

Interpret The tangent line to $f(x) = x^2$ at $x = 1$ has slope 2 and is parallel to the chord joining $A(0, 0)$ and $B(2, 4)$ (Figure 4.12).

Now try Exercise 1.

EXAMPLE 2 Exploring the Mean Value Theorem

Explain why each of the following functions fails to satisfy the conditions of the Mean Value Theorem on the interval [–1, 1].

(a) $f(x) = \sqrt{x^2} + 1$ **(b)** $f(x) = \begin{cases} x^3 + 3 & \text{for } x < 1 \\ x^2 + 1 & \text{for } x \geq 1 \end{cases}$

SOLUTION

(a) Note that $\sqrt{x^2} + 1 = |x| + 1$, so this is just a vertical shift of the absolute value function, which has a nondifferentiable "corner" at $x = 0$. (See Section 3.2.) The function f is not differentiable on (–1, 1).

(b) Since $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^3 + 3 = 4$ and $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x^2 + 1 = 2$, the function has a discontinuity at $x = 1$. The function f is not continuous on [–1, 1].

If the two functions given had satisfied the necessary conditions, the *conclusion* of the Mean Value Theorem would have guaranteed the existence of a number c in (– 1, 1) such that $f'(c) = \dfrac{f(1) - f(-1)}{1 - (-1)} = 0$. Such a number c does not exist for the function in part (a), but one happens to exist for the function in part (b) (Figure 4.13).

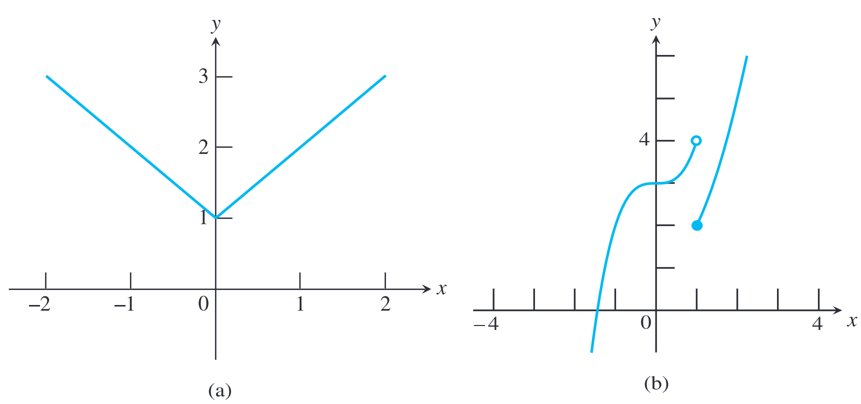


Figure 4.13 For both functions in Example 2, $\dfrac{f(1) - f(-1)}{1 - (-1)} = 0$ but neither function satisfies the conditions of the Mean Value Theorem on the interval [– 1, 1]. For the function in Example 2(a), there is no number c such that $f'(c) = 0$. It happens that $f'(0) = 0$ in Example 2(b).

Now try Exercise 3.

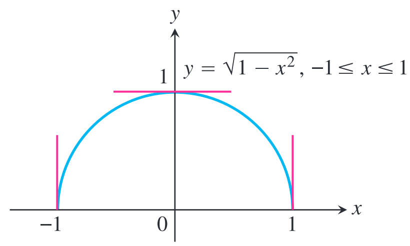


Figure 4.14 (Example 3)

EXAMPLE 3 Applying the Mean Value Theorem

Let $f(x) = \sqrt{1 - x^2}$, $A = (-1, f(-1))$, and $B = (1, f(1))$. Find a tangent to f in the interval $(-1, 1)$ that is parallel to the secant AB.

SOLUTION

The function f (Figure 4.14) is continuous on the interval [–1, 1] and

$$f'(x) = \frac{-x}{\sqrt{1 - x^2}}$$

is defined on the interval $(-1, 1)$. The function is not differentiable at $x = -1$ and $x = 1$, but it does not need to be for the theorem to apply. Since $f(-1) = f(1) = 0$, the tangent we are looking for is horizontal. We find that $f' = 0$ at $x = 0$, where the graph has the horizontal tangent $y = 1$. *Now try Exercise 9.*

Physical Interpretation

If we think of the difference quotient $(f(b) - f(a))/(b - a)$ as the average change in f over $[a, b]$ and $f'(c)$ as an instantaneous change, then the Mean Value Theorem says that the instantaneous change at some interior point must equal the average change over the entire interval.

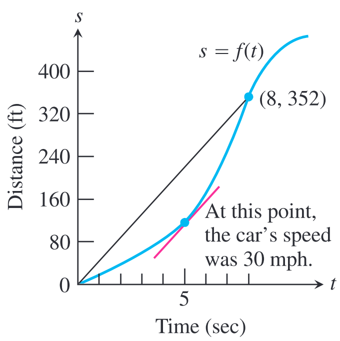


Figure 4.15 (Example 4)

EXAMPLE 4 Interpreting the Mean Value Theorem

If a car accelerating from zero takes 8 sec to go 352 ft, its average velocity for the 8-sec interval is $352/8 = 44$ ft/sec, or 30 mph. At some point during the acceleration, the theorem says, the speedometer must read exactly 30 mph (Figure 4.15).

Now try Exercise 11.

Increasing and Decreasing Functions

Our first use of the Mean Value Theorem will be its application to increasing and decreasing functions.

DEFINITIONS Increasing Function, Decreasing Function

Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1. f **increases** on I if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.

2. f **decreases** on I if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

Monotonic Functions

A function that is always increasing on an interval or always decreasing on an interval is said to be **monotonic** there.

The Mean Value Theorem allows us to identify exactly where graphs rise and fall. Functions with positive derivatives are increasing functions; functions with negative derivatives are decreasing functions.

COROLLARY 1 Increasing and Decreasing Functions

Let f be continuous on $[a, b]$ and differentiable on (a, b).

1. If $f' > 0$ at each point of (a, b), then f increases on $[a, b]$.

2. If $f' < 0$ at each point of (a, b), then f decreases on $[a, b]$.

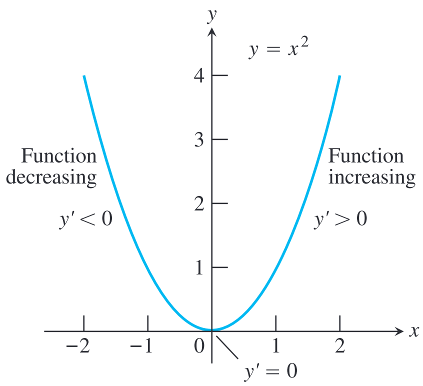


Figure 4.16 (Example 5)

Proof Let x_1 and x_2 be any two points in $[a, b]$ with $x_1 < x_2$. The Mean Value Theorem applied to f on $[x_1, x_2]$ gives

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

for some c between x_1 and x_2. The sign of the right-hand side of this equation is the same as the sign of $f'(c)$ because $x_2 - x_1$ is positive. Therefore,

(a) $f(x_1) < f(x_2)$ if $f' > 0$ on (a, b) (f is increasing), or

(b) $f(x_1) > f(x_2)$ if $f' < 0$ on (a, b) (f is decreasing). ■

EXAMPLE 5 Determining Where Graphs Rise or Fall

The function $y = x^2$ (Figure 4.16) is

(a) decreasing on $(-\infty, 0]$ because $y' = 2x < 0$ on $(-\infty, 0)$.

(b) increasing on $[0, \infty)$ because $y' = 2x > 0$ on $(0, \infty)$. *Now try Exercise 15.*

What's Happening at Zero?

Note that 0 appears in both intervals in Example 5, which is consistent both with the definition and with Corollary 1. Does this mean that the function $y = x^2$ is both increasing and decreasing at $x = 0$? No! This is because a function can only be described as increasing or decreasing on an *interval* with more than one point (see the definition). Saying that $y = x^2$ is "increasing at $x = 2$" is not really proper either, but you will often see that statement used as a short way of saying $y = x^2$ is "increasing on an interval containing 2."

EXAMPLE 6 Determining Where Graphs Rise or Fall

Where is the function $f(x) = x^3 - 4x$ increasing and where is it decreasing?

SOLUTION

Solve Graphically The graph of f in Figure 4.17 suggests that f is increasing from $-\infty$ to the x-coordinate of the local maximum, decreasing between the two local extrema, and increasing again from the x-coordinate of the local minimum to ∞. This information is supported by the superimposed graph of $f'(x) = 3x^2 - 4$.

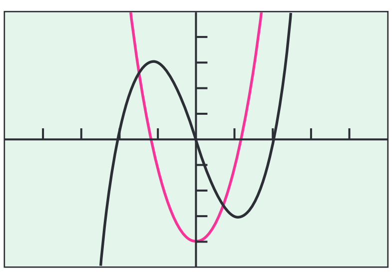
Confirm Analytically The function is increasing where $f'(x) > 0$.

$$3x^2 - 4 > 0$$
$$x^2 > \frac{4}{3}$$
$$x < -\sqrt{\frac{4}{3}} \quad \text{or} \quad x > \sqrt{\frac{4}{3}}$$

The function is decreasing where $f'(x) < 0$.

$$3x^2 - 4 < 0$$
$$x^2 < \frac{4}{3}$$
$$-\sqrt{\frac{4}{3}} < x < \sqrt{\frac{4}{3}}$$

In interval notation, f is increasing on $(-\infty, -\sqrt{4/3}]$, decreasing on $[-\sqrt{4/3}, \sqrt{4/3}]$, and increasing on $[\sqrt{4/3}, \infty)$. *Now try Exercise 27.*



[–5, 5] by [–5, 5]

Figure 4.17 By comparing the graphs of $f(x) = x^3 - 4x$ and $f'(x) = 3x^2 - 4$ we can relate the increasing and decreasing behavior of f to the sign of f'. (Example 6)

Other Consequences

We know that constant functions have the zero function as their derivative. We can now use the Mean Value Theorem to show conversely that the only functions with the zero function as derivative are constant functions.

COROLLARY 2 Functions with $f' = 0$ are Constant

If $f'(x) = 0$ at each point of an interval I, then there is a constant C for which $f(x) = C$ for all x in I.

Proof Our plan is to show that $f(x_1) = f(x_2)$ for any two points x_1 and x_2 in I. We can assume the points are numbered so that $x_1 < x_2$. Since f is differentiable at every point of $[x_1, x_2]$, it is continuous at every point as well. Thus, f satisfies the hypotheses of the Mean Value Theorem on $[x_1, x_2]$. Therefore, there is a point c between x_1 and x_2 for which

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Because $f'(c) = 0$, it follows that $f(x_1) = f(x_2)$. ■

We can use Corollary 2 to show that if two functions have the same derivative, they differ by a constant.

COROLLARY 3 Functions with the Same Derivative Differ by a Constant

If $f'(x) = g'(x)$ at each point of an interval I, then there is a constant C such that $f(x) = g(x) + C$ for all x in I.

Proof Let $h = f - g$. Then for each point x in I,

$$h'(x) = f'(x) - g'(x) = 0.$$

It follows from Corollary 2 that there is a constant C such that $h(x) = C$ for all x in I. Thus, $h(x) = f(x) - g(x) = C$, or $f(x) = g(x) + C$. ■

We know that the derivative of $f(x) = x^2$ is $2x$ on the interval $(-\infty, \infty)$. So, any other function $g(x)$ with derivative $2x$ on $(-\infty, \infty)$ must have the formula $g(x) = x^2 + C$ for some constant C.

EXAMPLE 7 Applying Corollary 3

Find the function $f(x)$ whose derivative is $\sin x$ and whose graph passes through the point $(0, 2)$.

SOLUTION

Since f has the same derivative as $g(x) = -\cos x$, we know that $f(x) = -\cos x + C$, for some constant C. To identify C, we use the condition that the graph must pass through $(0, 2)$. This is equivalent to saying that

$$\begin{aligned} f(0) &= 2 \\ -\cos(0) + C &= 2 \\ -1 + C &= 2 \\ C &= 3. \end{aligned}$$

The formula for f is $f(x) = -\cos x + 3$. *Now try Exercise 35.*

In Example 7 we were given a derivative and asked to find a function with that derivative. This type of function is so important that it has a name.

DEFINITION Antiderivative

A function $F(x)$ is an **antiderivative** of a function $f(x)$ if $F'(x) = f(x)$ for all x in the domain of f. The process of finding an antiderivative is **antidifferentiation.**

We know that if f has one antiderivative F then it has infinitely many antiderivatives, each differing from F by a constant. Corollary 3 says these are all there are. In Example 7, we found the particular antiderivative of sin x whose graph passed through the point (0, 2).

EXAMPLE 8 Finding Velocity and Position

Find the velocity and position functions of a body falling freely from a height of 0 meters under each of the following sets of conditions:

(a) The acceleration is 9.8 m/sec^2 and the body falls from rest.

(b) The acceleration is 9.8 m/sec^2 and the body is propelled downward with an initial velocity of 1 m/sec.

SOLUTION

(a) ***Falling from rest.*** We measure distance fallen in meters and time in seconds, and assume that the body is released from rest at time $t = 0$.

Velocity: We know that the velocity $v(t)$ is an antiderivative of the constant function 9.8. We also know that $g(t) = 9.8t$ is an antiderivative of 9.8. By Corollary 3,

$$v(t) = 9.8t + C$$

for some constant C. Since the body falls from rest, $v(0) = 0$. Thus,

$$9.8(0) + C = 0 \quad \text{and} \quad C = 0.$$

The body's velocity function is $v(t) = 9.8t$.

Position: We know that the position $s(t)$ is an antiderivative of $9.8t$. We also know that $h(t) = 4.9t^2$ is an antiderivative of $9.8t$. By Corollary 3,

$$s(t) = 4.9t^2 + C$$

for some constant C. Since $s(0) = 0$,

$$4.9(0)^2 + C = 0 \quad \text{and} \quad C = 0.$$

The body's position function is $s(t) = 4.9t^2$.

(b) ***Propelled downward.*** We measure distance fallen in meters and time in seconds, and assume that the body is propelled downward with velocity of 1 m/sec at time $t = 0$.

Velocity: The velocity function still has the form $9.8t + C$, but instead of being zero, the initial velocity (velocity at $t = 0$) is now 1 m/sec. Thus,

$$9.8(0) + C = 1 \quad \text{and} \quad C = 1.$$

The body's velocity function is $v(t) = 9.8t + 1$.

Position: We know that the position $s(t)$ is an antiderivative of $9.8t + 1$. We also know that $k(t) = 4.9t^2 + t$ is an antiderivative of $9.8t + 1$. By Corollary 3,

$$s(t) = 4.9t^2 + t + C$$

for some constant C. Since $s(0) = 0$,

$$4.9(0)^2 + 0 + C = 0 \quad \text{and} \quad C = 0.$$

The body's position function is $s(t) = 4.9t^2 + t$. ***Now try Exercise 43.***

Quick Review 4.2 *(For help, go to Sections 1.2, 2.3, and 3.2.)*

In Exercises 1 and 2, find exact solutions to the inequality.

1. $2x^2 - 6 < 0$

2. $3x^2 - 6 > 0$

In Exercises 3–5, let $f(x) = \sqrt{8 - 2x^2}$.

3. Find the domain of f.

4. Where is f continuous?

5. Where is f differentiable?

In Exercises 6–8, let $f(x) = \dfrac{x}{x^2 - 1}$.

6. Find the domain of f.

7. Where is f continuous?

8. Where is f differentiable?

In Exercises 9 and 10, find C so that the graph of the function f passes through the specified point.

9. $f(x) = -2x + C, \quad (-2, 7)$

10. $g(x) = x^2 + 2x + C, \quad (1, -1)$

Section 4.2 Exercises

In Exercises 1–8, **(a)** state whether or not the function satisfies the hypotheses of the Mean Value Theorem on the given interval, and **(b)** if it does, find each value of c in the interval (a, b) that satisfies the equation

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

1. $f(x) = x^2 + 2x - 1$ on $[0, 1]$

2. $f(x) = x^{2/3}$ on $[0, 1]$

3. $f(x) = x^{1/3}$ on $[-1,1]$

4. $f(x) = |x - 1|$ on $[0, 4]$

5. $f(x) = \sin^{-1}x$ on $[-1, 1]$

6. $f(x) = \ln(x - 1)$ on $[2, 4]$

7. $f(x) = \begin{cases} \cos x, & 0 \le x < \pi/2 \\ \sin x, & \pi/2 \le x \le \pi \end{cases}$ on $[0, \pi]$

8. $f(x) = \begin{cases} \sin^{-1} x, & -1 \le x < 1 \\ x/2+1, & 1 \le x \le 3 \end{cases}$ on $[-1, 3]$

In Exercises 9 and 10, the interval $a \le x \le b$ is given. Let $A = (a, f(a))$ and $B = (b, f(b))$. Write an equation for

(a) the secant line AB.

(b) a tangent line to f in the interval (a, b) that is parallel to AB.

9. $f(x) = x + \dfrac{1}{x}, \quad 0.5 \le x \le 2$

10. $f(x) = \sqrt{x - 1}, \quad 1 \le x \le 3$

11. ***Speeding*** A trucker handed in a ticket at a toll booth showing that in 2 h she had covered 159 mi on a toll road with speed limit 65 mph. The trucker was cited for speeding. Why?

12. ***Temperature Change*** It took 20 sec for the temperature to rise from 0°F to 212°F when a thermometer was taken from a freezer and placed in boiling water. Explain why at some moment in that interval the mercury was rising at exactly 10.6°F/sec.

13. ***Triremes*** Classical accounts tell us that a 170-oar trireme (ancient Greek or Roman warship) once covered 184 sea miles in 24 h. Explain why at some point during this feat the trireme's speed exceeded 7.5 knots (sea miles per hour).

14. ***Running a Marathon*** A marathoner ran the 26.2-mi New York City Marathon in 2.2 h. Show that at least twice, the marathoner was running at exactly 11 mph.

In Exercises 15–22, use analytic methods to find **(a)** the local extrema, **(b)** the intervals on which the function is increasing, and **(c)** the intervals on which the function is decreasing.

15. $f(x) = 5x - x^2$

16. $g(x) = x^2 - x - 12$

17. $h(x) = \dfrac{2}{x}$

18. $k(x) = \dfrac{1}{x^2}$

19. $f(x) = e^{2x}$

20. $f(x) = e^{-0.5x}$

21. $y = 4 - \sqrt{x + 2}$

22. $y = x^4 - 10x^2 + 9$

In Exercises 23–28, find **(a)** the local extrema, **(b)** the intervals on which the function is increasing, and **(c)** the intervals on which the function is decreasing.

23. $f(x) = x\sqrt{4 - x}$

24. $g(x) = x^{1/3}(x + 8)$

25. $h(x) = \dfrac{-x}{x^2 + 4}$

26. $k(x) = \dfrac{x}{x^2 - 4}$

27. $f(x) = x^3 - 2x - 2\cos x$

28. $g(x) = 2x + \cos x$

In Exercises 29–34, find all possible functions f with the given derivative.

29. $f'(x) = x$

30. $f'(x) = 2$

31. $f'(x) = 3x^2 - 2x + 1$

32. $f'(x) = \sin x$

33. $f'(x) = e^x$

34. $f'(x) = \dfrac{1}{x - 1}, \quad x > 1$

In Exercises 35–38, find the function with the given derivative whose graph passes through the point P.

35. $f'(x) = -\dfrac{1}{x^2}, \quad x > 0, \quad P(2, 1)$

36. $f'(x) = \dfrac{1}{4x^{3/4}}, \quad P(1, -2)$

37. $f'(x) = \dfrac{1}{x + 2}, \quad x > -2, \quad P(-1, 3)$

38. $f'(x) = 2x + 1 - \cos x, \quad P(0, 3)$

Group Activity In Exercises 39–42, sketch a graph of a differentiable function $y = f(x)$ that has the given properties.

39. **(a)** local minimum at (1, 1), local maximum at (3, 3)

(b) local minima at (1, 1) and (3, 3)

(c) local maxima at (1, 1) and (3, 3)

40. $f(2) = 3, \quad f'(2) = 0,$ and

(a) $f'(x) > 0$ for $x < 2$, $f'(x) < 0$ for $x > 2$.

(b) $f'(x) < 0$ for $x < 2$, $f'(x) > 0$ for $x > 2$.

(c) $f'(x) < 0$ for $x \neq 2$.

(d) $f'(x) > 0$ for $x \neq 2$.

41. $f'(-1) = f'(1) = 0, \quad f'(x) > 0$ on $(-1, 1)$, $f'(x) < 0$ for $x < -1$, $f'(x) > 0$ for $x > 1$.

42. A local minimum value that is greater than one of its local maximum values.

43. ***Free Fall*** On the moon, the acceleration due to gravity is 1.6 m/sec^2.

(a) If a rock is dropped into a crevasse, how fast will it be going just before it hits bottom 30 sec later?

(b) How far below the point of release is the bottom of the crevasse?

(c) If instead of being released from rest, the rock is thrown into the crevasse from the same point with a downward velocity of 4 m/sec, when will it hit the bottom and how fast will it be going when it does?

44. ***Diving*** **(a)** With what velocity will you hit the water if you step off from a 10-m diving platform?

(b) With what velocity will you hit the water if you dive off the platform with an upward velocity of 2 m/sec?

45. **Writing to Learn** The function

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}$$

is zero at $x = 0$ and at $x = 1$. Its derivative is equal to 1 at every point between 0 and 1, so f' is never zero between 0 and 1, and the graph of f has no tangent parallel to the chord from (0, 0) to (1, 0). Explain why this does not contradict the Mean Value Theorem.

46. **Writing to Learn** Explain why there is a zero of $y = \cos x$ between every two zeros of $y = \sin x$.

47. ***Unique Solution*** Assume that f is continuous on $[a, b]$ and differentiable on (a, b). Also assume that $f(a)$ and $f(b)$ have opposite signs and $f' \neq 0$ between a and b. Show that $f(x) = 0$ exactly once between a and b.

In Exercises 48 and 49, show that the equation has exactly one solution in the interval. [*Hint:* See Exercise 47.]

48. $x^4 + 3x + 1 = 0, \quad -2 \le x \le -1$

49. $x + \ln(x + 1) = 0, \quad 0 \le x \le 3$

50. ***Parallel Tangents*** Assume that f and g are differentiable on $[a, b]$ and that $f(a) = g(a)$ and $f(b) = g(b)$. Show that there is at least one point between a and b where the tangents to the graphs of f and g are parallel or the same line. Illustrate with a sketch.

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

51. **True or False** If f is differentiable and increasing on (a, b), then $f'(c) > 0$ for every c in (a, b). Justify your answer.

52. **True or False** If f is differentiable and $f'(c) > 0$ for every c in (a, b), then f is increasing on (a, b). Justify your answer.

53. Multiple Choice If $f(x) = \cos x$, then the Mean Value Theorem guarantees that somewhere between 0 and $\pi/3$, $f'(x) =$

(A) $-\frac{3}{2\pi}$ **(B)** $-\frac{\sqrt{3}}{2}$ **(C)** $-\frac{1}{2}$ **(D)** 0 **(E)** $\frac{1}{2}$

54. Multiple Choice On what interval is the function $g(x) = e^{x^3-6x^2+8}$ decreasing?

(A) $(-\infty, 2]$ **(B)** $[0, 4]$ **(C)** $[2, 4]$ **(D)** $(4, \infty)$ **(E)** no interval

55. Multiple Choice Which of the following functions is an antiderivative of $\frac{1}{\sqrt{x}}$?

(A) $-\frac{1}{\sqrt{2x^3}}$ **(B)** $-\frac{2}{\sqrt{x}}$ **(C)** $\frac{\sqrt{x}}{2}$ **(D)** $\sqrt{x} + 5$ **(E)** $2\sqrt{x} - 10$

56. Multiple Choice All of the following functions satisfy the conditions of the Mean Value Theorem on the interval $[-1, 1]$ *except*

(A) $\sin x$ **(B)** $\sin^{-1} x$ **(C)** $x^{5/3}$ **(D)** $x^{3/5}$ **(E)** $\frac{x}{x-2}$

Explorations

57. *Analyzing Derivative Data* Assume that f is continuous on $[-2, 2]$ and differentiable on $(-2, 2)$. The table gives some values of $f'(x)$.

x	$f'(x)$	x	$f'(x)$
−2	7	0.25	−4.81
−1.75	4.19	0.5	−4.25
−1.5	1.75	0.75	−3.31
−1.25	−0.31	1	−2
−1	−2	1.25	−0.31
−0.75	−3.31	1.5	1.75
−0.5	−4.25	1.75	4.19
−0.25	−4.81	2	7
0	−5		

(a) Estimate where f is increasing, decreasing, and has local extrema.

(b) Find a quadratic regression equation for the data in the table and superimpose its graph on a scatter plot of the data.

(c) Use the model in part (b) for f' and find a formula for f that satisfies $f(0) = 0$.

58. *Analyzing Motion Data* Priya's distance D in meters from a motion detector is given by the data in Table 4.1.

Table 4.1 Motion Detector Data

t (sec)	D (m)	t (sec)	D (m)
0.0	3.36	4.5	3.59
0.5	2.61	5.0	4.15
1.0	1.86	5.5	3.99
1.5	1.27	6.0	3.37
2.0	0.91	6.5	2.58
2.5	1.14	7.0	1.93
3.0	1.69	7.5	1.25
3.5	2.37	8.0	0.67
4.0	3.01		

(a) Estimate when Priya is moving toward the motion detector; away from the motion detector.

(b) Writing to Learn Give an interpretation of any local extreme values in terms of this problem situation.

(c) Find a cubic regression equation $D = f(t)$ for the data in Table 4.1 and superimpose its graph on a scatter plot of the data.

(d) Use the model in (c) for f to find a formula for f'. Use this formula to estimate the answers to (a).

Extending the Ideas

59. *Geometric Mean* The **geometric mean** of two positive numbers a and b is $\sqrt{ab}$. Show that for $f(x) = 1/x$ on any interval $[a, b]$ of positive numbers, the value of c in the conclusion of the Mean Value Theorem is $c = \sqrt{ab}$.

60. *Arithmetic Mean* The **arithmetic mean** of two numbers a and b is $(a + b)/2$. Show that for $f(x) = x^2$ on any interval $[a, b]$, the value of c in the conclusion of the Mean Value Theorem is $c = (a + b)/2$.

61. *Upper Bounds* Show that for any numbers a and b, $|\sin b - \sin a| \le |b - a|$.

62. *Sign of f′* Assume that f is differentiable on $a \le x \le b$ and that $f(b) < f(a)$. Show that f' is negative at some point between a and b.

63. *Monotonic Functions* Show that monotonic increasing and decreasing functions are one-to-one.

4.3 Connecting f' and f'' with the Graph of f

What you'll learn about

- First Derivative Test for Local Extrema
- Concavity
- Points of Inflection
- Second Derivative Test for Local Extrema
- Learning about Functions from Derivatives

. . . and why

Differential calculus is a powerful problem-solving tool precisely because of its usefulness for analyzing functions.

First Derivative Test for Local Extrema

As we see once again in Figure 4.18, a function f may have local extrema at some critical points while failing to have local extrema at others. The key is the sign of f' in a critical point's immediate vicinity. As x moves from left to right, the values of f increase where $f' > 0$ and decrease where $f' < 0$.

At the points where f has a minimum value, we see that $f' < 0$ on the interval immediately to the left and $f' > 0$ on the interval immediately to the right. (If the point is an endpoint, there is only the interval on the appropriate side to consider.) This means that the curve is falling (values decreasing) on the left of the minimum value and rising (values increasing) on its right. Similarly, at the points where f has a maximum value, $f' > 0$ on the interval immediately to the left and $f' < 0$ on the interval immediately to the right. This means that the curve is rising (values increasing) on the left of the maximum value and falling (values decreasing) on its right.

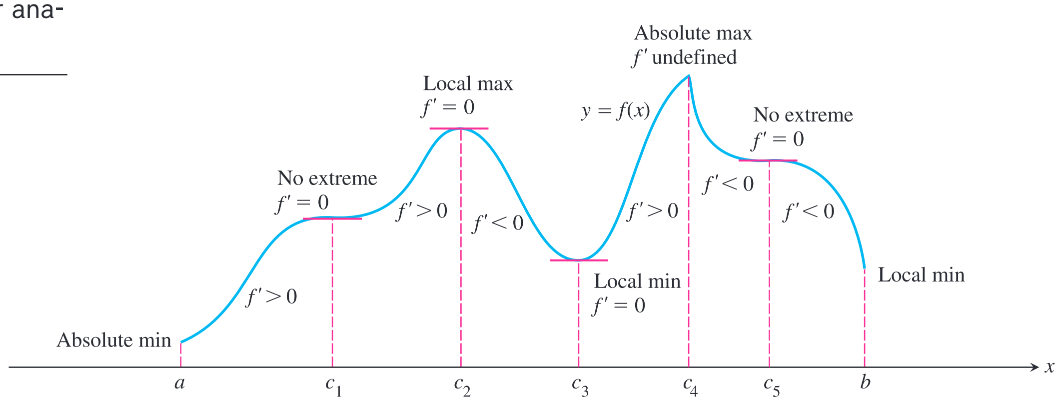


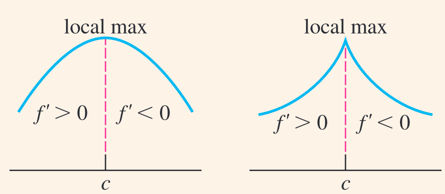
Figure 4.18 A function's first derivative tells how the graph rises and falls.

THEOREM 4 First Derivative Test for Local Extrema

The following test applies to a continuous function $f(x)$.

At a critical point c:

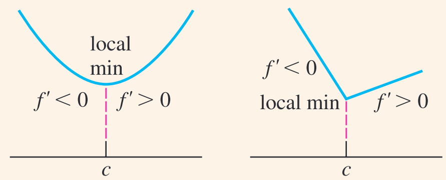
1. If f' changes sign from positive to negative at c ($f' > 0$ for $x < c$ and $f' < 0$ for $x > c$), then f has a local maximum value at c.



(a) $f'(c) = 0$ (b) $f'(c)$ undefined

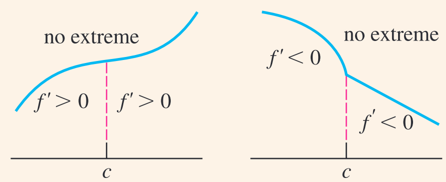
continued

2. If f' changes sign from negative to positive at c $(f' < 0$ for $x < c$ and $f' > 0$ for $x > c)$, then f has a local minimum value at c.



(a) $f'(c) = 0$ (b) $f'(c)$ undefined

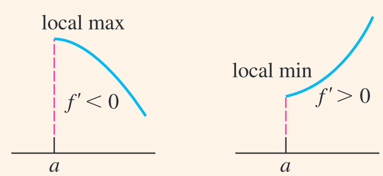
3. If f' does not change sign at c (f' has the same sign on both sides of c), then f has no local extreme value at c.



(a) $f'(c) = 0$ (b) $f'(c)$ undefined

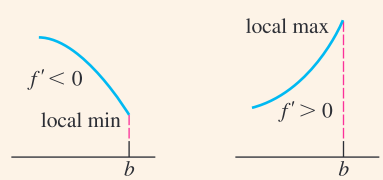
At a left endpoint *a*:

If $f' < 0$ $(f' > 0)$ for $x > a$, then f has a local maximum (minimum) value at a.



At a right endpoint *b*:

If $f' < 0$ $(f' > 0)$ for $x < b$, then f has a local minimum (maximum) value at b.



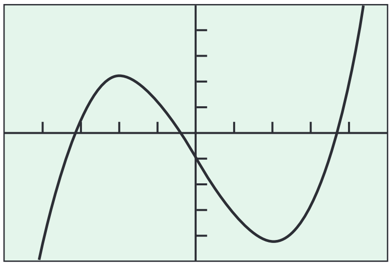
Here is how we apply the First Derivative Test to find the local extrema of a function. The critical points of a function f partition the x-axis into intervals on which f' is either positive or negative. We determine the sign of f' in each interval by evaluating f' for one value of x in the interval. Then we apply Theorem 4 as shown in Examples 1 and 2.

EXAMPLE 1 Using the First Derivative Test

For each of the following functions, use the First Derivative Test to find the local extreme values. Identify any absolute extrema.

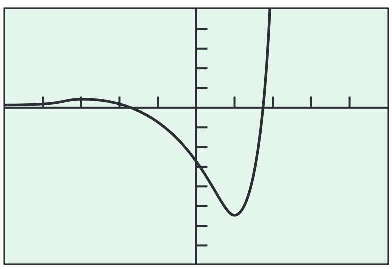
(a) $f(x) = x^3 - 12x - 5$ **(b)** $g(x) = (x^2 - 3)e^x$

continued



[–5, 5] by [–25, 25]

Figure 4.19 The graph of $f(x) = x^3 - 12x - 5$.

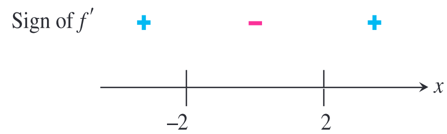


[–5, 5] by [–8, 5]

Figure 4.20 The graph of $g(x) = (x^2 - 3)e^x$.

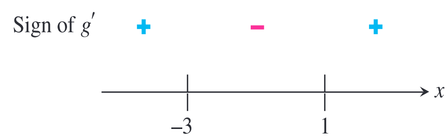
SOLUTION

(a) Since f is differentiable for all real numbers, the only possible critical points are the zeros of f'. Solving $f'(x) = 3x^2 - 12 = 0$, we find the zeros to be $x = 2$ and $x = -2$. The zeros partition the x-axis into three intervals, as shown below:



Using the First Derivative Test, we can see from the sign of f' on each interval that there is a local maximum at $x = -2$ and a local minimum at $x = 2$. The local maximum value is $f(-2) = 11$, and the local minimum value is $f(2) = -21$. There are no absolute extrema, as the function has range $(-\infty, \infty)$ (Figure 4.19).

(b) Since g is differentiable for all real numbers, the only possible critical points are the zeros of g'. Since $g'(x) = (x^2 - 3) \cdot e^x + (2x) \cdot e^x = (x^2 + 2x - 3) \cdot e^x$, we find the zeros of g' to be $x = 1$ and $x = -3$. The zeros partition the x-axis into three intervals, as shown below:



Using the First Derivative Test, we can see from the sign of f' on each interval that there is a local maximum at $x = -3$ and a local minimum at $x = 1$. The local maximum value is $g(-3) = 6e^{-3} \approx 0.299$, and the local minimum value is $g(1) = -2e \approx -5.437$. Although this function has the same increasing–decreasing–increasing pattern as f, its left end behavior is quite different. We see that $\lim_{x\to-\infty} g(x) = 0$, so the graph approaches the y-axis asymptotically and is therefore bounded below. This makes $g(1)$ an *absolute* minimum. Since $\lim_{x\to\infty} g(x) = \infty$, there is no absolute maximum (Figure 4.20).

Now try Exercise 3.

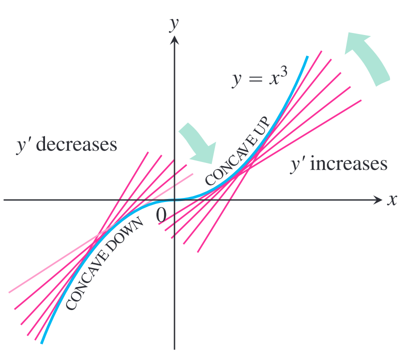


Figure 4.21 The graph of $y = x^3$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$.

Concavity

As you can see in Figure 4.21, the function $y = x^3$ rises as x increases, but the portions defined on the intervals $(-\infty, 0)$ and $(0, \infty)$ *turn* in different ways. Looking at tangents as we scan from left to right, we see that the slope y' of the curve decreases on the interval $(-\infty, 0)$ and then increases on the interval $(0, \infty)$. The curve $y = x^3$ is *concave down* on $(-\infty, 0)$ and *concave up* on $(0, \infty)$. The curve lies below the tangents where it is concave down, and above the tangents where it is concave up.

DEFINITION Concavity

The graph of a differentiable function $y = f(x)$ is

(a) **concave up** on an open interval I if y' is increasing on I.

(b) **concave down** on an open interval I if y' is decreasing on I.

If a function $y = f(x)$ has a second derivative, then we can conclude that y' increases if $y'' > 0$ and y' decreases if $y'' < 0$.

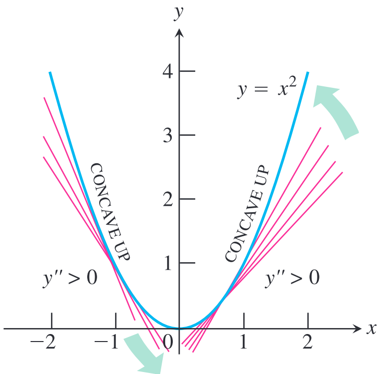


Figure 4.22 The graph of $y = x^2$ is concave up on any interval. (Example 2)

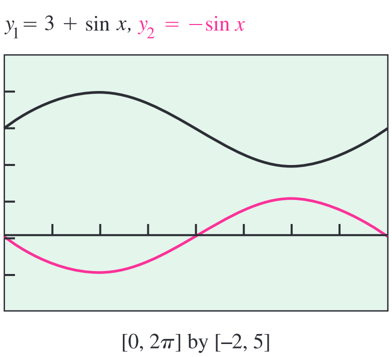


Figure 4.23 Using the graph of y'' to determine the concavity of y. (Example 2)

Concavity Test

The graph of a twice-differentiable function $y = f(x)$ is

(a) concave up on any interval where $y'' > 0$.

(b) concave down on any interval where $y'' < 0$.

EXAMPLE 2 Determining Concavity

Use the Concavity Test to determine the concavity of the given functions on the given intervals:

(a) $y = x^2$ on $(3, 10)$ **(b)** $y = 3 + \sin x$ on $(0, 2\pi)$

SOLUTION

(a) Since $y'' = 2$ is always positive, the graph of $y = x^2$ is concave up on *any* interval. In particular, it is concave up on $(3, 10)$ (Figure 4.22).

(b) The graph of $y = 3 + \sin x$ is concave down on $(0, \pi)$, where $y'' = -\sin x$ is negative. It is concave up on $(\pi, 2\pi)$, where $y'' = -\sin x$ is positive (Figure 4.23).

Now try Exercise 7.

Points of Inflection

The curve $y = 3 + \sin x$ in Example 2 changes concavity at the point $(\pi, 3)$. We call $(\pi, 3)$ a *point of inflection* of the curve.

DEFINITION Point of Inflection

A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection.**

A point on a curve where y'' is positive on one side and negative on the other is a point of inflection. At such a point, y'' is either zero (because derivatives have the intermediate value property) or undefined. If y is a twice differentiable function, $y'' = 0$ at a point of inflection and y' has a local maximum or minimum.

EXAMPLE 3 Finding Points of Inflection

Find all points of inflection of the graph of $y = e^{-x^2}$.

SOLUTION

First we find the second derivative, recalling the Chain and Product Rules:

$$y = e^{-x^2}$$
$$y' = e^{-x^2} \cdot (-2x)$$
$$y'' = e^{-x^2} \cdot (-2x) \cdot (-2x) + e^{-x^2} \cdot (-2)$$
$$= e^{-x^2}(4x^2 - 2)$$

The factor e^{-x^2} is always positive, while the factor $(4x^2 - 2)$ changes sign at $-\sqrt{1/2}$ and at $\sqrt{1/2}$. Since y'' must also change sign at these two numbers, the points of inflection are $(-\sqrt{1/2}, 1/\sqrt{e})$ and $(\sqrt{1/2}, 1/\sqrt{e})$. We confirm our solution graphically by observing the changes of curvature in Figure 4.24.

Now try Exercise 13.

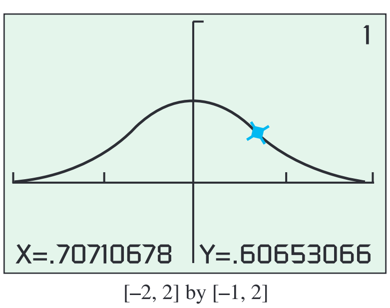


Figure 4.24 Graphical confirmation that the graph of $y = e^{-x^2}$ has a point of inflection at $x = \sqrt{1/2}$ (and hence also at $x = -\sqrt{1/2}$). (Example 3)

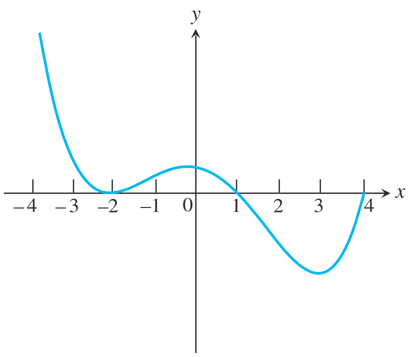


Figure 4.25 The graph of f', the derivative of f, on the interval $[-4, 4]$.

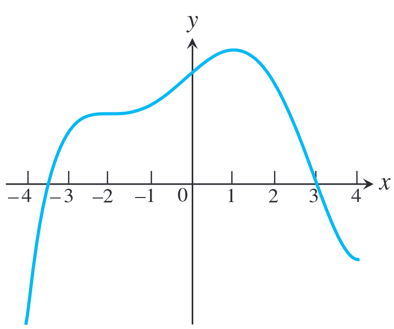
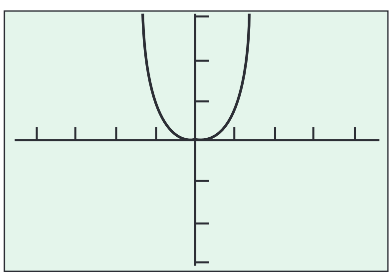
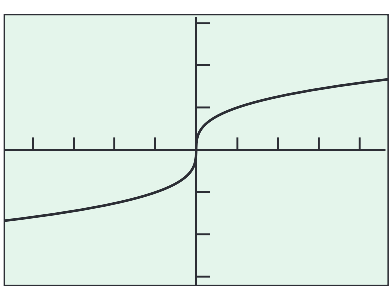


Figure 4.26 A possible graph of f. (Example 4)



[– 4.7, 4.7] by [–3.1, 3.1]

Figure 4.27 The function $f(x) = x^4$ does not have a point of inflection at the origin, even though $f''(0) = 0$.



[–4.7, 4.7] by [–3.1, 3.1]

Figure 4.28 The function $f(x) = \sqrt[3]{x}$ has a point of inflection at the origin, even though $f''(0) \neq 0$.

EXAMPLE 4 Reading the Graph of the Derivative

The graph of the *derivative* of a function f on the interval $[-4, 4]$ is shown in Figure 4.25. Answer the following questions about f, justifying each answer with information obtained from the graph of f'.

(a) On what intervals is f increasing?

(b) On what intervals is the graph of f concave up?

(c) At which x-coordinates does f have local extrema?

(d) What are the x-coordinates of all inflection points of the graph of f?

(e) Sketch a possible graph of f on the interval $[-4, 4]$.

SOLUTION

(a) Since $f' > 0$ on the intervals $[-4, -2)$ and $(-2, 1)$, the function f must be increasing on the entire interval $[-4, 1]$ with a horizontal tangent at $x = -2$ (a "shelf point").

(b) The graph of f is concave up on the intervals where f' is increasing. We see from the graph that f' is increasing on the intervals $(-2, 0)$ and $(3, 4)$.

(c) By the First Derivative Test, there is a local maximum at $x = 1$ because the sign of f' changes from positive to negative there. Note that there is no extremum at $x = -2$, since f' does not change sign. Because the function increases from the left endpoint and decreases to the right endpoint, there are local minima at the endpoints $x = -4$ and $x = 4$.

(d) The inflection points of the graph of f have the same x-coordinates as the turning points of the graph of f', namely -2, 0, and 3.

(e) A possible graph satisfying all the conditions is shown in Figure 4.26.

Now try Exercise 23.

Caution: It is tempting to oversimplify a point of inflection as a point where the second derivative is zero, but that can be wrong for two reasons:

1. *The second derivative can be zero at a noninflection point*. For example, consider the function $f(x) = x^4$ (Figure 4.27). Since $f''(x) = 12x^2$, we have $f''(0) = 0$; however, $(0, 0)$ is not an inflection point. Note that f'' does not *change sign* at $x = 0$.
2. *The second derivative need not be zero at an inflection point*. For example, consider the function $f(x) = \sqrt[3]{x}$ (Figure 4.28). The concavity changes at $x = 0$, but there is a *vertical* tangent line, so both $f'(0)$ and $f''(0)$ fail to exist.

Therefore, the only safe way to test algebraically for a point of inflection is to confirm a sign change of the second derivative. This *could* occur at a point where the second derivative is zero, but it also could occur at a point where the second derivative fails to exist.

To study the motion of a body moving along a line, we often graph the body's position as a function of time. One reason for doing so is to reveal where the body's acceleration, given by the second derivative, changes sign. On the graph, these are the points of inflection.

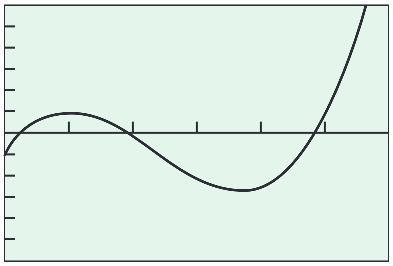
EXAMPLE 5 Studying Motion along a Line

A particle is moving along the x-axis with position function

$$x(t) = 2t^3 - 14t^2 + 22t - 5, \qquad t \geq 0.$$

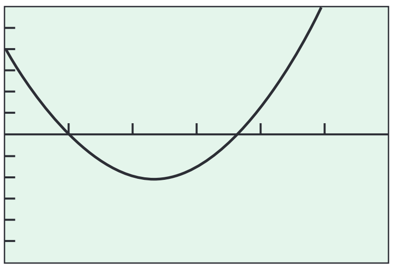
Find the velocity and acceleration, and describe the motion of the particle.

continued



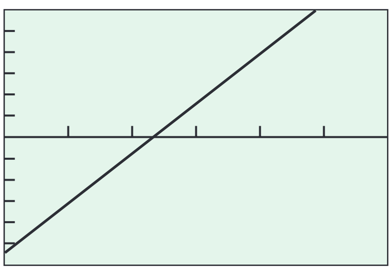
[0, 6] by [–30, 30]

(a)



[0, 6] by [–30, 30]

(b)



[0, 6] by [–30, 30]

(c)

Figure 4.29 The graph of
(a) $x(t) = 2t^3 - 14t^2 + 22t - 5$, $t \geq 0$,
(b) $x'(t) = 6t^2 - 28t + 22$, and
(c) $x''(t) = 12t - 28$. (Example 5)

SOLUTION

Solve Analytically

The velocity is

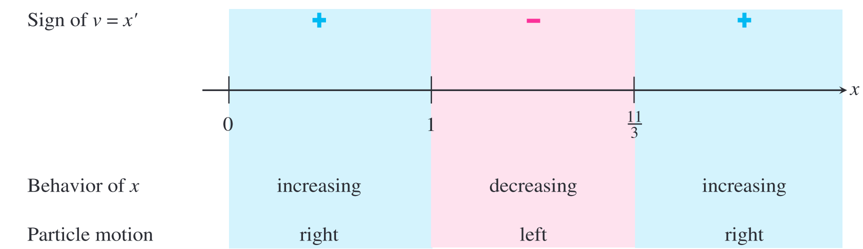
$$v(t) = x'(t) = 6t^2 - 28t + 22 = 2(t - 1)(3t - 11),$$

and the acceleration is

$$a(t) = v'(t) = x''(t) = 12t - 28 = 4(3t - 7).$$

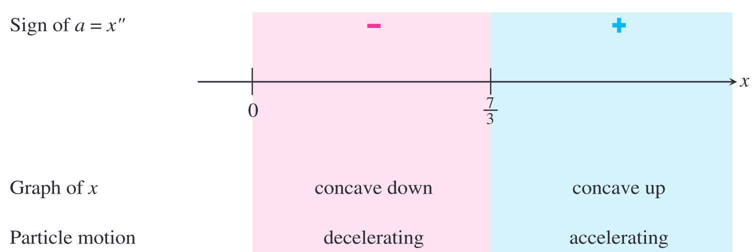
When the function $x(t)$ is increasing, the particle is moving to the right on the x-axis; when $x(t)$ is decreasing, the particle is moving to the left. Figure 4.29 shows the graphs of the position, velocity, and acceleration of the particle.

Notice that the first derivative ($v = x'$) is zero when $t = 1$ and $t = 11/3$. These zeros partition the t-axis into three intervals, as shown in the sign graph of v below:



The particle is moving to the right in the time intervals [0, 1) and (11/3, ∞) and moving to the left in (1, 11/3).

The acceleration $a(t) = 12t - 28$ has a single zero at $t = 7/3$. The sign graph of the acceleration is shown below:



The accelerating force is directed toward the left during the time interval [0, 7/3], is momentarily zero at $t = 7/3$, and is directed toward the right thereafter.

Now try Exercise 25.

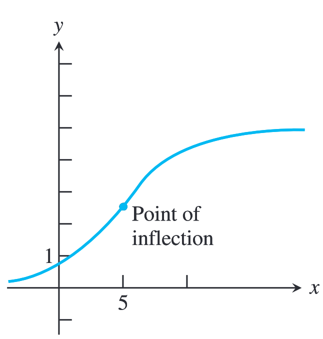


Figure 4.30 A logistic curve

$$y = \frac{c}{1 + ae^{-bx}}.$$

The growth of an individual company, of a population, in sales of a new product, or of salaries often follows a *logistic* or *life cycle curve* like the one shown in Figure 4.30. For example, sales of a new product will generally grow slowly at first, then experience a period of rapid growth. Eventually, sales growth slows down again. The function f in Figure 4.30 is increasing. Its rate of increase, f', is at first increasing ($f'' > 0$) up to the point of inflection, and then its rate of increase, f', is decreasing ($f'' < 0$). This is, in a sense, the opposite of what happens in Figure 4.21.

Some graphers have the logistic curve as a built-in regression model. We use this feature in Example 6.

Table 4.2 Population of Alaska

Years since 1900	Population
20	55,036
30	59,278
40	75,524
50	128,643
60	226,167
70	302,583
80	401,851
90	550,043
100	626,932

Source: Bureau of the Census, U.S. Chamber of Commerce.

EXAMPLE 6 Population Growth in Alaska

Table 4.2 shows the population of Alaska in each 10-year census between 1920 and 2000.

(a) Find the logistic regression for the data.

(b) Use the regression equation to predict the Alaskan population in the 2020 census.

(c) Find the inflection point of the regression equation. What significance does the inflection point have in terms of population growth in Alaska?

(d) What does the regression equation indicate about the population of Alaska in the long run?

SOLUTION

(a) Using years since 1900 as the independent variable and population as the dependent variable, the logistic regression equation is approximately

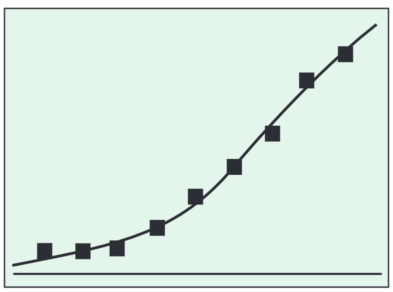
$$y = \frac{895598}{1 + 71.57e^{-0.0516x}}.$$

Its graph is superimposed on a scatter plot of the data in Figure 4.31(a). Store the regression equation as Y1 in your calculator.

(b) The calculator reports Y1(120) to be approximately 781,253. (Given the uncertainty of this kind of extrapolation, it is probably more reasonable to say "approximately 781,200.")

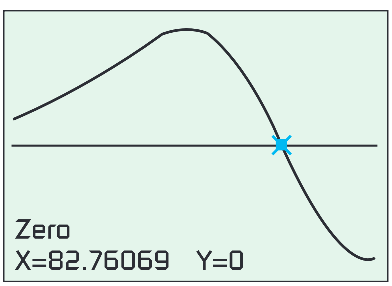
(c) The inflection point will occur where y'' changes sign. Finding y'' algebraically would be tedious, but we can graph the numerical derivative of the numerical derivative and find the zero graphically. Figure 4.31(b) shows the graph of y'', which is nDeriv(nDeriv(Y1,X,X),X,X) in calculator syntax. The zero is approximately 83, so the inflection point occurred in 1983, when the population was about 450,570 and growing the fastest.

(d) Notice that $\lim_{x\to\infty} \frac{895598}{1+ 71.57e^{-0.0516x}} = 895598$, so the regression equation indicates that the population of Alaska will stabilize at about 895,600 in the long run. Do not put too much faith in this number, however, as human population is dependent on too many variables that can, and will, change over time. ***Now try Exercise 31.***



[12, 108] by [0, 730000]

(a)



[12, 108] by [–250, 250]

(b)

Figure 4.31 (a) The logistic regression curve

$$y = \frac{895598}{1 + 71.57e^{-0.0516x}}$$

superimposed on the population data from Table 4.2, and (b) the graph of y'' showing a zero at about $x = 83$.

Second Derivative Test for Local Extrema

Instead of looking for sign changes in y' at critical points, we can sometimes use the following test to determine the presence of local extrema.

THEOREM 5 Second Derivative Test for Local Extrema

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.

This test requires us to know f'' *only at c itself* and not in an interval about c. This makes the test easy to apply. That's the good news. The bad news is that the test fails if $f''(c) = 0$ or if $f''(c)$ fails to exist. When this happens, go back to the First Derivative Test for local extreme values.

In Example 7, we apply the Second Derivative Test to the function in Example 1.

EXAMPLE 7 Using the Second Derivative Test

Find the local extreme values of $f(x) = x^3 - 12x - 5$.

SOLUTION

We have

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4)$$
$$f''(x) = 6x.$$

Testing the critical points $x = \pm 2$ (there are no endpoints), we find

$$f''(-2) = -12 < 0 \Rightarrow f \text{ has a local maximum at } x = -2 \text{ and}$$
$$f''(2) = 12 > 0 \Rightarrow f \text{ has a local minimum at } x = 2.$$

Now try Exercise 35.

EXAMPLE 8 Using f' and f'' to Graph f

Let $f'(x) = 4x^3 - 12x^2$.

(a) Identify where the extrema of f occur.

(b) Find the intervals on which f is increasing and the intervals on which f is decreasing.

(c) Find where the graph of f is concave up and where it is concave down.

(d) Sketch a possible graph for f.

SOLUTION

f is continuous since f' exists. The domain of f' is $(-\infty, \infty)$, so the domain of f is also $(-\infty, \infty)$. Thus, the critical points of f occur only at the zeros of f'. Since

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3),$$

the first derivative is zero at $x = 0$ and $x = 3$.

Intervals	$x < 0$	$0 < x < 3$	$3 < x$
Sign of f'	$-$	$-$	$+$
Behavior of f	decreasing	decreasing	increasing

(a) Using the First Derivative Test and the table above we see that there is no extremum at $x = 0$ and a local minimum at $x = 3$.

(b) Using the table above we see that f is decreasing in $(-\infty, 0]$ and $[0, 3]$, and increasing in $[3, \infty)$.

(c) $f''(x) = 12x^2 - 24x = 12x(x - 2)$ is zero at $x = 0$ and $x = 2$.

Intervals	$x < 0$	$0 < x < 2$	$2 < x$
Sign of f''	$+$	$-$	$+$
Behavior of f	concave up	concave down	concave up

We see that f is concave up on the intervals $(-\infty, 0)$ and $(2, \infty)$, and concave down on $(0, 2)$.

Note

The *Second Derivative Test* does not apply at $x = 0$ because $f''(0) = 0$. We need the First Derivative Test to see that there is no local extremum at $x = 0$.

continued

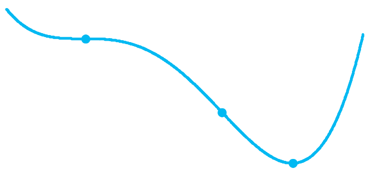


Figure 4.32 The graph for f has no extremum but has points of inflection where $x = 0$ and $x = 2$, and a local minimum where $x = 3$. (Example 8)

(d) Summarizing the information in the two tables above we obtain

$x < 0$	$0 < x < 2$	$2 < x < 3$	$x < 3$
decreasing	decreasing	decreasing	increasing
concave up	concave down	concave up	concave up

Figure 4.32 shows one possibility for the graph of f. *Now try Exercise 39.*

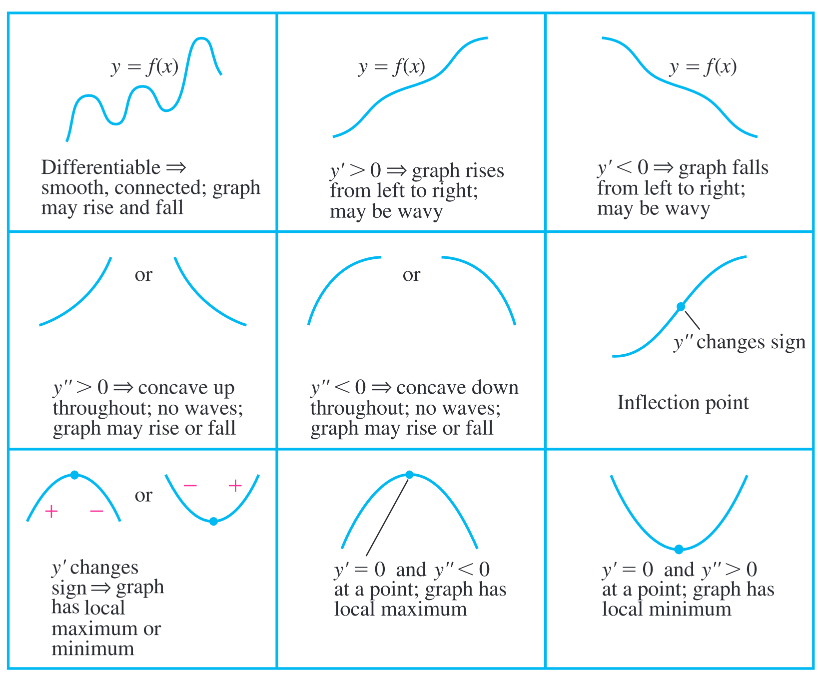
EXPLORATION 1 **Finding f from f'**

Let $f'(x) = 4x^3 - 12x^2$.

1. Find three different functions with derivative equal to $f'(x)$. How are the graphs of the three functions related?
2. Compare their behavior with the behavior found in Example 8.

Learning about Functions from Derivatives

We have seen in Example 8 and Exploration 1 that we are able to recover almost everything we need to know about a differentiable function $y = f(x)$ by examining y'. We can find where the graph rises and falls and where any local extrema are assumed. We can differentiate y' to learn how the graph bends as it passes over the intervals of rise and fall. We can determine the shape of the function's graph. The only information we cannot get from the derivative is how to place the graph in the xy-plane. As we discovered in Section 4.2, the only additional information we need to position the graph is the value of f at one point.



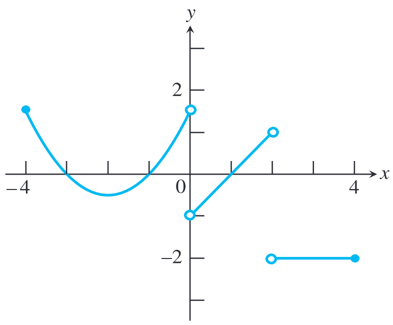


Figure 4.33 The graph of f', a discontinuous derivative.

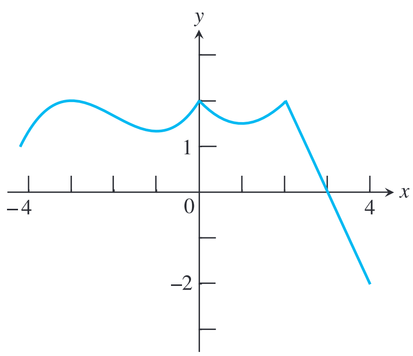


Figure 4.34 A possible graph of f. (Example 9)

Remember also that a function can be continuous and still have points of nondifferentiability (cusps, corners, and points with vertical tangent lines). Thus, a noncontinuous graph of f' could lead to a continuous graph of f, as Example 9 shows.

EXAMPLE 9 Analyzing a Discontinuous Derivative

A function f is continuous on the interval $[-4, 4]$. The discontinuous function f', with domain $[-4, 0) \cup (0, 2) \cup (2, 4]$, is shown in the graph to the right (Figure 4.33).

(a) Find the x-coordinates of all local extrema and points of inflection of f.

(b) Sketch a possible graph of f.

SOLUTION

(a) For extrema, we look for places where f' changes sign. There are local maxima at $x = -3$, 0, and 2 (where f' goes from positive to negative) and local minima at $x = -1$ and 1 (where f' goes from negative to positive). There are also local minima at the two endpoints $x = -4$ and 4, because f' starts positive at the left endpoint and ends negative at the right endpoint.

For points of inflection, we look for places where f'' changes sign, that is, where the graph of f' changes direction. This occurs only at $x = -2$.

(b) A possible graph of f is shown in Figure 4.34. The derivative information determines the shape of the three components, and the continuity condition determines that the three components must be linked together. ***Now try Exercises 49 and 53.***

EXPLORATION 2 Finding f from f' and f''

A function f is continuous on its domain $[-2, 4]$, $f(-2) = 5$, $f(4) = 1$, and f' and f'' have the following properties.

x	$-2 < x < 0$	$x = 0$	$0 < x < 2$	$x = 2$	$2 < x < 4$
f'	+	does not exist	−	0	−
f''	+	does not exist	+	0	−

1. Find where all absolute extrema of f occur.
2. Find where the points of inflection of f occur.
3. Sketch a possible graph of f.

Quick Review 4.3 *(For help, go to Sections 1.3, 2.2, 3.3, and 3.9.)*

In Exercises 1 and 2, factor the expression and use sign charts to solve the inequality.

1. $x^2 - 9 < 0$

2. $x^3 - 4x > 0$

In Exercises 3–6, find the domains of f and f'.

3. $f(x) = xe^x$

4. $f(x) = x^{3/5}$

5. $f(x) = \dfrac{x}{x-2}$

6. $f(x) = x^{2/5}$

In Exercises 7–10, find the horizontal asymptotes of the function's graph.

7. $y = (4 - x^2)e^x$

8. $y = (x^2 - x)e^{-x}$

9. $y = \dfrac{200}{1 + 10e^{-0.5x}}$

10. $y = \dfrac{750}{2 + 5e^{-0.1x}}$

Section 4.3 Exercises

In Exercises 1–6, use the First Derivative Test to determine the local extreme values of the function, and identify any absolute extrema. Support your answers graphically.

1. $y = x^2 - x - 1$

2. $y = -2x^3 + 6x^2 - 3$

3. $y = 2x^4 - 4x^2 + 1$

4. $y = xe^{1/x}$

5. $y = x\sqrt{8 - x^2}$

6. $y = \begin{cases} 3 - x^2, & x < 0 \\ x^2 + 1, & x \geq 0 \end{cases}$

In Exercises 7–12, use the Concavity Test to determine the intervals on which the graph of the function is **(a)** concave up and **(b)** concave down.

7. $y = 4x^3 + 21x^2 + 36x - 20$

8. $y = -x^4 + 4x^3 - 4x + 1$

9. $y = 2x^{1/5} + 3$

10. $y = 5 - x^{1/3}$

11. $y = \begin{cases} 2x, & x < 1 \\ 2 - x^2, & x \geq 1 \end{cases}$

12. $y = e^x, \quad 0 \leq x \leq 2\pi$

In Exercises 13–20, find all points of inflection of the function.

13. $y = xe^x$

14. $y = x\sqrt{9 - x^2}$

15. $y = \tan^{-1} x$

16. $y = x^3(4 - x)$

17. $y = x^{1/3}(x - 4)$

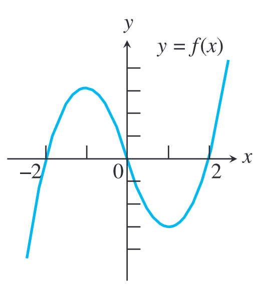
18. $y = x^{1/2}(x + 3)$

19. $y = \dfrac{x^3 - 2x^2 + x - 1}{x - 2}$

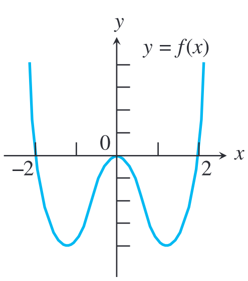
20. $y = \dfrac{x}{x^2 + 1}$

In Exercises 21 and 22, use the graph of the function f to estimate where **(a)** f' and **(b)** f'' are 0, positive, and negative.

21.

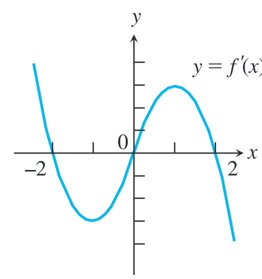


22.

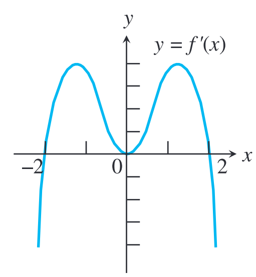


In Exercises 23 and 24, use the graph of the function f' to estimate the intervals on which the function f is **(a)** increasing or **(b)** decreasing. Also, **(c)** estimate the x-coordinates of all local extreme values.

23.



24.



In Exercises 25–28, a particle is moving along the x-axis with position function $x(t)$. Find the **(a)** velocity and **(b)** acceleration, and **(c)** describe the motion of the particle for $t \geq 0$.

25. $x(t) = t^2 - 4t + 3$

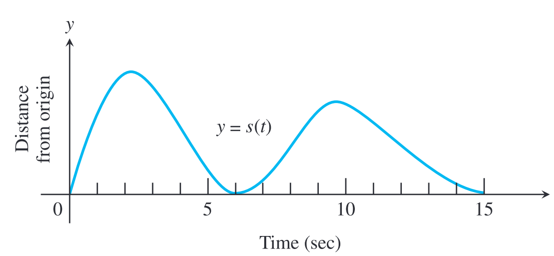
26. $x(t) = 6 - 2t - t^2$

27. $x(t) = t^3 - 3t + 3$

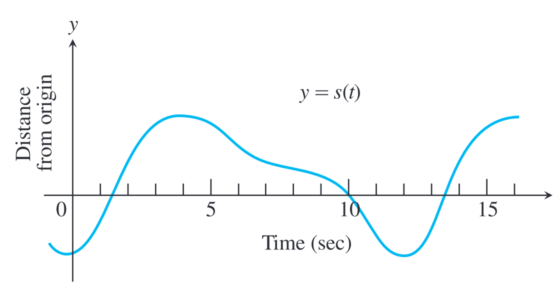
28. $x(t) = 3t^2 - 2t^3$

In Exercises 29 and 30, the graph of the position function $y = s(t)$ of a particle moving along a line is given. At approximately what times is the particle's **(a)** velocity equal to zero? **(b)** acceleration equal to zero?

29.



30.



31. Table 4.3 shows the population of Pennsylvania in each 10-year census between 1830 and 1950.

Table 4.3 Population of Pennsylvania

Years since 1820	Population in thousands
10	1348
20	1724
30	2312
40	2906
50	3522
60	4283
70	5258
80	6302
90	7665
100	8720
110	9631
120	9900
130	10,498

Source: Bureau of the Census, U.S. Chamber of Commerce.

(a) Find the logistic regression for the data.

(b) Graph the data in a scatter plot and superimpose the regression curve.

(c) Use the regression equation to predict the Pennsylvania population in the 2000 census.

(d) In what year was the Pennsylvania population growing the fastest? What significant behavior does the graph of the regression equation exhibit at that point?

(e) What does the regression equation indicate about the population of Pennsylvania in the long run?

32. In 1977, there were 12,168,450 basic cable television subscribers in the U.S. Table 4.4 shows the cumulative number of subscribers added to that baseline number from 1978 to 1985.

Table 4.4 Growth of Cable Television

Years since 1977	Added Subscribers since 1977
1	1,391,910
2	2,814,380
3	5,671,490
4	11,219,200
5	17,340,570
6	22,113,790
7	25,290,870
8	27,872,520

Source: Nielsen Media Research, as reported in *The World Almanac and Book of Facts 2004.*

(a) Find the logistic regression for the data.

(b) Graph the data in a scatter plot and superimpose the regression curve. Does it fit the data well?

(c) In what year between 1977 and 1985 were basic cable TV subscriptions growing the fastest? What significant behavior does the graph of the regression equation exhibit at that point?

(d) What does the regression equation indicate about the number of basic cable television subscribers in the long run? (Be sure to add the baseline 1977 number.)

(e) Writing to Learn In fact, the long-run number of basic cable subscribers predicted by the regression equation falls short of the actual 2002 number by more than 32 million. What circumstances changed to render the earlier model so ineffective?

In Exercises 33–38, use the Second Derivative Test to find the local extrema for the function.

33. $y = 3x - x^3 + 5$

34. $y = x^5 - 80x + 100$

35. $y = x^3 + 3x^2 - 2$

36. $y = 3x^5 - 25x^3 + 60x + 20$

37. $y = xe^x$

38. $y = xe^{-x}$

In Exercises 39 and 40, use the derivative of the function $y = f(x)$ to find the points at which f has a

(a) local maximum, **(b)** local minimum, or

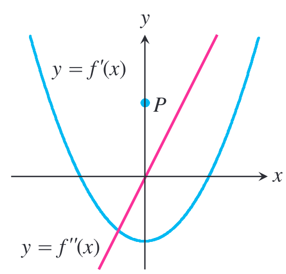
(c) point of inflection.

39. $y' = (x - 1)^2(x - 2)$

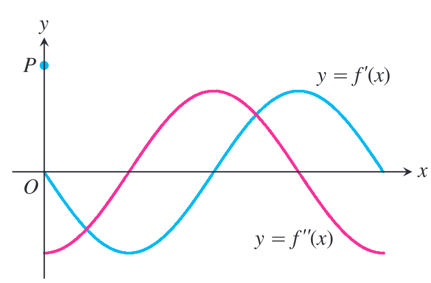
40. $y' = (x - 1)^2(x - 2)(x - 4)$

Exercises 41 and 42 show the graphs of the first and second derivatives of a function $y = f(x)$. Copy the figure and add a sketch of a possible graph of f that passes through the point P.

41.



42.



43. Writing to Learn If $f(x)$ is a differentiable function and $f'(c) = 0$ at an interior point c of f's domain, must f have a local maximum or minimum at $x = c$? Explain.

44. Writing to Learn If $f(x)$ is a twice-differentiable function and $f''(c) = 0$ at an interior point c of f's domain, must f have an inflection point at $x = c$? Explain.

45. *Connecting f and f′* Sketch a smooth curve $y = f(x)$ through the origin with the properties that $f'(x) < 0$ for $x < 0$ and $f'(x) > 0$ for $x > 0$.

46. *Connecting f and f″* Sketch a smooth curve $y = f(x)$ through the origin with the properties that $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$.

47. *Connecting f, f′, and f″* Sketch a continuous curve $y = f(x)$ with the following properties. Label coordinates where possible.

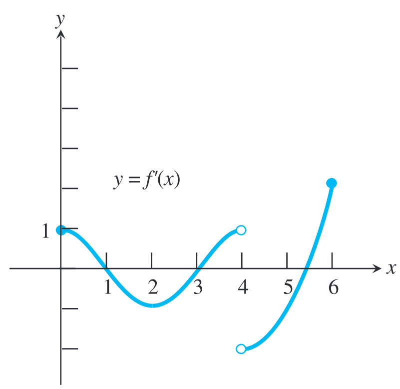
$f(-2) = 8$	$f'(x) > 0$ for $\|x\| > 2$
$f(0) = 4$	$f'(x) < 0$ for $\|x\| < 2$
$f(2) = 0$	$f''(x) < 0$ for $x < 0$
$f'(2) = f'(-2) = 0$	$f''(x) > 0$ for $x > 0$

48. *Using Behavior to Sketch* Sketch a continuous curve $y = f(x)$ with the following properties. Label coordinates where possible.

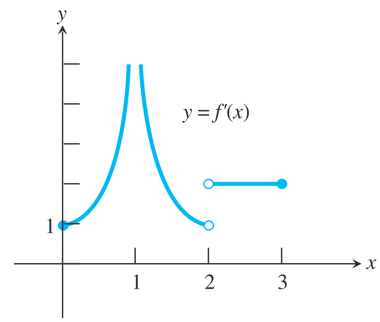
x	y	Curve
$x < 2$		falling, concave up
2	1	horizontal tangent
$2 < x < 4$		rising, concave up
4	4	inflection point
$4 < x < 6$		rising, concave down
6	7	horizontal tangent
$x > 6$		falling, concave down

In Exercises 49 and 50, use the graph of f' to estimate the intervals on which the function f is **(a)** increasing or **(b)** decreasing. Also, **(c)** estimate the x-coordinates of all local extreme values. (Assume that the function f is continuous, even at the points where f' is undefined.)

49. The domain of f' is $[0, 4) \cup (4, 6]$.



50. The domain of f' is $[0, 1) \cup (1, 2) \cup (2, 3]$.



Group Activity In Exercises 51 and 52, do the following.

(a) Find the absolute extrema of f and where they occur.

(b) Find any points of inflection.

(c) Sketch a possible graph of f.

51. f is continuous on $[0, 3]$ and satisfies the following.

x	0	1	2	3
f	0	2	0	-2
f'	3	0	does not exist	-3
f''	0	-1	does not exist	0

x	$0 < x < 1$	$1 < x < 2$	$2 < x < 3$
f	+	+	−
f'	+	−	−
f''	−	−	−

52. f is an even function, continuous on $[-3, 3]$, and satisfies the following.

x	0	1	2
f	2	0	-1
f'	does not exist	0	does not exist
f''	does not exist	0	does not exist

x	$0 < x < 1$	$1 < x < 2$	$2 < x < 3$
f	+	−	−
f'	−	−	+
f''	+	−	−

(d) What can you conclude about $f(3)$ and $f(-3)$?

Group Activity In Exercises 53 and 54, sketch a possible graph of a continuous function f that has the given properties.

53. Domain $[0, 6]$, graph of f' given in Exercise 49, and $f(0) = 2$.

54. Domain $[0, 3]$, graph of f' given in Exercise 50, and $f(0) = -3$.

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

55. True or False If $f''(c) = 0$, then $(c, f(c))$ is a point of inflection. Justify your answer.

56. True or False If $f'(c) = 0$ and $f''(c) < 0$, then $f(c)$ is a local maximum. Justify your answer.

57. Multiple Choice If $a < 0$, the graph of $y = ax^3 + 3x^2 + 4x + 5$ is concave up on

(A) $\left(-\infty, -\frac{1}{a}\right)$ **(B)** $\left(-\infty, \frac{1}{a}\right)$ **(C)** $\left(-\frac{1}{a}, \infty\right)$

(D) $\left(\frac{1}{a}, \infty\right)$ **(E)** $(-\infty, -1)$

58. Multiple Choice If $f(0) = f'(0) = f''(0) = 0$, which of the following *must be true*?

(A) There is a local maximum of f at the origin.
(B) There is a local minimum of f at the origin.
(C) There is no local extremum of f at the origin.
(D) There is a point of inflection of the graph of f at the origin.
(E) There is a horizontal tangent to the graph of f at the origin.

59. Multiple Choice The x-coordinates of the points of inflection of the graph of $y = x^5 - 5x^4 + 3x + 7$ are

(A) 0 only **(B)** 1 only **(C)** 3 only **(D)** 0 and 3 **(E)** 0 and 1

60. Multiple Choice Which of the following conditions would enable you to conclude that the graph of f has a point of inflection at $x = c$?

(A) There is a local maximum of f' at $x = c$.
(B) $f''(c) = 0$.
(C) $f''(c)$ does not exist.
(D) The sign of f' changes at $x = c$.
(E) f is a cubic polynomial and $c = 0$.

Exploration

61. *Graphs of Cubics* There is almost no leeway in the locations of the inflection point and the extrema of $f(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$, because the one inflection point occurs at $x = -b/(3a)$ and the extrema, if any, must be located symmetrically about this value of x. Check this out by examining **(a)** the cubic in Exercise 7 and **(b)** the cubic in Exercise 2. Then **(c)** prove the general case.

Extending the Ideas

In Exercises 62 and 63, feel free to use a CAS (computer algebra system), if you have one, to solve the problem.

62. *Logistic Functions* Let $f(x) = c/(1 + ae^{-bx})$ with $a > 0$, $abc \neq 0$.

(a) Show that f is increasing on the interval $(-\infty, \infty)$ if $abc > 0$, and decreasing if $abc < 0$.

(b) Show that the point of inflection of f occurs at $x = (\ln|a|)/b$.

63. *Quartic Polynomial Functions* Let $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ with $a \neq 0$.

(a) Show that the graph of f has 0 or 2 points of inflection.

(b) Write a condition that must be satisfied by the coefficients if the graph of f has 0 or 2 points of inflection.

Quick Quiz for AP* Preparation: Sections 4.1–4.3

You should solve these problems without using a graphing calculator.

1. Multiple Choice How many critical points does the function $f(x) = (x-2)^5(x+3)^4$ have?

(A) One **(B)** Two **(C)** Three **(D)** Five **(E)** Nine

2. Multiple Choice For what value of x does the function $f(x) = (x-2)(x-3)^2$ have a relative maximum?

(A) -3 **(B)** $-\frac{7}{3}$ **(C)** $-\frac{5}{2}$ **(D)** $\frac{7}{3}$ **(E)** $\frac{5}{2}$

3. Multiple Choice If g is a differentiable function such that $g(x) < 0$ for all real numbers x, and if $f'(x) = (x^2 - 9)g(x)$, which of the following is true?

(A) f has a relative maximum at $x = -3$ and a relative minimum at $x = 3$.

(B) f has a relative minimum at $x = -3$ and a relative maximum at $x = 3$.

(C) f has relative minima at $x = -3$ and at $x = 3$.

(D) f has relative maxima at $x = -3$ and at $x = 3$.

(E) It cannot be determined if f has any relative extrema.

4. Free Response Let f be the function given by $f(x) = 3\ln(x^2 + 2) - 2x$ with domain $[-2, 4]$.

(a) Find the coordinate of each relative maximum point and each relative minimum point of f. Justify your answer.

(b) Find the x-coordinate of each point of inflection of the graph of f.

(c) Find the absolute maximum value of $f(x)$.

4.4 Modeling and Optimization

What you'll learn about

- Examples from Mathematics
- Examples from Business and Industry
- Examples from Economics
- Modeling Discrete Phenomena with Differentiable Functions

. . . and why

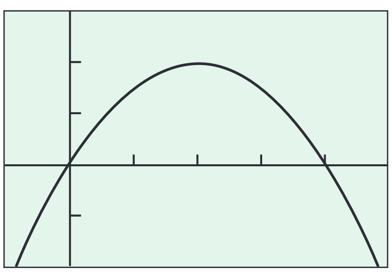
Historically, optimization problems were among the earliest applications of what we now call differential calculus.

Examples from Mathematics

While today's graphing technology makes it easy to find extrema without calculus, the algebraic methods of differentiation were understandably more practical, and certainly more accurate, when graphs had to be rendered by hand. Indeed, one of the oldest applications of what we now call "differential calculus" (pre-dating Newton and Leibniz) was to find maximum and minimum values of functions by finding where horizontal tangent lines might occur. We will use both algebraic and graphical methods in this section to solve "max-min" problems in a variety of contexts, but the emphasis will be on the *modeling* process that both methods have in common. Here is a strategy you can use:

Strategy for Solving Max-Min Problems

1. **Understand the Problem** Read the problem carefully. Identify the information you need to solve the problem.
2. **Develop a Mathematical Model of the Problem** Draw pictures and label the parts that are important to the problem. Introduce a variable to represent the quantity to be maximized or minimized. Using that variable, write a function whose extreme value gives the information sought.
3. **Graph the Function** Find the domain of the function. Determine what values of the variable make sense in the problem.
4. **Identify the Critical Points and Endpoints** Find where the derivative is zero or fails to exist.
5. **Solve the Mathematical Model** If unsure of the result, support or confirm your solution with another method.
6. **Interpret the Solution** Translate your mathematical result into the problem setting and decide whether the result makes sense.



[–5, 25] by [–100, 150]

Figure 4.35 The graph of $f(x) = x(20 - x)$ with domain $(-\infty, \infty)$ has an absolute maximum of 100 at $x = 10$. (Example 1)

EXAMPLE 1 Using the Strategy

Find two numbers whose sum is 20 and whose product is as large as possible.

SOLUTION

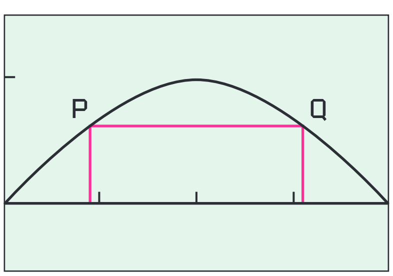
Model If one number is x, the other is $(20 - x)$, and their product is $f(x) = x(20 - x)$.

Solve Graphically We can see from the graph of f in Figure 4.35 that there is a maximum. From what we know about parabolas, the maximum occurs at $x = 10$.

Interpret The two numbers we seek are $x = 10$ and $20 - x = 10$.

Now try Exercise 1.

Sometimes we find it helpful to use both analytic and graphical methods together, as in Example 2.



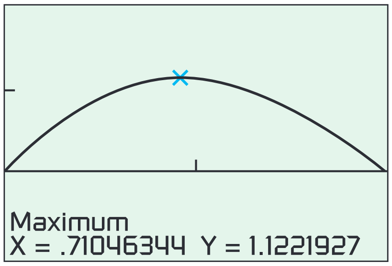
[0, π] by [–0.5, 1.5]

Figure 4.36 A rectangle inscribed under one arch of $y = \sin x$. (Example 2)

EXAMPLE 2 Inscribing Rectangles

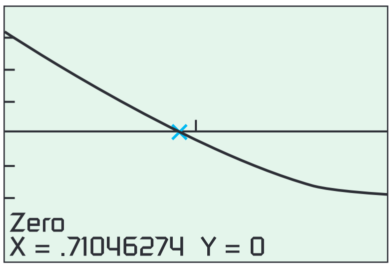
A rectangle is to be inscribed under one arch of the sine curve (Figure 4.36). What is the largest area the rectangle can have, and what dimensions give that area?

continued



[0, $\pi/2$] by [–1, 2]

(a)



[0, $\pi/2$] by [–4, 4]

(b)

Figure 4.37 The graph of (a) $A(x) = (\pi - 2x)\sin x$ and (b) A' in the interval $0 \le x \le \pi/2$. (Example 2)

SOLUTION

Model Let $(x, \sin x)$ be the coordinates of point P in Figure 4.36. From what we know about the sine function the x-coordinate of point Q is $(\pi - x)$. Thus,

$$\pi - 2x = \text{length of rectangle}$$

and

$$\sin x = \text{height of rectangle.}$$

The area of the rectangle is

$$A(x) = (\pi - 2x)\sin x.$$

Solve Analytically and Graphically We can assume that $0 \le x \le \pi/2$. Notice that $A = 0$ at the endpoints $x = 0$ and $x = \pi/2$. Since A is differentiable, the only critical points occur at the zeros of the first derivative,

$$A'(x) = -2\sin x + (\pi - 2x)\cos x.$$

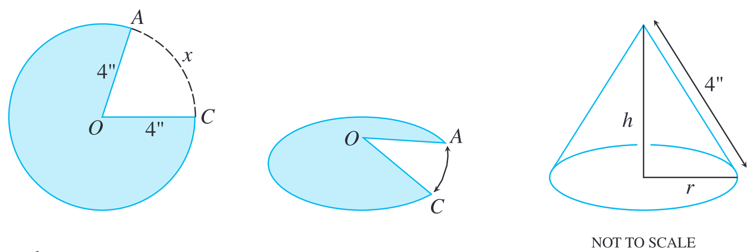
It is not possible to solve the equation $A'(x) = 0$ using algebraic methods. We can use the graph of A (Figure 4.37a) to find the maximum value and where it occurs. Or, we can use the graph of A' (Figure 4.37b) to find where the derivative is zero, and then evaluate A at this value of x to find the maximum value. The two x-values appear to be the same, as they should.

Interpret The rectangle has a maximum area of about 1.12 square units when $x \approx 0.71$. At this point, the rectangle is $\pi - 2x \approx 1.72$ units long by $\sin x \approx 0.65$ unit high.

Now try Exercise 5.

EXPLORATION 1 Constructing Cones

A cone of height h and radius r is constructed from a flat, circular disk of radius 4 in. by removing a sector AOC of arc length x in. and then connecting the edges OA and OC. What arc length x will produce the cone of maximum volume, and what is that volume?



1. Show that

$$r = \frac{8\pi - x}{2\pi}, \quad h = \sqrt{16 - r^2}, \quad \text{and}$$

$$V(x) = \frac{\pi}{3}\left(\frac{8\pi - x}{2\pi}\right)^2 \sqrt{16 - \left(\frac{8\pi - x}{2\pi}\right)^2}.$$

2. Show that the natural domain of V is $0 \le x \le 16\pi$. Graph V over this domain.
3. Explain why the restriction $0 \le x \le 8\pi$ makes sense in the problem situation. Graph V over this domain.
4. Use graphical methods to find where the cone has its maximum volume, and what that volume is.
5. Confirm your findings in part 4 analytically. [*Hint:* Use $V(x) = (1/3)\pi r^2 h$, $h^2 + r^2 = 16$, and the Chain Rule.]

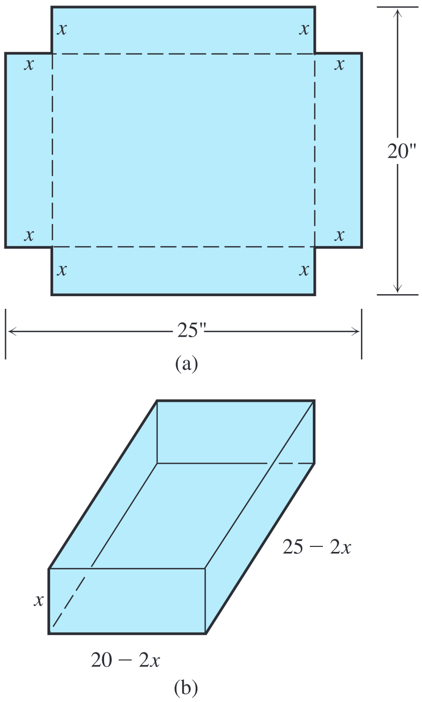
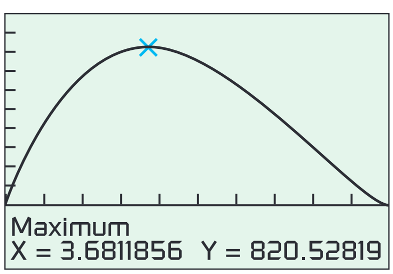


Figure 4.38 An open box made by cutting the corners from a piece of tin. (Example 3)

$y = x(20 - 2x)(25 - 2x)$



[0, 10] by [–300, 1000]

Figure 4.39 We chose the -300 in $-300 \le y \le 1000$ so that the coordinates of the local maximum at the bottom of the screen would not interfere with the graph. (Example 3)

Examples from Business and Industry

To *optimize* something means to maximize or minimize some aspect of it. What is the size of the most profitable production run? What is the least expensive shape for an oil can? What is the stiffest rectangular beam we can cut from a 12-inch log? We usually answer such questions by finding the greatest or smallest value of some function that we have used to model the situation.

EXAMPLE 3 Fabricating a Box

An open-top box is to be made by cutting congruent squares of side length x from the corners of a 20- by 25-inch sheet of tin and bending up the sides (Figure 4.38). How large should the squares be to make the box hold as much as possible? What is the resulting maximum volume?

SOLUTION

Model The height of the box is x, and the other two dimensions are $(20 - 2x)$ and $(25 - 2x)$. Thus, the volume of the box is

$$V(x) = x(20 - 2x)(25 - 2x).$$

Solve Graphically Because $2x$ cannot exceed 20, we have $0 \le x \le 10$. Figure 4.39 suggests that the maximum value of V is about 820.53 and occurs at $x \approx 3.68$.

Confirm Analytically Expanding, we obtain $V(x) = 4x^3 - 90x^2 + 500x$. The first derivative of V is

$$V'(x) = 12x^2 - 180x + 500.$$

The two solutions of the quadratic equation $V'(x) = 0$ are

$$c_1 = \frac{180 - \sqrt{180^2 - 48(500)}}{24} \approx 3.68 \qquad \text{and}$$

$$c_2 = \frac{180 + \sqrt{180^2 - 48(500)}}{24} \approx 11.32.$$

Only c_1 is in the domain $[0, 10]$ of V. The values of V at this one critical point and the two endpoints are

Critical point value: $V(c_1) \approx 820.53$

Endpoint values: $V(0) = 0, \quad V(10) = 0.$

Interpret Cutout squares that are about 3.68 in. on a side give the maximum volume, about 820.53 in^3.

Now try Exercise 7.

EXAMPLE 4 Designing a Can

You have been asked to design a one-liter oil can shaped like a right circular cylinder (see Figure 4.40 on the next page). What dimensions will use the least material?

continued

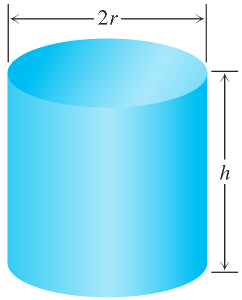
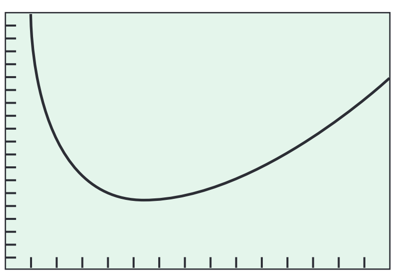


Figure 4.40 This one-liter can uses the least material when $h = 2r$. (Example 4)



[0, 15] by [0, 2000]

Figure 4.41 The graph of $A = 2\pi r^2 + 2000/r$, $r > 0$. (Example 4)

SOLUTION

Volume of can: If r and h are measured in centimeters, then the volume of the can in cubic centimeters is

$$\pi r^2 h = 1000.$$

Surface area of can: $A = \underbrace{2\pi r^2} + \underbrace{2\pi rh}$

How can we interpret the phrase "least material"? One possibility is to ignore the thickness of the material and the waste in manufacturing. Then we ask for dimensions r and h that make the total surface area as small as possible while satisfying the constraint $\pi r^2 h = 1000$. (Exercise 17 describes one way to take waste into account.)

Model To express the surface area as a function of one variable, we solve for one of the variables in $\pi r^2 h = 1000$ and substitute that expression into the surface area formula. Solving for h is easier,

$$h = \frac{1000}{\pi r^2}.$$

Thus,

$$\begin{aligned} A &= 2\pi r^2 + 2\pi rh \\ &= 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right) \\ &= 2\pi r^2 + \frac{2000}{r}. \end{aligned}$$

Solve Analytically Our goal is to find a value of $r > 0$ that minimizes the value of A. Figure 4.41 suggests that such a value exists.

Notice from the graph that for small r (a tall thin container, like a piece of pipe), the term $2000/r$ dominates and A is large. For large r (a short wide container, like a pizza pan), the term $2\pi r^2$ dominates and A again is large.

Since A is differentiable on $r > 0$, an interval with no endpoints, it can have a minimum value only where its first derivative is zero.

$$\begin{aligned} \frac{dA}{dr} &= 4\pi r - \frac{2000}{r^2} \\ 0 &= 4\pi r - \frac{2000}{r^2} \\ 4\pi r^3 &= 2000 \\ r &= \sqrt[3]{\frac{500}{\pi}} \approx 5.42 \end{aligned}$$

Something happens at $r = \sqrt[3]{500/\pi}$, but what?

If the domain of A were a closed interval, we could find out by evaluating A at this critical point and the endpoints and comparing the results. But the domain is an open interval, so we must learn what is happening at $r = \sqrt[3]{500/\pi}$ by referring to the shape of A's graph. The second derivative

$$\frac{d^2A}{dr^2} = 4\pi + \frac{4000}{r^3}$$

is positive throughout the domain of A. The graph is therefore concave up and the value of A at $r = \sqrt[3]{500/\pi}$ an absolute minimum.

continued

The corresponding value of h (after a little algebra) is

$$h = \frac{1000}{\pi r^2} = 2\sqrt[3]{\frac{500}{\pi}} = 2r.$$

Interpret The one-liter can that uses the least material has height equal to the diameter, with $r \approx 5.42$ cm and $h \approx 10.84$ cm. ***Now try Exercise 11.***

Examples from Economics

Here we want to point out two more places where calculus makes a contribution to economic theory. The first has to do with maximizing profit. The second has to do with minimizing average cost.

Suppose that

$$r(x) = \text{the revenue from selling } x \text{ items},$$
$$c(x) = \text{the cost of producing the } x \text{ items},$$
$$p(x) = r(x) - c(x) = \text{the profit from selling } x \text{ items}.$$

The marginal revenue, marginal cost, and marginal profit at this production level (x items) are

$$\frac{dr}{dx} = \text{marginal revenue}, \qquad \frac{dc}{dx} = \text{marginal cost}, \qquad \frac{dp}{dx} = \text{marginal profit}.$$

The first observation is about the relationship of p to these derivatives.

Marginal Analysis

Because differentiable functions are locally linear, we can use the marginals to approximate the extra revenue, cost, or profit resulting from selling or producing one more item. Using these approximations is referred to as *marginal analysis*.

THEOREM 6 Maximum Profit

Maximum profit (if any) occurs at a production level at which marginal revenue equals marginal cost.

Proof We assume that $r(x)$ and $c(x)$ are differentiable for all $x > 0$, so if $p(x) = r(x) - c(x)$ has a maximum value, it occurs at a production level at which $p'(x) = 0$. Since $p'(x) = r'(x) - c'(x)$, $p'(x) = 0$ implies that

$$r'(x) - c'(x) = 0 \quad \text{or} \quad r'(x) = c'(x). \qquad \blacksquare$$

Figure 4.42 gives more information about this situation.

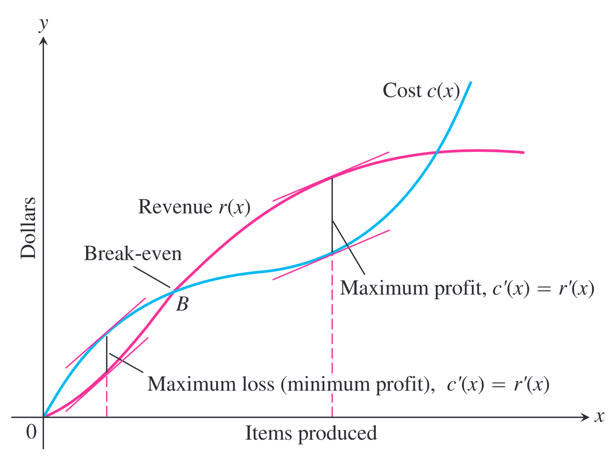


Figure 4.42 The graph of a typical cost function starts concave down and later turns concave up. It crosses the revenue curve at the break-even point B. To the left of B, the company operates at a loss. To the right, the company operates at a profit, the maximum profit occurring where $r'(x) = c'(x)$. Farther to the right, cost exceeds revenue (perhaps because of a combination of market saturation and rising labor and material costs) and production levels become unprofitable again.

What guidance do we get from this observation? We know that a production level at which $p'(x) = 0$ need not be a level of maximum profit. It might be a level of minimum profit, for example. But if we are making financial projections for our company, we should look for production levels at which marginal cost seems to equal marginal revenue. If there is a most profitable production level, it will be one of these.

EXAMPLE 5 Maximizing Profit

Suppose that $r(x) = 9x$ and $c(x) = x^3 - 6x^2 + 15x$, where x represents thousands of units. Is there a production level that maximizes profit? If so, what is it?

SOLUTION

Notice that $r'(x) = 9$ and $c'(x) = 3x^2 - 12x + 15$.

$$3x^2 - 12x + 15 = 9$$

$$3x^2 - 12x + 6 = 0$$

The two solutions of the quadratic equation are

$$x_1 = \frac{12 - \sqrt{72}}{6} = 2 - \sqrt{2} \approx 0.586 \quad \text{and}$$

$$x_2 = \frac{12 + \sqrt{72}}{6} = 2 + \sqrt{2} \approx 3.414.$$

The possible production levels for maximum profit are $x \approx 0.586$ thousand units or $x \approx 3.414$ thousand units. The graphs in Figure 4.43 show that maximum profit occurs at about $x = 3.414$ and maximum loss occurs at about $x = 0.586$.

Another way to look for optimal production levels is to look for levels that minimize the average cost of the units produced. Theorem 7 helps us find them. ***Now try Exercise 23.***

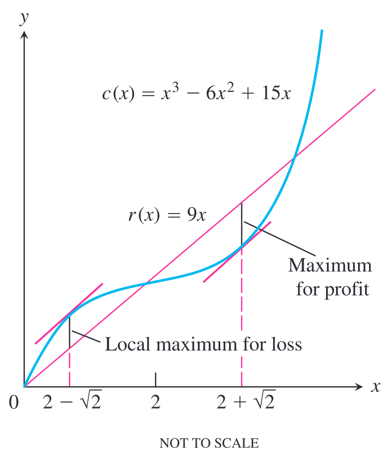


Figure 4.43 The cost and revenue curves for Example 5.

THEOREM 7 Minimizing Average Cost

The production level (if any) at which average cost is smallest is a level at which the average cost equals the marginal cost.

Proof We assume that $c(x)$ is differentiable.

$$c(x) = \text{cost of producing } x \text{ items}, \quad x > 0$$

$$\frac{c(x)}{x} = \text{average cost of producing } x \text{ items}$$

If the average cost can be minimized, it will be a production level at which

$$\frac{d}{dx}\left(\frac{c(x)}{x}\right) = 0$$

$$\frac{xc'(x) - c(x)}{x^2} = 0$$

$$xc'(x) - c(x) = 0$$

$$\underbrace{c'(x)} = \underbrace{\frac{c(x)}{x}}.$$

■

Again we have to be careful about what Theorem 7 does and does not say. It does not say that there is a production level of minimum average cost—it says where to look to see if there is one. Look for production levels at which average cost and marginal cost are equal. Then check to see if any of them gives a minimum average cost.

EXAMPLE 6 Minimizing Average Cost

Suppose $c(x) = x^3 - 6x^2 + 15x$, where x represents thousands of units. Is there a production level that minimizes average cost? If so, what is it?

SOLUTION

We look for levels at which average cost equals marginal cost.

$$\text{Marginal cost:} \quad c'(x) = 3x^2 - 12x + 15$$

$$\text{Average cost:} \quad \frac{c(x)}{x} = x^2 - 6x + 15$$

$$3x^2 - 12x + 15 = x^2 - 6x + 15$$

$$2x^2 - 6x = 0$$

$$2x(x - 3) = 0$$

$$x = 0 \quad \text{or} \quad x = 3$$

Since $x > 0$, the only production level that might minimize average cost is $x = 3$ thousand units.

We use the second derivative test.

$$\frac{d}{dx}\left(\frac{c(x)}{x}\right) = 2x - 6$$

$$\frac{d^2}{dx^2}\left(\frac{c(x)}{x}\right) = 2 > 0$$

The second derivative is positive for all $x > 0$, so $x = 3$ gives an absolute minimum. ***Now try Exercise 25.***

Modeling Discrete Phenomena with Differentiable Functions

In case you are wondering how we can use differentiable functions $c(x)$ and $r(x)$ to describe the cost and revenue that comes from producing a number of items x that can only be an integer, here is the rationale.

When x is large, we can reasonably fit the cost and revenue data with smooth curves $c(x)$ and $r(x)$ that are defined not only at integer values of x but at the values in between just as we do when we use regression equations. Once we have these differentiable functions, which are supposed to behave like the real cost and revenue when x is an integer, we can apply calculus to draw conclusions about their values. We then translate these mathematical conclusions into inferences about the real world that we hope will have predictive value. When they do, as is the case with the economic theory here, we say that the functions give a good model of reality.

What do we do when our calculus tells us that the best production level is a value of x that isn't an integer, as it did in Example 5? We use the nearest convenient integer. For $x \approx 3.414$ thousand units in Example 5, we might use 3414, or perhaps 3410 or 3420 if we ship in boxes of 10.

Quick Review 4.4 *(For help, go to Sections 1.6, 4.1, and Appendix A.1.)*

1. Use the first derivative test to identify the local extrema of $y = x^3 - 6x^2 + 12x - 8$.

2. Use the second derivative test to identify the local extrema of $y = 2x^3 + 3x^2 - 12x - 3$.

3. Find the volume of a cone with radius 5 cm and height 8 cm.

4. Find the dimensions of a right circular cylinder with volume 1000 cm^3 and surface area 600 cm^2.

In Exercises 5–8, rewrite the expression as a trigonometric function of the angle α.

5. $\sin(-\alpha)$

6. $\cos(-\alpha)$

7. $\sin(\pi - \alpha)$

8. $\cos(\pi - \alpha)$

In Exercises 9 and 10, use substitution to find the exact solutions of the system of equations.

9. $$\begin{cases} x^2 + y^2 = 4 \\ y = \sqrt{3}x \end{cases}$$

10. $$\begin{cases} \dfrac{x^2}{4} + \dfrac{y^2}{9} = 1 \\ y = x + 3 \end{cases}$$

Section 4.4 Exercises

In Exercises 1–10, solve the problem analytically. Support your answer graphically.

1. *Finding Numbers* The sum of two nonnegative numbers is 20. Find the numbers if

(a) the sum of their squares is as large as possible; as small as possible.

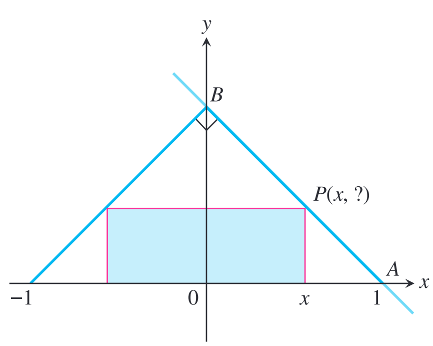
(b) one number plus the square root of the other is as large as possible; as small as possible.

2. *Maximizing Area* What is the largest possible area for a right triangle whose hypotenuse is 5 cm long, and what are its dimensions?

3. *Maximizing Perimeter* What is the smallest perimeter possible for a rectangle whose area is 16 in^2, and what are its dimensions?

4. *Finding Area* Show that among all rectangles with an 8-m perimeter, the one with largest area is a square.

5. *Inscribing Rectangles* The figure shows a rectangle inscribed in an isosceles right triangle whose hypotenuse is 2 units long.



(a) Express the y-coordinate of P in terms of x. [*Hint:* Write an equation for the line AB.]

(b) Express the area of the rectangle in terms of x.

(c) What is the largest area the rectangle can have, and what are its dimensions?

6. *Largest Rectangle* A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have, and what are its dimensions?

7. *Optimal Dimensions* You are planning to make an open rectangular box from an 8- by 15-in. piece of cardboard by cutting congruent squares from the corners and folding up the sides. What are the dimensions of the box of largest volume you can make this way, and what is its volume?

8. *Closing Off the First Quadrant* You are planning to close off a corner of the first quadrant with a line segment 20 units long running from $(a, 0)$ to $(0, b)$. Show that the area of the triangle enclosed by the segment is largest when $a = b$.

9. *The Best Fencing Plan* A rectangular plot of farmland will be bounded on one side by a river and on the other three sides by a single-strand electric fence. With 800 m of wire at your disposal, what is the largest area you can enclose, and what are its dimensions?

10. *The Shortest Fence* A 216-m^2 rectangular pea patch is to be enclosed by a fence and divided into two equal parts by another fence parallel to one of the sides. What dimensions for the outer rectangle will require the smallest total length of fence? How much fence will be needed?

11. *Designing a Tank* Your iron works has contracted to design and build a 500-ft^3, square-based, open-top, rectangular steel holding tank for a paper company. The tank is to be made by welding thin stainless steel plates together along their edges. As the production engineer, your job is to find dimensions for the base and height that will make the tank weigh as little as possible.

(a) What dimensions do you tell the shop to use?

(b) Writing to Learn Briefly describe how you took weight into account.

12. *Catching Rainwater* A 1125-ft^3 open-top rectangular tank with a square base x ft on a side and y ft deep is to be built with its top flush with the ground to catch runoff water. The costs associated with the tank involve not only the material from which the tank is made but also an excavation charge proportional to the product xy.

(a) If the total cost is

$$c = 5(x^2 + 4xy) + 10xy,$$

what values of x and y will minimize it?

(b) Writing to Learn Give a possible scenario for the cost function in (a).

13. *Designing a Poster* You are designing a rectangular poster to contain 50 in^2 of printing with a 4-in. margin at the top and bottom and a 2-in. margin at each side. What overall dimensions will minimize the amount of paper used?

14. *Vertical Motion* The height of an object moving vertically is given by

$$s = -16t^2 + 96t + 112,$$

with s in ft and t in sec. Find **(a)** the object's velocity when $t = 0$, **(b)** its maximum height and when it occurs, and **(c)** its velocity when $s = 0$.

15. *Finding an Angle* Two sides of a triangle have lengths a and b, and the angle between them is θ. What value of θ will maximize the triangle's area? [*Hint:* $A = (1/2)\, ab \sin \theta$.]

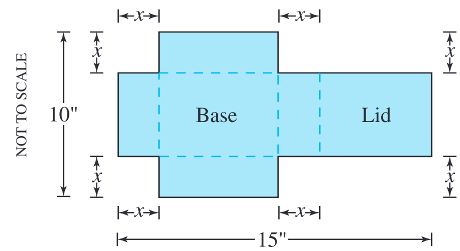
16. *Designing a Can* What are the dimensions of the lightest open-top right circular cylindrical can that will hold a volume of 1000 cm^3? Compare the result here with the result in Example 4.

17. *Designing a Can* You are designing a 1000-cm^3 right circular cylindrical can whose manufacture will take waste into account. There is no waste in cutting the aluminum for the side, but the top and bottom of radius r will be cut from squares that measure $2r$ units on a side. The total amount of aluminum used up by the can will therefore be

$$A = 8r^2 + 2\pi rh$$

rather than the $A = 2\pi r^2 + 2\pi rh$ in Example 4. In Example 4 the ratio of h to r for the most economical can was 2 to 1. What is the ratio now?

18. *Designing a Box with Lid* A piece of cardboard measures 10- by 15-in. Two equal squares are removed from the corners of a 10-in. side as shown in the figure. Two equal rectangles are removed from the other corners so that the tabs can be folded to form a rectangular box with lid.



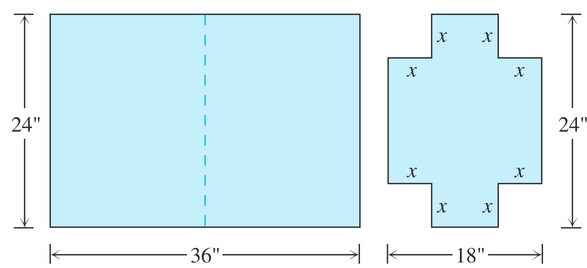
(a) Write a formula $V(x)$ for the volume of the box.

(b) Find the domain of V for the problem situation and graph V over this domain.

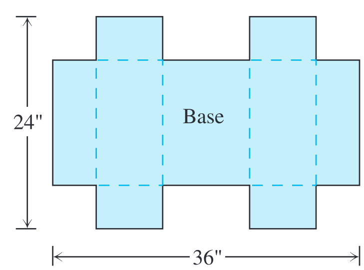
(c) Use a graphical method to find the maximum volume and the value of x that gives it.

(d) Confirm your result in part (c) analytically.

19. *Designing a Suitcase* A 24- by 36-in. sheet of cardboard is folded in half to form a 24- by 18-in. rectangle as shown in the figure. Then four congruent squares of side length x are cut from the corners of the folded rectangle. The sheet is unfolded, and the six tabs are folded up to form a box with sides and a lid.



The sheet is then unfolded.



(a) Write a formula $V(x)$ for the volume of the box.

(b) Find the domain of V for the problem situation and graph V over this domain.

(c) Use a graphical method to find the maximum volume and the value of x that gives it.

(d) Confirm your result in part (c) analytically.

(e) Find a value of x that yields a volume of 1120 in^3.

(f) Writing to Learn Write a paragraph describing the issues that arise in part (b).

20. *Quickest Route* Jane is 2 mi offshore in a boat and wishes to reach a coastal village 6 mi down a straight shoreline from the point nearest the boat. She can row 2 mph and can walk 5 mph. Where should she land her boat to reach the village in the least amount of time?

21. *Inscribing Rectangles* A rectangle is to be inscribed under the arch of the curve $y = 4\cos(0.5x)$ from $x = -\pi$ to $x = \pi$. What are the dimensions of the rectangle with largest area, and what is the largest area?

22. *Maximizing Volume* Find the dimensions of a right circular cylinder of maximum volume that can be inscribed in a sphere of radius 10 cm. What is the maximum volume?

23. *Maximizing Profit* Suppose $r(x) = 8\sqrt{x}$ represents revenue and $c(x) = 2x^2$ represents cost, with x measured in thousands of units. Is there a production level that maximizes profit? If so, what is it?

24. *Maximizing Profit* Suppose $r(x) = x^2/(x^2 + 1)$ represents revenue and $c(x) = (x - 1)^3/3 - 1/3$ represents cost, with x measured in thousands of units. Is there a production level that maximizes profit? If so, what is it?

25. *Minimizing Average Cost* Suppose $c(x) = x^3 - 10x^2 - 30x$, where x is measured in thousands of units. Is there a production level that minimizes average cost? If so, what is it?

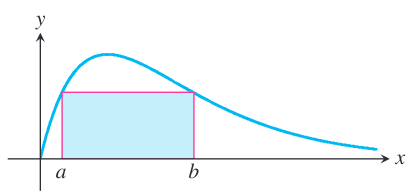
26. *Minimizing Average Cost* Suppose $c(x) = xe^x - 2x^2$, where x is measured in thousands of units. Is there a production level that minimizes average cost? If so, what is it?

27. *Tour Service* You operate a tour service that offers the following rates:

- \$200 per person if 50 people (the minimum number to book the tour) go on the tour.
- For each additional person, up to a maximum of 80 people total, the rate per person is reduced by \$2.

It costs \$6000 (a fixed cost) plus \$32 per person to conduct the tour. How many people does it take to maximize your profit?

28. *Group Activity* The figure shows the graph of $f(x) = xe^{-x}$, $x \geq 0$.



(a) Find where the absolute maximum of f occurs.

(b) Let $a > 0$ and $b > 0$ be given as shown in the figure. Complete the following table where A is the area of the rectangle in the figure.

a	b	A
0.1		
0.2		
0.3		
⋮		
1		

(c) Draw a scatter plot of the data (a, A).

(d) Find the quadratic, cubic, and quartic regression equations for the data in part (b), and superimpose their graphs on a scatter plot of the data.

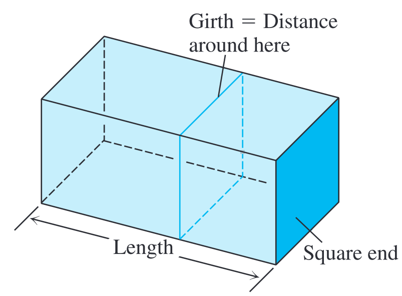
(e) Use each of the regression equations in part (d) to estimate the maximum possible value of the area of the rectangle.

29. *Cubic Polynomial Functions*
Let $f(x) = ax^3 + bx^2 + cx + d,\ a \neq 0$.

(a) Show that f has either 0 or 2 local extrema.

(b) Give an example of each possibility in part (a).

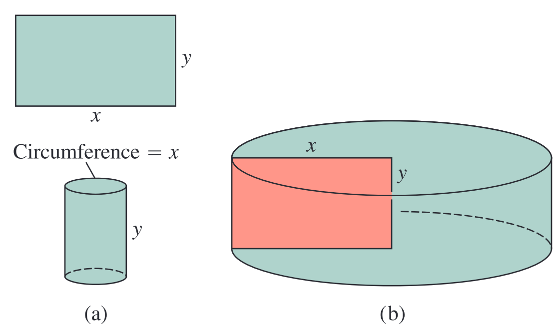
30. *Shipping Packages* The U.S. Postal Service will accept a box for domestic shipment only if the sum of its length and girth (distance around), as shown in the figure, does not exceed 108 in. What dimensions will give a box with a square end the largest possible volume?



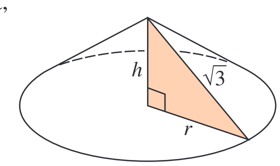
31. *Constructing Cylinders* Compare the answers to the following two construction problems.

(a) A rectangular sheet of perimeter 36 cm and dimensions x cm by y cm is to be rolled into a cylinder as shown in part (a) of the figure. What values of x and y give the largest volume?

(b) The same sheet is to be revolved about one of the sides of length y to sweep out the cylinder as shown in part (b) of the figure. What values of x and y give the largest volume?



32. *Constructing Cones* A right triangle whose hypotenuse is $\sqrt{3}$ m long is revolved about one of its legs to generate a right circular cone. Find the radius, height, and volume of the cone of greatest volume that can be made this way.

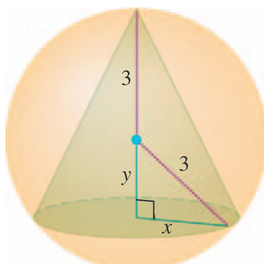


33. ***Finding Parameter Values*** What value of a makes $f(x) = x^2 + (a/x)$ have **(a)** a local minimum at $x = 2$? **(b)** a point of inflection at $x = 1$?

34. ***Finding Parameter Values*** Show that $f(x) = x^2 + (a/x)$ cannot have a local maximum for any value of a.

35. ***Finding Parameter Values*** What values of a and b make $f(x) = x^3 + ax^2 + bx$ have **(a)** a local maximum at $x = -1$ and a local minimum at $x = 3$? **(b)** a local minimum at $x = 4$ and a point of inflection at $x = 1$?

36. ***Inscribing a Cone*** Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3.

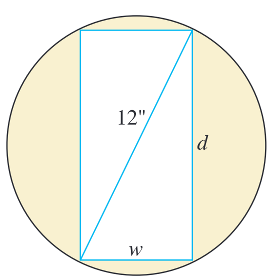


37. ***Strength of a Beam*** The strength S of a rectangular wooden beam is proportional to its width times the square of its depth.

(a) Find the dimensions of the strongest beam that can be cut from a 12-in. diameter cylindrical log.

(b) **Writing to Learn** Graph S as a function of the beam's width w, assuming the proportionality constant to be $k = 1$. Reconcile what you see with your answer in part (a).

(c) **Writing to Learn** On the same screen, graph S as a function of the beam's depth d, again taking $k = 1$. Compare the graphs with one another and with your answer in part (a). What would be the effect of changing to some other value of k? Try it.



38. ***Stiffness of a Beam*** The stiffness S of a rectangular beam is proportional to its width times the cube of its depth.

(a) Find the dimensions of the stiffest beam that can be cut from a 12-in. diameter cylindrical log.

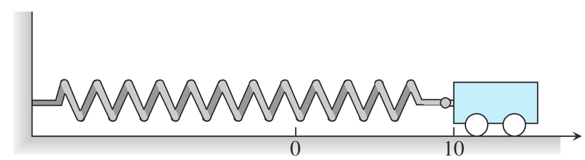
(b) **Writing to Learn** Graph S as a function of the beam's width w, assuming the proportionality constant to be $k = 1$. Reconcile what you see with your answer in part (a).

(c) **Writing to Learn** On the same screen, graph S as a function of the beam's depth d, again taking $k = 1$. Compare the graphs with one another and with your answer in part (a). What would be the effect of changing to some other value of k? Try it.

39. ***Frictionless Cart*** A small frictionless cart, attached to the wall by a spring, is pulled 10 cm from its rest position and released at time $t = 0$ to roll back and forth for 4 sec. Its position at time t is $s = 10 \cos \pi t$.

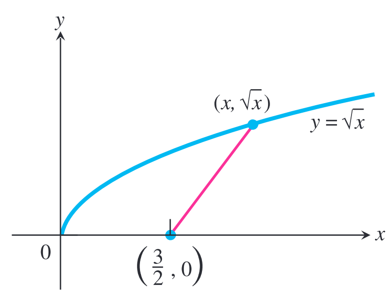
(a) What is the cart's maximum speed? When is the cart moving that fast? Where is it then? What is the magnitude of the acceleration then?

(b) Where is the cart when the magnitude of the acceleration is greatest? What is the cart's speed then?



40. ***Electrical Current*** Suppose that at any time t (sec) the current i (amp) in an alternating current circuit is $i = 2 \cos t + 2 \sin t$. What is the peak (largest magnitude) current for this circuit?

41. ***Calculus and Geometry*** How close does the curve $y = \sqrt{x}$ come to the point $(3/2, 0)$? [*Hint:* If you minimize the *square* of the distance, you can avoid square roots.]



42. ***Calculus and Geometry*** How close does the semicircle $y = \sqrt{16 - x^2}$ come to the point $(1, \sqrt{3})$?

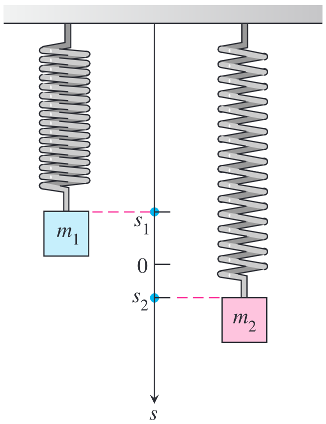
43. **Writing to Learn** Is the function $f(x) = x^2 - x + 1$ ever negative? Explain.

44. **Writing to Learn** You have been asked to determine whether the function $f(x) = 3 + 4 \cos x + \cos 2x$ is ever negative.

(a) Explain why you need to consider values of x only in the interval $[0, 2\pi]$.

(b) Is f ever negative? Explain.

45. ***Vertical Motion*** Two masses hanging side by side from springs have positions $s_1 = 2 \sin t$ and $s_2 = \sin 2t$, respectively, with s_1 and s_2 in meters and t in seconds.



(a) At what times in the interval $t > 0$ do the masses pass each other? [*Hint:* $\sin 2t = 2 \sin t \cos t$.]

(b) When in the interval $0 \le t \le 2\pi$ is the vertical distance between the masses the greatest? What is this distance? (*Hint:* $\cos 2t = 2\cos^2 t - 1$.)

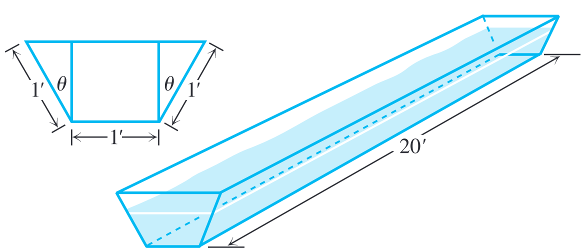
46. ***Motion on a Line*** The positions of two particles on the s-axis are $s_1 = \sin t$ and $s_2 = \sin(t + \pi/3)$, with s_1 and s_2 in meters and t in seconds.

(a) At what time(s) in the interval $0 \le t \le 2\pi$ do the particles meet?

(b) What is the farthest apart that the particles ever get?

(c) When in the interval $0 \le t \le 2\pi$ is the distance between the particles changing the fastest?

47. ***Finding an Angle*** The trough in the figure is to be made to the dimensions shown. Only the angle θ can be varied. What value of θ will maximize the trough's volume?

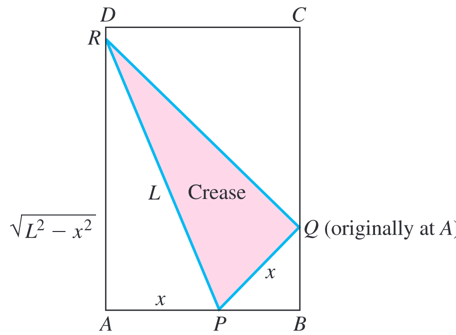


48. **Group Activity** ***Paper Folding*** A rectangular sheet of 8 1/2- by 11-in. paper is placed on a flat surface. One of the corners is placed on the opposite longer edge, as shown in the figure, and held there as the paper is smoothed flat. The problem is to make the length of the crease as small as possible. Call the length L. Try it with paper.

(a) Show that $L^2 = 2x^3/(2x - 8.5)$.

(b) What value of x minimizes L^2?

(c) What is the minimum value of L?



49. ***Sensitivity to Medicine*** *(continuation of Exercise 48, Section 3.3)* Find the amount of medicine to which the body is most sensitive by finding the value of M that maximizes the derivative dR/dM.

50. ***Selling Backpacks*** It costs you c dollars each to manufacture and distribute backpacks. If the backpacks sell at x dollars each, the number sold is given by

$$n = \frac{a}{x - c} + b(100 - x),$$

where a and b are certain positive constants. What selling price will bring a maximum profit?

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

51. **True or False** A continuous function on a closed interval must attain a maximum value on that interval. Justify your answer.

52. **True or False** If $f'(c) = 0$ and $f(c)$ is not a local maximum, then $f(c)$ is a local minimum. Justify your answer.

53. **Multiple Choice** Two positive numbers have a sum of 60. What is the maximum product of one number times the square of the second number?

(A) 3481

(B) 3600

(C) 27,000

(D) 32,000

(E) 36,000

54. **Multiple Choice** A continuous function f has domain [1, 25] and range [3, 30]. If $f'(x) < 0$ for all x between 1 and 25, what is $f(25)$?

(A) 1

(B) 3

(C) 25

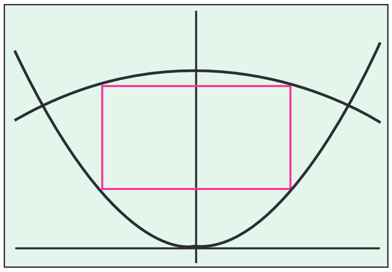
(D) 30

(E) impossible to determine from the information given

55. Multiple Choice What is the maximum area of a right triangle with hypotenuse 10?

(A) 24 **(B)** 25 **(C)** $25\sqrt{2}$ **(D)** 48 **(E)** 50

56. Multiple Choice A rectangle is inscribed between the parabolas $y = 4x^2$ and $y = 30 - x^2$ as shown below:



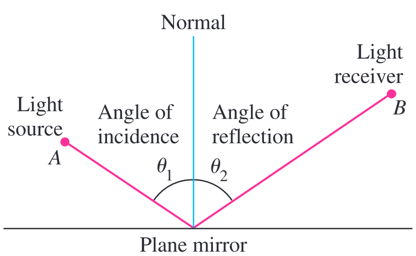
[–3, 3] by [–2, 40]

What is the maximum area of such a rectangle ?

(A) $20\sqrt{2}$ **(B)** 40 **(C)** $30\sqrt{2}$ **(D)** 50 **(E)** $40\sqrt{2}$

Explorations

57. *Fermat's Principle in Optics* Fermat's principle in optics states that light always travels from one point to another along a path that minimizes the travel time. Light from a source A is reflected by a plane mirror to a receiver at point B, as shown in the figure. Show that for the light to obey Fermat's principle, the angle of incidence must equal the angle of reflection, both measured from the line normal to the reflecting surface. (This result can also be derived without calculus. There is a purely geometric argument, which you may prefer.)



58. *Tin Pest* When metallic tin is kept below 13.2°C, it slowly becomes brittle and crumbles to a gray powder. Tin objects eventually crumble to this gray powder spontaneously if kept in a cold climate for years. The Europeans who saw tin organ pipes in their churches crumble away years ago called the change *tin pest* because it seemed to be contagious. And indeed it was, for the gray powder is a catalyst for its own formation.

A *catalyst* for a chemical reaction is a substance that controls the rate of reaction without undergoing any permanent change in itself. An *autocatalytic reaction* is one whose product is a catalyst for its own formation. Such a reaction may proceed slowly at first if the amount of catalyst present is small and slowly again at the end, when most of the original substance is used up. But in between, when both the substance and its catalyst product are abundant, the reaction proceeds at a faster pace.

In some cases it is reasonable to assume that the rate $v = dx/dt$ of the reaction is proportional both to the amount of the original substance present and to the amount of product. That is, v may be considered to be a function of x alone, and

$$v = kx(a - x) = kax - kx^2,$$

where

x = the amount of product,

a = the amount of substance at the beginning,

k = a positive constant.

At what value of x does the rate v have a maximum? What is the maximum value of v?

59. *How We Cough* When we cough, the trachea (windpipe) contracts to increase the velocity of the air going out. This raises the question of how much it should contract to maximize the velocity and whether it really contracts that much when we cough.

Under reasonable assumptions about the elasticity of the tracheal wall and about how the air near the wall is slowed by friction, the average flow velocity v (in cm/sec) can be modeled by the equation

$$v = c(r_0 - r)r^2, \quad \frac{r_0}{2} \le r \le r_0,$$

where r_0 is the rest radius of the trachea in cm and c is a positive constant whose value depends in part on the length of the trachea.

(a) Show that v is greatest when $r = (2/3)r_0$, that is, when the trachea is about 33% contracted. The remarkable fact is that X-ray photographs confirm that the trachea contracts about this much during a cough.

(b) Take r_0 to be 0.5 and c to be 1, and graph v over the interval $0 \le r \le 0.5$. Compare what you see to the claim that v is a maximum when $r = (2/3)r_0$.

60. *Wilson Lot Size Formula* One of the formulas for inventory management says that the average weekly cost of ordering, paying for, and holding merchandise is

$$A(q) = \frac{km}{q} + cm + \frac{hq}{2},$$

where q is the quantity you order when things run low (shoes, radios, brooms, or whatever the item might be), k is the cost of placing an order (the same, no matter how often you order), c is the cost of one item (a constant), m is the number of items sold each week (a constant), and h is the weekly holding cost per item (a constant that takes into account things such as space, utilities, insurance, and security).

(a) Your job, as the inventory manager for your store, is to find the quantity that will minimize $A(q)$. What is it? (The formula you get for the answer is called the *Wilson lot size formula.*)

(b) Shipping costs sometimes depend on order size. When they do, it is more realistic to replace k by $k + bq$, the sum of k and a constant multiple of q. What is the most economical quantity to order now?

61. ***Production Level*** Show that if $r(x) = 6x$ and $c(x) = x^3 - 6x^2 + 15x$ are your revenue and cost functions, then the best you can do is break even (have revenue equal cost).

62. ***Production Level*** Suppose $c(x) = x^3 - 20x^2 + 20{,}000x$ is the cost of manufacturing x items. Find a production level that will minimize the average cost of making x items.

Extending the Ideas

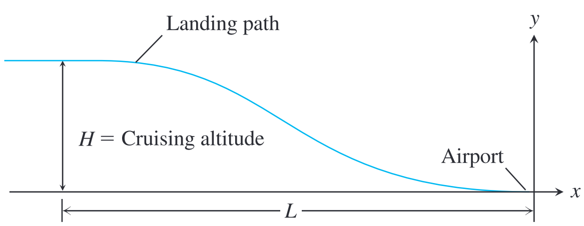
63. ***Airplane Landing Path*** An airplane is flying at altitude H when it begins its descent to an airport runway that is at horizontal ground distance L from the airplane, as shown in the figure. Assume that the landing path of the airplane is the graph of a cubic polynomial function $y = ax^3 + bx^2 + cx + d$ where $y(-L) = H$ and $y(0) = 0$.

(a) What is dy/dx at $x = 0$?

(b) What is dy/dx at $x = -L$?

(c) Use the values for dy/dx at $x = 0$ and $x = -L$ together with $y(0) = 0$ and $y(-L) = H$ to show that

$$y(x) = H\left[2\left(\frac{x}{L}\right)^3 + 3\left(\frac{x}{L}\right)^2\right].$$



In Exercises 64 and 65, you might find it helpful to use a CAS.

64. ***Generalized Cone Problem*** A cone of height h and radius r is constructed from a flat, circular disk of radius a in. as described in Exploration 1.

(a) Find a formula for the volume V of the cone in terms of x and a.

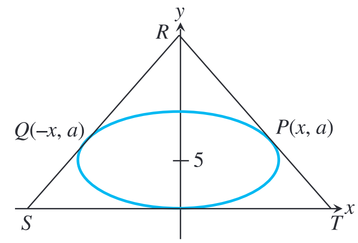
(b) Find r and h in the cone of maximum volume for $a = 4, 5, 6, 8$.

(c) **Writing to Learn** Find a simple relationship between r and h that is independent of a for the cone of maximum volume. Explain how you arrived at your relationship.

65. ***Circumscribing an Ellipse*** Let $P(x, a)$ and $Q(-x, a)$ be two points on the upper half of the ellipse

$$\frac{x^2}{100} + \frac{(y-5)^2}{25} = 1$$

centered at (0, 5). A triangle RST is formed by using the tangent lines to the ellipse at Q and P as shown in the figure.



(a) Show that the area of the triangle is

$$A(x) = -f'(x)\left[x - \frac{f(x)}{f'(x)}\right]^2,$$

where $y = f(x)$ is the function representing the upper half of the ellipse.

(b) What is the domain of A? Draw the graph of A. How are the asymptotes of the graph related to the problem situation?

(c) Determine the height of the triangle with minimum area. How is it related to the y-coordinate of the center of the ellipse?

(d) Repeat parts (a)–(c) for the ellipse

$$\frac{x^2}{C^2} + \frac{(y-B)^2}{B^2} = 1$$

centered at $(0, B)$. Show that the triangle has minimum area when its height is $3B$.

4.5 Linearization and Newton's Method

What you'll learn about

- Linear Approximation
- Newton's Method
- Differentials
- Estimating Change with Differentials
- Absolute, Relative, and Percentage Change
- Sensitivity to Change

. . . and why

Engineering and science depend on approximations in most practical applications; it is important to understand how approximation techniques work.

Linear Approximation

In our study of the derivative we have frequently referred to the "tangent line to the curve" at a point. What makes that tangent line so important mathematically is that it provides a *useful representation of the curve itself* if we stay close enough to the point of tangency. We say that differentiable curves are always **locally linear,** a fact that can best be appreciated graphically by zooming in at a point on the curve, as Exploration 1 shows.

EXPLORATION 1 Appreciating Local Linearity

The function $f(x) = (x^2 + 0.0001)^{1/4} + 0.9$ is differentiable at $x = 0$ and hence "locally linear" there. Let us explore the significance of this fact with the help of a graphing calculator.

1. Graph $y = f(x)$ in the "ZoomDecimal" window. What appears to be the behavior of the function at the point (0, 1)?

2. Show algebraically that f is differentiable at $x = 0$. What is the equation of the tangent line at (0, 1)?

3. Now zoom in repeatedly, keeping the cursor at (0, 1). What is the long-range outcome of repeated zooming?

4. The graph of $y = f(x)$ eventually looks like the graph of a line. What line is it?

We hope that this exploration gives you a new appreciation for the tangent line. As you zoom in on a differentiable function, its graph at that point actually seems to *become* the graph of the tangent line! This observation—that even the most complicated differentiable curve behaves locally like the simplest graph of all, a straight line—is the basis for most of the applications of differential calculus. It is what allows us, for example, to refer to the derivative as the "slope of the curve" or as "the velocity at time t_0."

Algebraically, the principle of local linearity means that the *equation* of the tangent line defines a function that can be used to *approximate* a differentiable function near the point of tangency. In recognition of this fact, we give the equation of the tangent line a new name: the *linearization of f at a*. Recall that the tangent line at $(a, f(a))$ has point-slope equation $y - f(a) = f'(x)(x - a)$ (Figure 4.44).

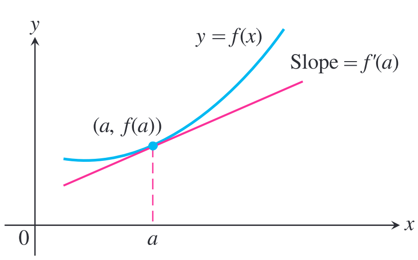


Figure 4.44 The tangent to the curve $y = f(x)$ at $x = a$ is the line $y = f(a) + f'(a)(x - a)$.

DEFINITION Linearization

If f is differentiable at $x = a$, then the equation of the tangent line,

$$L(x) = f(a) + f'(a)(x - a),$$

defines the **linearization of *f* at *a*.** The approximation $f(x) \approx L(x)$ is the **standard linear approximation of *f* at *a*.** The point $x = a$ is the **center** of the approximation.

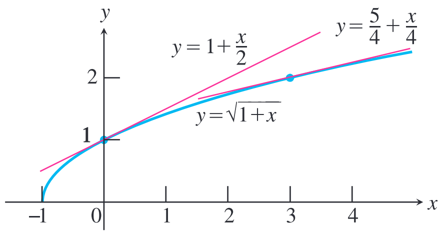


Figure 4.45 The graph of $f(x) = \sqrt{1+x}$ and its linearization at $x = 0$ and $x = 3$. (Example 1)

EXAMPLE 1 Finding a Linearization

Find the linearization of $f(x) = \sqrt{1+x}$ at $x = 0$, and use it to approximate $\sqrt{1.02}$ without a calculator. Then use a calculator to determine the accuracy of the approximation.

SOLUTION

Since $f(0) = 1$, the point of tangency is (0, 1). Since $f'(x) = \frac{1}{2}(1+x)^{-1/2}$, the slope of the tangent line is $f'(0) = \frac{1}{2}$. Thus

$$L(x) = 1 + \frac{1}{2}(x - 0) = 1 + \frac{x}{2}. \quad \text{(Figure 4.45)}$$

To approximate $\sqrt{1.02}$, we use $x = 0.02$:

$$\sqrt{1.02} = f(0.02) \approx L(0.02) = 1 + \frac{0.02}{2} = 1.01$$

The calculator gives $\sqrt{1.02} = 1.009950494$, so the approximation error is $|1.009950494 - 1.01| \approx 4.05 \times 10^{-5}$. We report that the error is less than 10^{-4}.

Now try Exercise 1.

Why not just use a calculator?

We readily admit that linearization will never replace a calculator when it comes to finding square roots. Indeed, historically it was the other way around. Understanding linearization, however, brings you one step closer to understanding how the calculator finds those square roots so easily. You will get many steps closer when you study Taylor polynomials in Chapter 9. (A linearization is just a Taylor polynomial of degree 1.)

Look at how accurate the approximation $\sqrt{1+x} \approx 1 + \frac{x}{2}$ is for values of x near 0.

Approximation	|True Value – Approximation|
$\sqrt{1.002} \approx 1 + \frac{0.002}{2} = 1.001$	$<10^{-6}$
$\sqrt{1.02} \approx 1 + \frac{0.02}{2} = 1.01$	$<10^{-4}$
$\sqrt{1.2} \approx 1 + \frac{0.2}{2} = 1.1$	$<10^{-2}$

As we move away from zero (the center of the approximation), we lose accuracy and the approximation becomes less useful. For example, using $L(2) = 2$ as an approximation for $f(2) = \sqrt{3}$ is not even accurate to one decimal place. We could do slightly better using $L(2)$ to approximate $f(2)$ if we were to use 3 as the center of our approximation (Figure 4.45).

EXAMPLE 2 Finding a Linearization

Find the linearization of $f(x) = \cos x$ at $x = \pi/2$ and use it to approximate cos 1.75 without a calculator. Then use a calculator to determine the accuracy of the approximation.

SOLUTION

Since $f(\pi/2) = \cos(\pi/2) = 0$, the point of tangency is $(\pi/2, 0)$. The slope of the tangent line is $f'(\pi/2) = -\sin(\pi/2) = -1$. Thus

$$L(x) = 0 + (-1)\left(x - \frac{\pi}{2}\right) = -x + \frac{\pi}{2}. \quad \text{(Figure 4.46)}$$

To approximate cos (1.75), we use $x = 1.75$:

$$\cos 1.75 = f(1.75) \approx L(1.75) = -1.75 + \frac{\pi}{2}$$

The calculator gives $\cos 1.75 = -0.1782460556$, so the approximation error is $|-0.1782460556 - (-1.75 + \pi/2)| \approx 9.57 \times 10^{-4}$. We report that the error is less than 10^{-3}.

Now try Exercise 5.

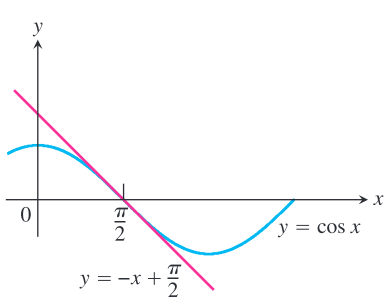


Figure 4.46 The graph of $f(x) = \cos x$ and its linearization at $x = \pi/2$. Near $x = \pi/2$, $\cos x \approx -x + (\pi/2)$. (Example 2)

EXAMPLE 3 Approximating Binomial Powers

Example 1 introduces a special case of a general linearization formula that applies to powers of $1 + x$ for small values of x:

$$(1 + x)^k \approx 1 + kx.$$

If k is a positive integer this follows from the Binomial Theorem, but the formula actually holds for *all* real values of k. (We leave the justification to you as Exercise 7.) Use this formula to find polynomials that will approximate the following functions for values of x close to zero:

(a) $\sqrt[3]{1-x}$ **(b)** $\dfrac{1}{1-x}$ **(c)** $\sqrt{1+5x^4}$ **(d)** $\dfrac{1}{\sqrt{1-x^2}}$

SOLUTION

We change each expression to the form $(1 + y)^k$, where k is a real number and y is a function of x that is close to 0 when x is close to zero. The approximation is then given by $1 + ky$.

(a) $\sqrt[3]{1-x} = (1 + (-x))^{1/3} \approx 1 + \dfrac{1}{3}(-x) = 1 - \dfrac{x}{3}$

(b) $\dfrac{1}{1-x} = (1 + (-x))^{-1} \approx 1 + (-1)(-x) = 1 + x$

(c) $\sqrt{1 + 5x^4} = ((1 + 5x^4))^{1/2} \approx 1 + \dfrac{1}{2}(5x^4) = 1 + \dfrac{5}{2}x^4$

(d) $\dfrac{1}{\sqrt{1-x^2}} = ((1 + (-x^2))^{-1/2} \approx 1 + \left(-\dfrac{1}{2}\right)(-x^2) = 1 + \dfrac{1}{2}x^2$

Now try Exercise 9.

EXAMPLE 4 Approximating Roots

Use linearizations to approximate **(a)** $\sqrt{123}$ and **(b)** $\sqrt[3]{123}$.

SOLUTION

Part of the analysis is to decide where to center the approximations.

(a) Let $f(x) = \sqrt{x}$. The closest perfect square to 123 is 121, so we center the linearization at $x = 121$. The tangent line at (121, 11) has slope

$$f'(121) = \frac{1}{2}(121)^{-1/2} = \frac{1}{2} \cdot \frac{1}{\sqrt{121}} = \frac{1}{22}.$$

So

$$\sqrt{121} \approx L(121) = 11 + \frac{1}{22}(123 - 121) = 11.\overline{09}.$$

(b) Let $f(x) = \sqrt[3]{x}$. The closest perfect cube to 123 is 125, so we center the linearization at $x = 125$. The tangent line at (125, 5) has slope

$$f'(125) = \frac{1}{3}(125)^{-2/3} = \frac{1}{3} \cdot \frac{1}{(\sqrt[3]{125})^2} = \frac{1}{75}.$$

So

$$\sqrt[3]{123} \approx L(123) = 5 + \frac{1}{75}(123 - 125) = 4.97\overline{3}.$$

A calculator shows both approximations to be within 10^{-3} of the actual values.

Now try Exercise 11.

Newton's Method

Newton's method is a numerical technique for approximating a zero of a function with zeros of its linearizations. Under favorable circumstances, the zeros of the linearizations

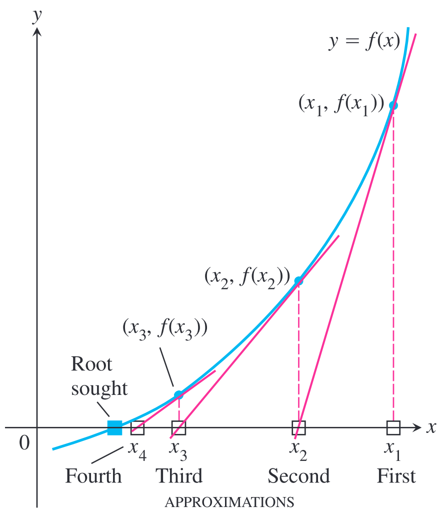


Figure 4.47 Usually the approximations rapidly approach an actual zero of $y = f(x)$.

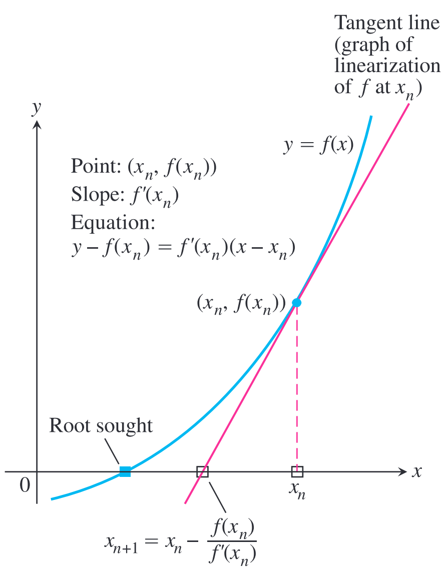


Figure 4.48 From x_n we go up to the curve and follow the tangent line down to find x_{n+1}.

converge rapidly to an accurate approximation. Many calculators use the method because it applies to a wide range of functions and usually gets results in only a few steps. Here is how it works.

To find a solution of an equation $f(x) = 0$, we begin with an initial estimate x_1, found either by looking at a graph or simply guessing. Then we use the tangent to the curve $y = f(x)$ at $(x_1, f(x_1))$ to approximate the curve (Figure 4.47). The point where the tangent crosses the x-axis is the next approximation x_2. The number x_2 is usually a better approximation to the solution than is x_1. The point where the tangent to the curve at $(x_2, f(x_2))$ crosses the x-axis is the next approximation x_3. We continue on, using each approximation to generate the next, until we are close enough to the zero to stop.

There is a formula for finding the $(n + 1)$st approximation x_{n+1} from the nth approximation x_n. The point-slope equation for the tangent to the curve at $(x_n, f(x_n))$ is

$$y - f(x_n) = f'(x_n)(x - x_n).$$

We can find where it crosses the x-axis by setting $y = 0$ (Figure 4.48).

$$\begin{aligned} 0 - f(x_n) &= f'(x_n)(x - x_n) \\ -f(x_n) &= f'(x_n) \cdot x - f'(x_n) \cdot x_n \\ f'(x_n) \cdot x &= f'(x_n) \cdot x_n - f(x_n) \\ x &= x_n - \frac{f(x_n)}{f'(x_n)} \end{aligned}$$

This value of x is the next approximation x_{n+1}. Here is a summary of Newton's method.

Procedure for Newton's Method

1. Guess a first approximation to a solution of the equation $f(x) = 0$. A graph of $y = f(x)$ may help.
2. Use the first approximation to get a second, the second to get a third, and so on, using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

EXAMPLE 5 Using Newton's Method

Use Newton's method to solve $x^3 + 3x + 1 = 0$.

SOLUTION

Let $f(x) = x^3 + 3x + 1$, then $f'(x) = 3x^2 + 3$ and

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^{\,3} + 3x_n + 1}{3x_n^{\,2} + 3}.$$

The graph of f in Figure 4.49 on the next page suggests that $x_1 = -0.3$ is a good first approximation to the zero of f in the interval $-1 \le x \le 0$. Then,

$$\begin{aligned} x_1 &= -0.3, \\ x_2 &= -0.322324159, \\ x_3 &= -0.3221853603, \\ x_4 &= -0.3221853546. \end{aligned}$$

The x_n for $n \ge 5$ all appear to equal x_4 on the calculator we used for our computations. We conclude that the solution to the equation $x^3 + 3x + 1 = 0$ is about -0.3221853546.

Now try Exercise 15.

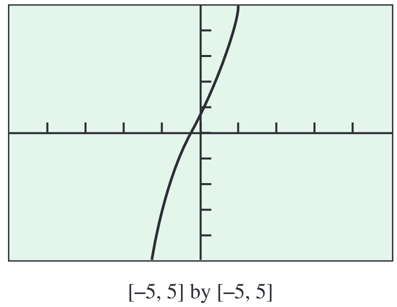


Figure 4.49 A calculator graph of $y = x^3 + 3x + 1$ suggests that -0.3 is a good first guess at the zero to begin Newton's method. (Example 5)

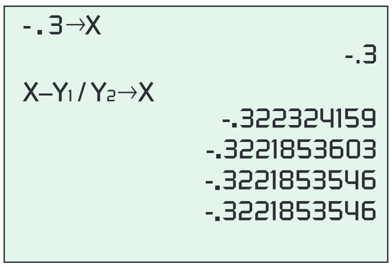


Figure 4.50 A graphing calculator does the computations for Newton's method. (Exploration 2)

EXPLORATION 2 **Using Newton's Method on Your Calculator**

Here is an easy way to get your calculator to perform the calculations in Newton's method. Try it with the function $f(x) = x^3 + 3x + 1$ from Example 5.

1. Enter the function in Y1 and its derivative in Y2.
2. On the home screen, store the initial guess into x. For example, using the initial guess in Example 5, you would type −.3→X.
3. Type X–Y1/Y2→X and press the ENTER key over and over. Watch as the numbers converge to the zero of f. When the values stop changing, it means that your calculator has found the zero to the extent of its displayed digits (Figure 4.50).
4. Experiment with different initial guesses and repeat Steps 2 and 3.
5. Experiment with different functions and repeat Steps 1 through 3. Compare each final value you find with the value given by your calculator's built-in zero-finding feature.

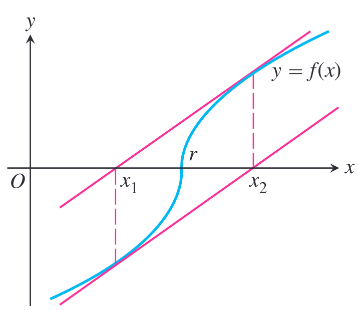


Figure 4.51 The graph of the function

$$f(x) = \begin{cases} -\sqrt{r - x}, & x < r \\ \sqrt{x - r}, & x \geq r. \end{cases}$$

If $x_1 = r - h$, then $x_2 = r + h$. Successive approximations go back and forth between these two values, and Newton's method fails to converge.

Newton's method does not work if $f'(x_1) = 0$. In that case, choose a new starting point.

Newton's method does not always converge. For instance (see Figure 4.51), successive approximations $r - h$ and $r + h$ can go back and forth between these two values, and no amount of iteration will bring us any closer to the zero r.

If Newton's method does converge, it converges to a zero of f. However, the method may converge to a zero that is different from the expected one if the starting value is not close enough to the zero sought. Figure 4.52 shows how this might happen.

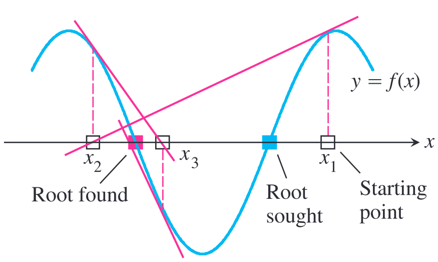


Figure 4.52 Newton's method may miss the zero you want if you start too far away.

Differentials

Leibniz used the notation dy/dx to represent the derivative of y with respect to x. The notation *looks* like a quotient of real numbers, but it is really a *limit* of quotients in which both numerator and denominator go to zero (without actually equaling zero). That makes it tricky to define dy and dx as separate entities. (See the margin note, "Leibniz and His Notation.") Since we really only need to define dy and dx as formal variables, we define them in terms of each other so that their quotient must be the derivative.

Leibniz and His Notation

Although Leibniz did most of his calculus using *dy* and *dx* as separable entities, he never quite settled the issue of what they were. To him, they were "infinitesimals"—nonzero numbers, but infinitesimally small. There was much debate about whether such things could exist in mathematics, but luckily for the early development of calculus it did not matter: thanks to the Chain Rule, *dy*/*dx* *behaved* like a quotient whether it was one or not.

DEFINITION Differentials

Let $y = f(x)$ be a differentiable function. The **differential dx** is an independent variable. The **differential dy** is

$$dy = f'(x)\,dx.$$

Unlike the independent variable dx, the variable dy is always a dependent variable. It depends on both x and dx.

EXAMPLE 6 Finding the Differential *dy*

Find the differential dy and evaluate dy for the given values of x and dx.

(a) $y = x^5 + 37x, \quad x = 1, \quad dx = 0.01$ **(b)** $y = \sin 3x, \quad x = \pi, \quad dx = -0.02$

(c) $x + y = xy, \quad x = 2, \quad dx = 0.05$

SOLUTION

(a) $dy = (5x^4 + 37)\,dx$. When $x = 1$ and $dx = 0.01$, $dy = (5 + 37)(0.01) = 0.42$.

(b) $dy = (3 \cos 3x)\,dx$. When $x = \pi$ and $dx = -0.02$, $dy = (3 \cos 3\pi)(-0.02) = 0.06$.

(c) We could solve explicitly for y before differentiating, but it is easier to use implicit differentiation:

$$\begin{aligned} d(x + y) &= d(xy) \\ dx + dy &= xdy + ydx \\ dy(1 - x) &= (y - 1)dx \\ dy &= \frac{(y - 1)dx}{1 - x} \end{aligned}$$

When $x = 2$ in the original equation, $2 + y = 2y$, so y is also 2. Therefore

$$dy = \frac{(2 - 1)(0.05)}{(1 - 2)} = -0.05.$$

Now try Exercise 19.

If $dx \neq 0$, then the quotient of the differential dy by the differential dx is equal to the derivative $f'(x)$ because

$$\frac{dy}{dx} = \frac{f'(x)\,dx}{dx} = f'(x).$$

We sometimes write

$$df = f'(x)\,dx$$

in place of $dy = f'(x)\,dx$, calling df the **differential of f.** For instance, if $f(x) = 3x^2 - 6$, then

$$df = d(3x^2 - 6) = 6x\,dx.$$

Every differentiation formula like

$$\frac{d(u + v)}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \text{or} \quad \frac{d(\sin u)}{dx} = \cos u \frac{du}{dx}$$

has a corresponding differential form like

$$d(u + v) = du + dv \quad \text{or} \quad d(\sin u) = \cos u\,du.$$

Fan Chung Graham
(1949–)

"Don't be intimidated!" is Dr. Fan Chung Graham's advice to young women considering careers in mathematics. Fan Chung Graham came to the U.S. from Taiwan to earn a Ph.D. in Mathematics from the University of Pennsylvania. She worked in the field of combinatorics at Bell Labs and Bellcore, and then, in 1994, returned to her alma mater as a Professor of Mathematics. Her research interests include spectral graph theory, discrete geometry, algorithms, and communication networks.

EXAMPLE 7 Finding Differentials of Functions

(a) $d(\tan 2x) = \sec^2(2x)\, d(2x) = 2\sec^2 2x\, dx$

(b) $d\left(\dfrac{x}{x+1}\right) = \dfrac{(x+1)\,dx - x\,d(x+1)}{(x+1)^2} = \dfrac{x\,dx + dx - x\,dx}{(x+1)^2} = \dfrac{dx}{(x+1)^2}$

Now try Exercise 27.

Estimating Change with Differentials

Suppose we know the value of a differentiable function $f(x)$ at a point a and we want to predict how much this value will change if we move to a nearby point $a + dx$. If dx is small, f and its linearization L at a will change by nearly the same amount (Figure 4.53). Since the values of L are simple to calculate, calculating the change in L offers a practical way to estimate the change in f.

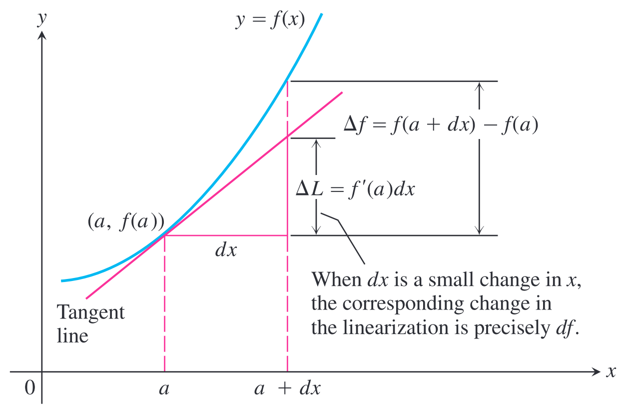


Figure 4.53 Approximating the change in the function f by the change in the linearization of f.

In the notation of Figure 4.53, the change in f is

$$\Delta f = f(a + dx) - f(a).$$

The corresponding change in L is

$$\begin{aligned}\Delta L &= L(a + dx) - L(a)\\ &= \underbrace{f(a) + f'(a)[(a + dx) - a]} - \underbrace{f(a)}\\ &= f'(a)\,dx.\end{aligned}$$

Thus, the differential $df = f'(x)\,dx$ has a geometric interpretation: The value of df at $x = a$ is ΔL, the change in the linearization of f corresponding to the change dx.

Differential Estimate of Change

Let $f(x)$ be differentiable at $x = a$. The approximate change in the value of f when x changes from a to $a + dx$ is

$$df = f'(a)\,dx.$$

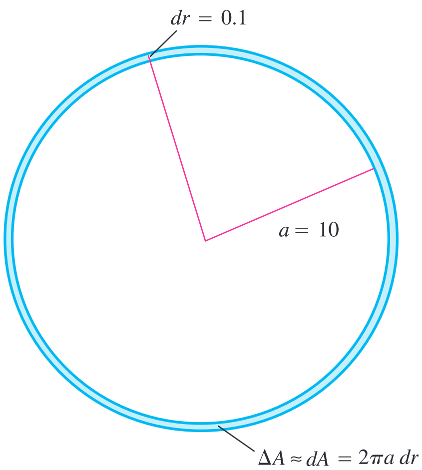


Figure 4.54 When dr is small compared with a, as it is when $dr = 0.1$ and $a = 10$, the differential $dA = 2\pi a\, dr$ gives a good estimate of ΔA. (Example 8)

EXAMPLE 8 Estimating Change With Differentials

The radius r of a circle increases from $a = 10$ m to 10.1 m (Figure 4.54). Use dA to estimate the increase in the circle's area A. Compare this estimate with the true change ΔA, and find the approximation error.

SOLUTION

Since $A = \pi r^2$, the estimated increase is

$$dA = A'(a)\, dr = 2\pi a\, dr = 2\pi(10)(0.1) = 2\pi \text{ m}^2.$$

The true change is

$$\Delta A = \pi(10.1)^2 - \pi(10)^2 = (102.01 - 100)\pi = 2.01\pi \text{ m}^2.$$

The approximation error is $\Delta A - dA = 2.01\pi - 2\pi = 0.01\pi$ m^2.

Now try Exercise 31.

Absolute, Relative, and Percentage Change

As we move from a to a nearby point $a + dx$, we can describe the change in f in three ways:

	True	Estimated
Absolute change	$\Delta f = f(a + dx) - f(a)$	$df = f'(a)\, dx$
Relative change	$\dfrac{\Delta f}{f(a)}$	$\dfrac{df}{f(a)}$
Percentage change	$\dfrac{\Delta f}{f(a)} \times 100$	$\dfrac{df}{f(a)} \times 100$

Why It's Easy to Estimate Change in Perimeter

Note that the *true* change in Example 9 is $P(13) - P(12) = 26\pi - 24\pi = 2\pi$, so the differential estimate in this case is perfectly accurate! Why? Since $P = 2\pi r$ is a linear function of r, the linearization of P is the same as P itself. It is useful to keep in mind that local linearity is what makes estimation by differentials work.

EXAMPLE 9 Changing Tires

Inflating a bicycle tire changes its radius from 12 inches to 13 inches. Use differentials to estimate the absolute change, the relative change, and the percentage change in the perimeter of the tire.

SOLUTION

Perimeter $P = 2\pi r$, so $\Delta P \approx dP = 2\pi\, dr = 2\pi(1) = 2\pi \approx 6.28$.

The absolute change is approximately 6.3 inches.

The relative change (when $P(12) = 24\pi$) is approximately $2\pi/24\pi \approx 0.08$.

The percentage change is approximately 8 percent. ***Now try Exercise 35.***

Another way to interpret the change in $f(x)$ resulting from a change in x is the effect that an error in estimating x has on the estimation of $f(x)$. We illustrate this in Example 10.

EXAMPLE 10 Estimating the Earth's Surface Area

Suppose the earth were a perfect sphere and we determined its radius to be 3959 ± 0.1 miles. What effect would the tolerance of ± 0.1 mi have on our estimate of the earth's surface area?

continued

SOLUTION

The surface area of a sphere of radius r is $S = 4\pi r^2$. The uncertainty in the calculation of S that arises from measuring r with a tolerance of dr miles is

$$dS = 8\pi r\, dr.$$

With $r = 3959$ and $dr = 0.1$, our estimate of S could be off by as much as

$$dS = 8\pi(3959)(0.1) \approx 9950 \text{ mi}^2,$$

to the nearest square mile, which is about the area of the state of Maryland.

Now try Exercise 41.

EXAMPLE 11 Determining Tolerance

About how accurately should we measure the radius r of a sphere to calculate the surface area $S = 4\pi r^2$ within 1% of its true value?

SOLUTION

We want any inaccuracy in our measurement to be small enough to make the corresponding increment ΔS in the surface area satisfy the inequality

$$|\Delta S| \le \frac{1}{100}S = \frac{4\pi r^2}{100}.$$

We replace ΔS in this inequality by its approximation

$$dS = \left(\frac{dS}{dr}\right) dr = 8\pi r\, dr.$$

This gives

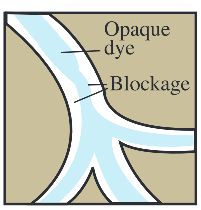
$$|8\pi r\, dr| \le \frac{4\pi r^2}{100}, \quad \text{or} \quad |dr| \le \frac{1}{8\pi r}\cdot\frac{4\pi r^2}{100} = \frac{1}{2}\cdot\frac{r}{100} = 0.005r.$$

We should measure r with an error dr that is no more than 0.5% of the true value.

Now try Exercise 49.

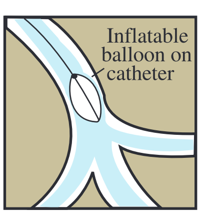
Angiography

An opaque dye is injected into a partially blocked artery to make the inside visible under X-rays. This reveals the location and severity of the blockage.



Angioplasty

A balloon-tipped catheter is inflated inside the artery to widen it at the blockage site.



EXAMPLE 12 Unclogging Arteries

In the late 1830s, the French physiologist Jean Poiseuille ("pwa-ZOY") discovered the formula we use today to predict how much the radius of a partially clogged artery has to be expanded to restore normal flow. His formula,

$$V = kr^4,$$

says that the volume V of fluid flowing through a small pipe or tube in a unit of time at a fixed pressure is a constant times the fourth power of the tube's radius r. How will a 10% increase in r affect V?

SOLUTION

The differentials of r and V are related by the equation

$$dV = \frac{dV}{dr}\,dr = 4kr^3\,dr.$$

The relative change in V is

$$\frac{dV}{V} = \frac{4kr^3 dr}{kr^4} = 4\frac{dr}{r}.$$

The relative change in V is 4 times the relative change in r, so a 10% increase in r will produce a 40% increase in the flow. *Now try Exercise 51.*

Sensitivity to Change

The equation $df = f'(x)\,dx$ tells how *sensitive* the output of f is to a change in input at different values of x. The larger the value of f' at x, the greater the effect of a given change dx.

EXAMPLE 13 Finding Depth of a Well

You want to calculate the depth of a well from the equation $s = 16t^2$ by timing how long it takes a heavy stone you drop to splash into the water below. How sensitive will your calculations be to a 0.1 sec error in measuring the time?

SOLUTION

The size of ds in the equation

$$ds = 32t\,dt$$

depends on how big t is. If $t = 2$ sec, the error caused by $dt = 0.1$ is only

$$ds = 32(2)(0.1) = 6.4 \text{ ft.}$$

Three seconds later at $t = 5$ sec, the error caused by the same dt is

$$ds = 32(5)(0.1) = 16 \text{ ft.}$$

Now try Exercise 53.

Quick Review 4.5 *(For help, go to Sections 3.3, 3.6, and 3.9.)*

In Exercises 1 and 2, find dy/dx.

1. $y = \sin(x^2 + 1)$ **2.** $y = \dfrac{x + \cos x}{x + 1}$

In Exercises 3 and 4, solve the equation graphically.

3. $xe^{-x} + 1 = 0$ **4.** $x^3 + 3x + 1 = 0$

In Exercises 5 and 6, let $f(x) = xe^{-x} + 1$. Write an equation for the line tangent to f at $x = c$.

5. $c = 0$ **6.** $c = -1$

7. Find where the tangent line in **(a)** Exercise 5 and **(b)** Exercise 6 crosses the x-axis.

8. Let $g(x)$ be the function whose graph is the tangent line to the graph of $f(x) = x^3 - 4x + 1$ at $x = 1$. Complete the table.

x	$f(x)$	$g(x)$
0.7		
0.8		
0.9		
1		
1.1		
1.2		
1.3		

In Exercises 9 and 10, graph $y = f(x)$ and its tangent line at $x = c$.

9. $c = 1.5, \quad f(x) = \sin x$

10. $c = 4, \quad f(x) = \begin{cases} -\sqrt{3 - x}, & x < 3 \\ \sqrt{x - 3}, & x \geq 3 \end{cases}$

Section 4.5 Exercises

In Exercises 1–6, **(a)** find the linearization $L(x)$ of $f(x)$ at $x = a$. **(b)** How accurate is the approximation $L(a + 0.1) \approx f(a + 0.1)$? See the comparisons following Example 1.

1. $f(x) = x^3 - 2x + 3, \quad a = 2$

2. $f(x) = \sqrt{x^2 + 9}, \quad a = -4$

3. $f(x) = x + \dfrac{1}{x}, \quad a = 1$

4. $f(x) = \ln(x + 1), \quad a = 0$

5. $f(x) = \tan x, \quad a = \pi$ **6.** $f(x) = \cos^{-1} x, \quad a = 0$

7. Show that the linearization of $f(x) = (1 + x)^k$ at $x = 0$ is $L(x) = 1 + kx$.

8. Use the linearization $(1 + x)^k \approx 1 + kx$ to approximate the following. State how accurate your approximation is.

(a) $(1.002)^{100}$ **(b)** $\sqrt[3]{1.009}$

In Exercises 9 and 10, use the linear approximation $(1 + x)^k \approx 1 + kx$ to find an approximation for the function $f(x)$ for values of x near zero.

9. **(a)** $f(x) = (1 - x)^6$ **(b)** $f(x) = \dfrac{2}{1 - x}$ **(c)** $f(x) = \dfrac{1}{\sqrt{1 + x}}$

10. **(a)** $f(x) = (4 + 3x)^{1/3}$ **(b)** $f(x) = \sqrt{2 + x^2}$

(c) $f(x) = \sqrt[3]{\left(1 - \dfrac{1}{2 + x}\right)^2}$

In Exercises 11–14, approximate the root by using a linearization centered at an appropriate nearby number.

11. $\sqrt{101}$ **12.** $\sqrt[3]{26}$

13. $\sqrt[3]{998}$ **14.** $\sqrt{80}$

In Exercises 15–18, use Newton's method to estimate all real solutions of the equation. Make your answers accurate to 6 decimal places.

15. $x^3 + x - 1 = 0$ **16.** $x^4 + x - 3 = 0$

17. $x^2 - 2x + 1 = \sin x$ **18.** $x^4 - 2 = 0$

In Exercises 19–26, **(a)** find dy, and **(b)** evaluate dy for the given value of x and dx.

19. $y = x^3 - 3x, \quad x = 2, \quad dx = 0.05$

20. $y = \dfrac{2x}{1 + x^2}, \quad x = -2, \quad dx = 0.1$

21. $y = x^2 \ln x, \quad x = 1, \quad dx = 0.01$

22. $y = x\sqrt{1 - x^2}, \quad x = 0, \quad dx = -0.2$

23. $y = e^{\sin x}, \quad x = \pi, \quad dx = -0.1$

24. $y = 3 \csc\left(1 - \dfrac{x}{3}\right), \quad x = 1, \quad dx = 0.1$

25. $y + xy - x = 0, \quad x = 0, \quad dx = 0.01$

26. $2y = x^2 - xy, \quad x = 2, \quad dx = -0.05$

In Exercises 27–30, find the differential.

27. $d(\sqrt{1 - x^2})$

28. $d(e^{5x} + x^5)$

29. $d(\arctan 4x)$

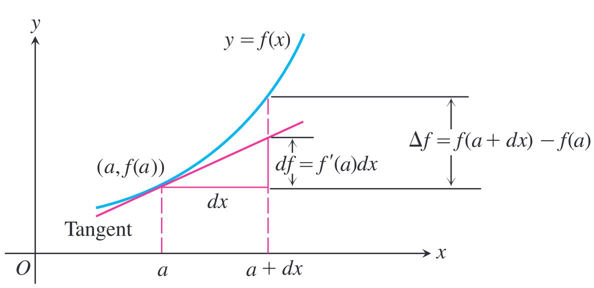
30. $d(8^x + x^8)$

In Exercises 31–34, the function f changes value when x changes from a to $a + dx$. Find

(a) the true change $\Delta f = f(a + dx) - f(a)$.

(b) the estimated change $df = f'(a)\,dx$.

(c) the approximation error $|\Delta f - df|$.



31. $f(x) = x^2 + 2x, \quad a = 0, \quad dx = 0.1$

32. $f(x) = x^3 - x, \quad a = 1, \quad dx = 0.1$

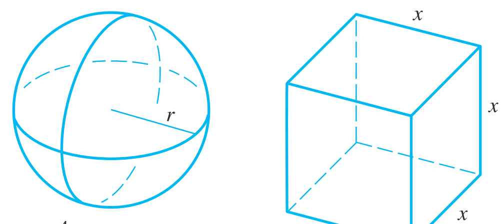
33. $f(x) = x^{-1}, \quad a = 0.5, \quad dx = 0.05$

34. $f(x) = x^4, \quad a = 1, \quad dx = 0.01$

In Exercises 35–40, write a differential formula that estimates the given change in volume or surface area. Then use the formula to estimate the change when the dependent variable changes from 10 cm to 10.05 cm.

35. ***Volume*** The change in the volume $V = (4/3)\pi r^3$ of a sphere when the radius changes from a to $a + dr$

36. ***Surface Area*** The change in the surface area $S = 4\pi r^2$ of a sphere when the radius changes from a to $a + dr$



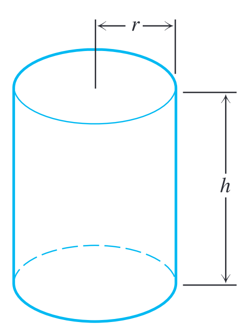
$V = \frac{4}{3}\pi r^3, \quad S = 4\pi r^2$ $\qquad V = x^3, \quad S = 6x^2$

37. ***Volume*** The change in the volume $V = x^3$ of a cube when the edge lengths change from a to $a + dx$

38. ***Surface Area*** The change in the surface area $S = 6x^2$ of a cube when the edge lengths change from a to $a + dx$

39. ***Volume*** The change in the volume $V = \pi r^2 h$ of a right circular cylinder when the radius changes from a to $a + dr$ and the height does not change

40. ***Surface Area*** The change in the lateral surface area $S = 2\pi rh$ of a right circular cylinder when the height changes from a to $a + dh$ and the radius does not change



$V = \pi r^2 h, \quad S = 2\pi rh$

In Exercises 41–44, use differentials to estimate the maximum error in measurement resulting from the tolerance of error in the dependent variable. Express answers to the nearest tenth, since that is the precision used to express the tolerance.

41. The area of a circle with radius 10 ± 0.1 in.

42. The volume of a sphere with radius 8 ± 0.3 in.

43. The volume of a cube with side 15 ± 0.2 cm.

44. The area of an equilateral triangle with side 20 ± 0.5 cm.

45. ***Linear Approximation*** Let f be a function with $f(0) = 1$ and $f'(x) = \cos(x^2)$.

(a) Find the linearization of f at $x = 0$.

(b) Estimate the value of f at $x = 0.1$.

(c) Writing to Learn Do you think the actual value of f at $x = 0.1$ is greater than or less than the estimate in part (b)? Explain.

46. ***Expanding Circle*** The radius of a circle is increased from 2.00 to 2.02 m.

(a) Estimate the resulting change in area.

(b) Estimate as a percentage of the circle's original area.

47. ***Growing Tree*** The diameter of a tree was 10 in. During the following year, the circumference increased 2 in. About how much did the tree's diameter increase? the tree's cross section area?

48. ***Percentage Error*** The edge of a cube is measured as 10 cm with an error of 1%. The cube's volume is to be calculated from this measurement. Estimate the percentage error in the volume calculation.

49. ***Tolerance*** About how accurately should you measure the side of a square to be sure of calculating the area to within 2% of its true value?

50. ***Tolerance*** **(a)** About how accurately must the interior diameter of a 10-m high cylindrical storage tank be measured to calculate the tank's volume to within 1% of its true value?

(b) About how accurately must the tank's exterior diameter be measured to calculate the amount of paint it will take to paint the side of the tank to within 5% of the true amount?

51. ***Minting Coins*** A manufacturer contracts to mint coins for the federal government. The coins must weigh within 0.1% of their ideal weight, so the volume must be within 0.1% of the ideal volume. Assuming the thickness of the coins does not change, what is the percentage change in the volume of the coin that would result from a 0.1% increase in the radius?

52. ***Tolerance*** The height and radius of a right circular cylinder are equal, so the cylinder's volume is $V = \pi h^3$. The volume is to be calculated with an error of no more than 1% of the true value. Find approximately the greatest error that can be tolerated in the measurement of h, expressed as a percentage of h.

53. ***Estimating Volume*** You can estimate the volume of a sphere by measuring its circumference with a tape measure, dividing by 2π to get the radius, then using the radius in the volume formula. Find how sensitive your volume estimate is to a 1/8 in. error in the circumference measurement by filling in the table below for spheres of the given sizes. Use differentials when filling in the last column.

Sphere Type	True Radius	Tape Error	Radius Error	Volume Error
Orange	2 in.	1/8 in.		
Melon	4 in.	1/8 in.		
Beach Ball	7 in.	1/8 in.		

54. ***Estimating Surface Area*** Change the heading in the last column of the table in Exercise 53 to "Surface Area Error" and find how sensitive the measure of surface area is to a 1/8 in. error in estimating the circumference of the sphere.

55. ***The Effect of Flight Maneuvers on the Heart*** The amount of work done by the heart's main pumping chamber, the left ventricle, is given by the equation

$$W = PV + \frac{V\delta v^2}{2g},$$

where W is the work per unit time, P is the average blood pressure, V is the volume of blood pumped out during the unit of time, δ ("delta") is the density of the blood, v is the average velocity of the exiting blood, and g is the acceleration of gravity.

When P, V, δ, and v remain constant, W becomes a function of g, and the equation takes the simplified form

$$W = a + \frac{b}{g} \ (a, b \text{ constant}).$$

As a member of NASA's medical team, you want to know how sensitive W is to apparent changes in g caused by flight maneuvers, and this depends on the initial value of g. As part of your investigation, you decide to compare the effect on W of a given change dg on the moon, where $g = 5.2$ ft/sec^2, with the effect the same change dg would have on Earth, where $g = 32$ ft/sec^2. Use the simplified equation above to find the ratio of dW_{moon} to dW_{Earth}.

56. ***Measuring Acceleration of Gravity*** When the length L of a clock pendulum is held constant by controlling its temperature, the pendulum's period T depends on the acceleration of gravity g. The period will therefore vary slightly as the clock is moved from place to place on the earth's surface, depending on the change in g. By keeping track of ΔT, we can estimate the variation in g from the equation $T = 2\pi(L/g)^{1/2}$ that relates T, g, and L.

(a) With L held constant and g as the independent variable, calculate dT and use it to answer parts (b) and (c).

(b) Writing to Learn If g increases, will T increase or decrease? Will a pendulum clock speed up or slow down? Explain.

(c) A clock with a 100-cm pendulum is moved from a location where $g = 980$ cm/sec^2 to a new location. This increases the period by $dT = 0.001$ sec. Find dg and estimate the value of g at the new location.

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

57. True or False Newton's method will not find the zero of $f(x) = x/(x^2 + 1)$ if the first guess is greater than 1. Justify your answer.

58. True or False If u and v are differentiable functions, then $d(uv) = du\,dv$. Justify your answer.

59. Multiple Choice What is the linearization of $f(x) = e^x$ at $x = 1$?

(A) $y = e$ **(B)** $y = ex$ **(C)** $y = e^x$
(D) $y = x - e$ **(E)** $y = e(x - 1)$

60. Multiple Choice If $y = \tan x$, $x = \pi$, and $dx = 0.5$, what does dy equal?

(A) -0.25 **(B)** -0.5 **(C)** 0 **(D)** 0.5 **(E)** 0.25

61. Multiple Choice If Newton's method is used to find the zero of $f(x) = x - x^3 + 2$, what is the third estimate if the first estimate is 1?

(A) $-\frac{3}{4}$ **(B)** $\frac{3}{2}$ **(C)** $\frac{8}{5}$ **(D)** $\frac{18}{11}$ **(E)** 3

62. Multiple Choice If the linearization of $y = \sqrt[3]{x}$ at $x = 64$ is used to approximate $\sqrt[3]{66}$, what is the percentage error?

(A) 0.01% **(B)** 0.04% **(C)** 0.4% **(D)** 1% **(E)** 4%

Explorations

63. *Newton's Method* Suppose your first guess in using Newton's method is lucky in the sense that x_1 is a root of $f(x) = 0$. What happens to x_2 and later approximations?

64. *Oscillation* Show that if $h > 0$, applying Newton's method to

$$f(x) = \begin{cases} \sqrt{x}, & x \ge 0 \\ \sqrt{-x}, & x < 0 \end{cases}$$

leads to $x_2 = -h$ if $x_1 = h$, and to $x_2 = h$ if $x_1 = -h$. Draw a picture that shows what is going on.

65. *Approximations that Get Worse and Worse* Apply Newton's method to $f(x) = x^{1/3}$ with $x_1 = 1$, and calculate x_2, x_3, x_4, and x_5. Find a formula for $|x_n|$. What happens to $|x_n|$ as $n \to \infty$? Draw a picture that shows what is going on.

66. *Quadratic Approximations*

(a) Let $Q(x) = b_0 + b_1(x - a) + b_2(x - a)^2$ be a quadratic approximation to $f(x)$ at $x = a$ with the properties:

i. $Q(a) = f(a)$,

ii. $Q'(a) = f'(a)$,

iii. $Q''(a) = f''(a)$.

Determine the coefficients b_0, b_1, and b_2.

(b) Find the quadratic approximation to $f(x) = 1/(1 - x)$ at $x = 0$.

(c) Graph $f(x) = 1/(1 - x)$ and its quadratic approximation at $x = 0$. Then zoom in on the two graphs at the point (0, 1). Comment on what you see.

(d) Find the quadratic approximation to $g(x) = 1/x$ at $x = 1$. Graph g and its quadratic approximation together. Comment on what you see.

(e) Find the quadratic approximation to $h(x) = \sqrt{1 + x}$ at $x = 0$. Graph h and its quadratic approximation together. Comment on what you see.

(f) What are the linearizations of f, g, and h at the respective points in parts (b), (d), and (e)?

67. *Multiples of Pi* Store any number as X in your calculator. Then enter the command X−tan(X)→X and press the ENTER key repeatedly until the displayed value stops changing. The result is always an integral multiple of π. Why is this so? [*Hint:* These are zeros of the sine function.]

Extending the Ideas

68. *Formulas for Differentials* Verify the following formulas.

(a) $d(c) = 0$ (c a constant)

(b) $d(cu) = c\,du$ (c a constant)

(c) $d(u + v) = du + dv$

(d) $d(u \cdot v) = u\,dv + v\,du$

(e) $d\left(\dfrac{u}{v}\right) = \dfrac{v\,du - u\,dv}{v^2}$

(f) $d(u^n) = nu^{n-1}\,du$

69. *Linearization* Show that the approximation of $\tan x$ by its linearization at the origin must improve as $x \to 0$ by showing that

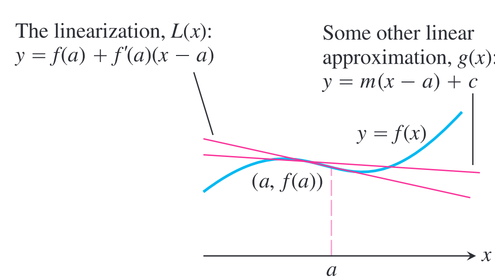
$$\lim_{x \to 0} \frac{\tan x}{x} = 1.$$

70. *The Linearization is the Best Linear Approximation* Suppose that $y = f(x)$ is differentiable at $x = a$ and that $g(x) = m(x - a) + c$ (m and c constants). If the error $E(x) = f(x) - g(x)$ were small enough near $x = a$, we might think of using g as a linear approximation of f instead of the linearization $L(x) = f(a) + f'(a)(x - a)$. Show that if we impose on g the conditions

i. $E(a) = 0$,

ii. $\lim_{x \to a} \dfrac{E(x)}{x - a} = 0$,

then $g(x) = f(a) + f'(a)(x - a)$. Thus, the linearization gives the only linear approximation whose error is both zero at $x = a$ and negligible in comparison with $(x - a)$.



71. Writing to Learn Find the linearization of $f(x) = \sqrt{x + 1} + \sin x$ at $x = 0$. How is it related to the individual linearizations for $\sqrt{x + 1}$ and $\sin x$?

4.6 Related Rates

What you'll learn about

- Related Rate Equations
- Solution Strategy
- Simulating Related Motion

. . . and why

Related rate problems are at the heart of Newtonian mechanics; it was essentially to solve such problems that calculus was invented.

Related Rate Equations

Suppose that a particle $P(x, y)$ is moving along a curve C in the plane so that its coordinates x and y are differentiable functions of time t. If D is the distance from the origin to P, then using the Chain Rule we can find an equation that relates dD/dt, dx/dt, and dy/dt.

$$D = \sqrt{x^2 + y^2}$$

$$\frac{dD}{dt} = \frac{1}{2}(x^2 + y^2)^{-1/2}\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt}\right)$$

Any equation involving two or more variables that are differentiable functions of time t can be used to find an equation that relates their corresponding rates.

EXAMPLE 1 Finding Related Rate Equations

(a) Assume that the radius r of a sphere is a differentiable function of t and let V be the volume of the sphere. Find an equation that relates dV/dt and dr/dt.

(b) Assume that the radius r and height h of a cone are differentiable functions of t and let V be the volume of the cone. Find an equation that relates dV/dt, dr/dt, and dh/dt.

SOLUTION

(a) $V = \frac{4}{3}\pi r^3$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

(b) $V = \frac{\pi}{3} r^2 h$

$$\frac{dV}{dt} = \frac{\pi}{3}\left(r^2 \cdot \frac{dh}{dt} + 2r\frac{dr}{dt} \cdot h\right) = \frac{\pi}{3}\left(r^2\frac{dh}{dt} + 2rh\frac{dr}{dt}\right)$$

Now try Exercise 3.

Solution Strategy

What has always distinguished calculus from algebra is its ability to deal with variables that change over time. Example 1 illustrates how easy it is to move from a formula relating static variables to a formula that relates their rates of change: simply differentiate the formula implicitly with respect to t. This introduces an important category of problems called *related rate problems* that still constitutes one of the most important applications of calculus.

We introduce a strategy for solving related rate problems, similar to the strategy we introduced for max-min problems earlier in this chapter.

Strategy for Solving Related Rate Problems

1. **Understand the problem.** In particular, identify the variable whose rate of change you *seek* and the variable (or variables) whose rate of change you *know*.
2. **Develop a mathematical model of the problem.** Draw a picture (many of these problems involve geometric figures) and label the parts that are important to the problem. *Be sure to distinguish constant quantities from variables that change over time.* Only constant quantities can be assigned numerical values at the start.

continued

3. **Write an equation relating the variable whose rate of change you seek with the variable(s) whose rate of change you know.** The formula is often geometric, but it could come from a scientific application.
4. **Differentiate both sides of the equation implicitly with respect to time t.** Be sure to follow all the differentiation rules. The Chain Rule will be especially critical, as you will be differentiating with respect to the parameter t.
5. **Substitute values for any quantities that depend on time.** Notice that it is only safe to do this *after* the differentiation step. Substituting too soon "freezes the picture" and makes changeable variables behave like constants, with zero derivatives.
6. **Interpret the solution.** Translate your mathematical result into the problem setting (with appropriate units) and decide whether the result makes sense.

We illustrate the strategy in Example 2.

EXAMPLE 2 A Rising Balloon

A hot-air balloon rising straight up from a level field is tracked by a range finder 500 feet from the lift-off point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 radians per minute. How fast is the balloon rising at that moment?

SOLUTION

We will carefully identify the six steps of the strategy in this first example.

Step 1: Let h be the height of the balloon and let θ be the elevation angle.

We seek: dh/dt

We know: $d\theta/dt = 0.14$ rad/min

Step 2: We draw a picture (Figure 4.55). We label the horizontal distance "500 ft" because it does not change over time. We label the height "h" and the angle of elevation "θ." Notice that we do not label the angle "$\pi/4$," as that would freeze the picture.

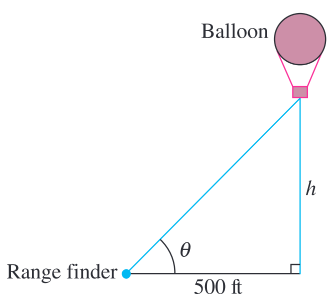


Figure 4.55 The picture shows how h and θ are related geometrically. We seek dh/dt when $\theta = \pi/4$ and $d\theta/dt = 0.14$ rad/min. (Example 2)

Step 3: We need a formula that relates h and θ. Since $\frac{h}{500} = \tan\theta$, we get $h = 500\tan\theta$.

Step 4: Differentiate implicitly:

$$\frac{d}{dt}(h) = \frac{d}{dt}(500\tan\theta)$$
$$\frac{dh}{dt} = 500\sec^2\theta\,\frac{d\theta}{dt}$$

Step 5: Let $d\theta/dt = 0.14$ and let $\theta = \pi/4$. (Note that it is now safe to specify our moment in time.)

$$\frac{dh}{dt} = 500\sec^2\left(\frac{\pi}{4}\right)(0.14) = 500(\sqrt{2})^2(0.14) = 140.$$

Step 6: At the moment in question, the balloon is rising at the rate of 140 ft/min.

Now try Exercise 11.

Unit Analysis in Example 2

A careful analysis of the units in Example 2 gives

$$dh/dt = (500 \text{ ft})(\sqrt{2})^2(0.14 \text{ rad/min}) = 140 \text{ ft} \cdot \text{rad/min}.$$

Remember that radian measure is actually dimensionless, adaptable to whatever unit is applied to the "unit" circle. The linear units in Example 2 are measured in feet, so "ft · rad" is simply "ft."

EXAMPLE 3 A Highway Chase

A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 mi north of the intersection and the car is 0.8 mi to the east, the police determine with radar that the distance between them and the car is increasing at 20 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?

continued

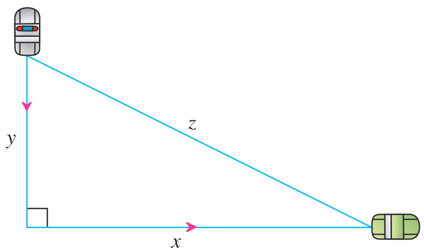


Figure 4.56 A sketch showing the variables in Example 3. We know dy/dt and dz/dt, and we seek dx/dt. The variables x, y, and z are related by the Pythagorean Theorem: $x^2 + y^2 = z^2$.

SOLUTION

We carry out the steps of the strategy.

Let x be the distance of the speeding car from the intersection, let y be the distance of the police cruiser from the intersection, and let z be the distance between the car and the cruiser. Distances x and z are increasing, but distance y is decreasing; so dy/dt is negative.

We seek: dx/dt

We know: $dz/dt = 20$ mph and $dy/dt = -60$ mph

A sketch (Figure 4.56) shows that x, y, and z form three sides of a right triangle. We need to relate those three variables, so we use the Pythagorean Theorem:

$$x^2 + y^2 = z^2$$

Differentiating implicitly with respect to t, we get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}, \text{ which reduces to } x\frac{dx}{dt} + y\frac{dy}{dt} = z\frac{dz}{dt}.$$

We now substitute the numerical values for x, y, dz/dt, dy/dt, and z (which equals $\sqrt{x^2 + y^2}$):

$$(0.8)\frac{dx}{dt} + (0.6)(-60) = \sqrt{(0.8)^2 + (0.6)^2}(20)$$

$$(0.8)\frac{dx}{dt} - 36 = (1)(20)$$

$$\frac{dx}{dt} = 70$$

At the moment in question, the car's speed is 70 mph. ***Now try Exercise 13.***

EXAMPLE 4 Filling a Conical Tank

Water runs into a conical tank at the rate of 9 ft^3/min. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?

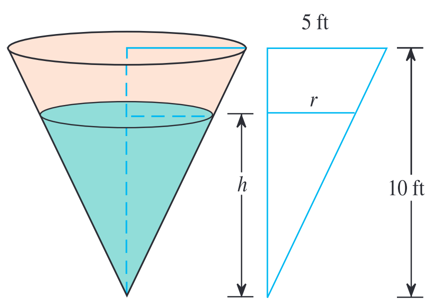


Figure 4.57 In Example 4, the cone of water is increasing in volume inside the reservoir. We know dV/dt and we seek dh/dt. Similar triangles enable us to relate V directly to h.

SOLUTION 1

We carry out the steps of the strategy. Figure 4.57 shows a partially filled conical tank. The tank itself does not change over time; what we are interested in is the changing cone of *water* inside the tank. Let V be the volume, r the radius, and h the height of the cone of water.

We seek: dh/dt

We know: $dV/dt = 9\ \text{ft}^3/\text{min}$

We need to relate V and h. The volume of the cone of water is $V = \frac{1}{3}\pi r^2 h$, but this formula also involves the variable r, whose rate of change is not given. We need to either find dr/dt (see Solution 2) or eliminate r from the equation, which we can do by using the similar triangles in Figure 4.57 to relate r and h:

$$\frac{r}{h} = \frac{5}{10}, \text{ or simply } r = \frac{h}{2}.$$

Therefore,

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3.$$

continued

Differentiate with respect to t:

$$\frac{dV}{dt} = \frac{\pi}{12} \cdot 3h^2 \frac{dh}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}.$$

Let $h = 6$ and $dV/dt = 9$; then solve for dh/dt:

$$9 = \frac{\pi}{4}(6)^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{1}{\pi} \approx 0.32$$

At the moment in question, the water level is rising at 0.32 ft/min.

SOLUTION 2

The similar triangle relationship

$$r = \frac{h}{2} \text{ also implies that } \frac{dr}{dt} = \frac{1}{2}\frac{dh}{dt}$$

and that $r = 3$ when $h = 6$. So, we could have left all three variables in the formula $V = \frac{1}{3}\pi r^2 h$ and proceeded as follows:

$$\frac{dV}{dt} = \frac{1}{3}\pi\left(2r\frac{dr}{dt}h + r^2\frac{dh}{dt}\right)$$

$$= \frac{1}{3}\pi\left(2r\left(\frac{1}{2}\frac{dh}{dt}\right)h + r^2\frac{dh}{dt}\right)$$

$$9 = \frac{1}{3}\pi\left(2(3)\left(\frac{1}{2}\frac{dh}{dt}\right)(6) + (3)^2\frac{dh}{dt}\right)$$

$$9 = 9\pi\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{1}{\pi}$$

This is obviously more complicated than the one-variable approach. In general, it is computationally easier to simplify expressions as much as possible *before* you differentiate.

Now try Exercise 17.

Simulating Related Motion

Parametric mode on a grapher can be used to simulate the motion of moving objects when the motion of each can be expressed as a function of time. In a classic related rate problem, the top end of a ladder slides vertically down a wall as the bottom end is pulled horizontally away from the wall at a steady rate. Exploration 1 shows how you can use your grapher to simulate the related movements of the two ends of the ladder.

EXPLORATION 1 The Sliding Ladder

A 10-foot ladder leans against a vertical wall. The base of the ladder is pulled away from the wall at a constant rate of 2 ft/sec.

1. Explain why the motion of the two ends of the ladder can be represented by the parametric equations given on the next page.

continued

$$X1T = 2T$$

$$Y1T = 0$$

$$X2T = 0$$

$$Y2T = \sqrt{10^2 - (2T)^2}$$

2. What minimum and maximum values of T make sense in this problem?
3. Put your grapher in parametric and simultaneous modes. Enter the parametric equations and change the graphing style to "0" (the little ball) if your grapher has this feature. Set Tmin=0, Tmax=5, Tstep=5/20, Xmin=−1, Xmax=17, Xscl=0, Ymin=−1, Ymax=11, and Yscl=0. You can speed up the action by making the denominator in the Tstep smaller or slow it down by making it larger.
4. Press GRAPH and watch the two ends of the ladder move as time changes. Do both ends seem to move at a constant rate?
5. To see the simulation again, enter "ClrDraw" from the DRAW menu.
6. If y represents the vertical height of the top of the ladder and x the distance of the bottom from the wall, relate y and x and find dy/dt in terms of x and y. (Remember that $dx/dt = 2$.)
7. Find dy/dt when $t = 3$ and interpret its meaning. Why is it negative?
8. In theory, how fast is the top of the ladder moving as it hits the ground?

Figure 4.58 shows you how to write a calculator program that animates the falling ladder as a line segment.

```
PROGRAM:LADDER
:For (A, 0, 5, .25)
:ClrDraw
:Line(2,2+√ (100–
(2A)²), 2+2A, 2)
:If A=0:Pause
:End
```

```
WINDOW
  Xmin=2
  Xmax=20
  Xscl=0
  Ymin=1
  Ymax=13
  Yscl=0
  Xres=1
```

Figure 4.58 This 5-step program (with the viewing window set as shown) will animate the ladder in Exploration 1. Be sure any functions in the "Y=" register are turned off. Run the program and the ladder appears against the wall; push ENTER to start the bottom moving away from the wall.
For an enhanced picture, you can insert the commands ":Pt-On(2,2+$\sqrt{(100-(2A)^2)}$,2)" and ":Pt-On(2+2A,2,2)" on either side of the middle line of the program.

Quick Review 4.6 *(For help, go to Sections 1.1, 1.4, and 3.7.)*

In Exercises 1 and 2, find the distance between the points A and B.

1. $A(0, 5)$, $B(7, 0)$
2. $A(0, a)$, $B(b, 0)$

In Exercises 3–6, find dy/dx.

3. $2xy + y^2 = x + y$
4. $x \sin y = 1 - xy$
5. $x^2 = \tan y$
6. $\ln(x + y) = 2x$

In Exercises 7 and 8, find a parametrization for the line segment with endpoints A and B.

7. $A(-2, 1)$, $B(4, -3)$
8. $A(0, -4)$, $B(5, 0)$

In Exercises 9 and 10, let $x = 2\cos t$, $y = 2\sin t$. Find a parameter interval that produces the indicated portion of the graph.

9. The portion in the second and third quadrants, including the points on the axes.
10. The portion in the fourth quadrant, including the points on the axes.

Section 4.6 Exercises

In Exercises 1–41, assume all variables are differentiable functions of t.

1. ***Area*** The radius r and area A of a circle are related by the equation $A = \pi r^2$. Write an equation that relates dA/dt to dr/dt.

2. ***Surface Area*** The radius r and surface area S of a sphere are related by the equation $S = 4\pi r^2$. Write an equation that relates dS/dt to dr/dt.

3. ***Volume*** The radius r, height h, and volume V of a right circular cylinder are related by the equation $V = \pi r^2 h$.

   (a) How is dV/dt related to dh/dt if r is constant?

   (b) How is dV/dt related to dr/dt if h is constant?

   (c) How is dV/dt related to dr/dt and dh/dt if neither r nor h is constant?

4. ***Electrical Power*** The power P (watts) of an electric circuit is related to the circuit's resistance R (ohms) and current I (amperes) by the equation $P = RI^2$.

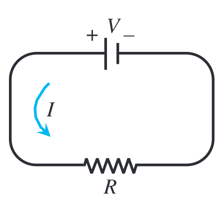
   (a) How is dP/dt related to dR/dt and dI/dt?

   (b) How is dR/dt related to dI/dt if P is constant?

5. ***Diagonals*** If x, y, and z are lengths of the edges of a rectangular box, the common length of the box's diagonals is $s = \sqrt{x^2 + y^2 + z^2}$. How is ds/dt related to dx/dt, dy/dt, and dz/dt?

6. ***Area*** If a and b are the lengths of two sides of a triangle, and θ the measure of the included angle, the area A of the triangle is $A = (1/2)\, ab \sin \theta$. How is dA/dt related to da/dt, db/dt, and $d\theta/dt$?

7. ***Changing Voltage*** The voltage V (volts), current I (amperes), and resistance R (ohms) of an electric circuit like the one shown here are related by the equation $V = IR$. Suppose that V is increasing at the rate of 1 volt/sec while I is decreasing at the rate of 1/3 amp/sec. Let t denote time in sec.



   (a) What is the value of dV/dt?

   (b) What is the value of dI/dt?

   (c) Write an equation that relates dR/dt to dV/dt and dI/dt.

   (d) Writing to Learn Find the rate at which R is changing when $V = 12$ volts and $I = 2$ amp. Is R increasing, or decreasing? Explain.

8. ***Heating a Plate*** When a circular plate of metal is heated in an oven, its radius increases at the rate of 0.01 cm/sec. At what rate is the plate's area increasing when the radius is 50 cm?

9. ***Changing Dimensions in a Rectangle*** The length ℓ of a rectangle is decreasing at the rate of 2 cm/sec while the width w is increasing at the rate of 2 cm/sec. When $\ell = 12$ cm and $w = 5$ cm, find the rates of change of

   (a) the area, **(b)** the perimeter, and

   (c) the length of a diagonal of the rectangle.

   (d) Writing to Learn Which of these quantities are decreasing, and which are increasing? Explain.

10. ***Changing Dimensions in a Rectangular Box*** Suppose that the edge lengths x, y, and z of a closed rectangular box are changing at the following rates:

$$\frac{dx}{dt} = 1 \text{ m/sec}, \quad \frac{dy}{dt} = -2 \text{ m/sec}, \quad \frac{dz}{dt} = 1 \text{ m/sec}.$$

   Find the rates at which the box's **(a)** volume, **(b)** surface area, and **(c)** diagonal length $s = \sqrt{x^2 + y^2 + z^2}$ are changing at the instant when $x = 4$, $y = 3$, and $z = 2$.

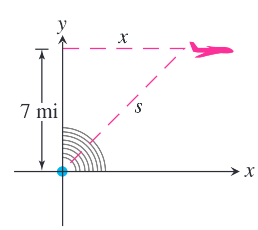
11. ***Inflating Balloon*** A spherical balloon is inflated with helium at the rate of 100π ft^3/min.

    (a) How fast is the balloon's radius increasing at the instant the radius is 5 ft?

    (b) How fast is the surface area increasing at that instant?

12. ***Growing Raindrop*** Suppose that a droplet of mist is a perfect sphere and that, through condensation, the droplet picks up moisture at a rate proportional to its surface area. Show that under these circumstances the droplet's radius increases at a constant rate.

13. ***Air Traffic Control*** An airplane is flying at an altitude of 7 mi and passes directly over a radar antenna as shown in the figure. When the plane is 10 mi from the antenna ($s = 10$), the radar detects that the distance s is changing at the rate of 300 mph. What is the speed of the airplane at that moment?



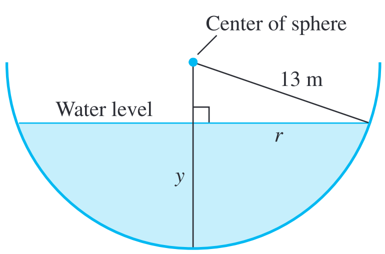
14. ***Flying a Kite*** Inge flies a kite at a height of 300 ft, the wind carrying the kite horizontally away at a rate of 25 ft/sec. How fast must she let out the string when the kite is 500 ft away from her?

15. ***Boring a Cylinder*** The mechanics at Lincoln Automotive are reboring a 6-in. deep cylinder to fit a new piston. The machine they are using increases the cylinder's radius one-thousandth of an inch every 3 min. How rapidly is the cylinder volume increasing when the bore (diameter) is 3.800 in.?

16. ***Growing Sand Pile*** Sand falls from a conveyor belt at the rate of 10 m^3/min onto the top of a conical pile. The height of the pile is always three-eighths of the base diameter. How fast are the **(a)** height and **(b)** radius changing when the pile is 4 m high? Give your answer in cm/min.

17. ***Draining Conical Reservoir*** Water is flowing at the rate of 50 m^3/min from a concrete conical reservoir (vertex down) of base radius 45 m and height 6 m. **(a)** How fast is the water level falling when the water is 5 m deep? **(b)** How fast is the radius of the water's surface changing at that moment? Give your answer in cm/min.

18. ***Draining Hemispherical Reservoir*** Water is flowing at the rate of 6 m^3/min from a reservoir shaped like a hemispherical bowl of radius 13 m, shown here in profile. Answer the following questions given that the volume of water in a hemispherical bowl of radius R is $V = (\pi/3)y^2(3R - y)$ when the water is y units deep.

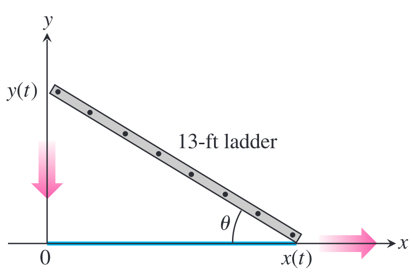


(a) At what rate is the water level changing when the water is 8 m deep?

(b) What is the radius r of the water's surface when the water is y m deep?

(c) At what rate is the radius r changing when the water is 8 m deep?

19. ***Sliding Ladder*** A 13-ft ladder is leaning against a house (see figure) when its base starts to slide away. By the time the base is 12 ft from the house, the base is moving at the rate of 5 ft/sec.

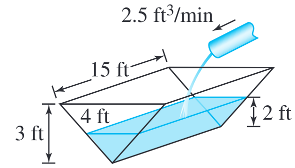


(a) How fast is the top of the ladder sliding down the wall at that moment?

(b) At what rate is the area of the triangle formed by the ladder, wall, and ground changing at that moment?

(c) At what rate is the angle θ between the ladder and the ground changing at that moment?

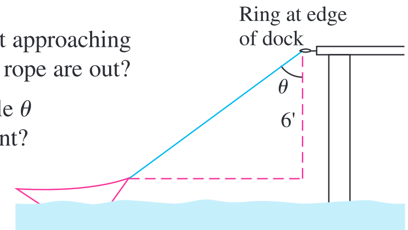
20. ***Filling a Trough*** A trough is 15 ft long and 4 ft across the top as shown in the figure. Its ends are isosceles triangles with height 3 ft. Water runs into the trough at the rate of 2.5 ft^3/min. How fast is the water level rising when it is 2 ft deep?



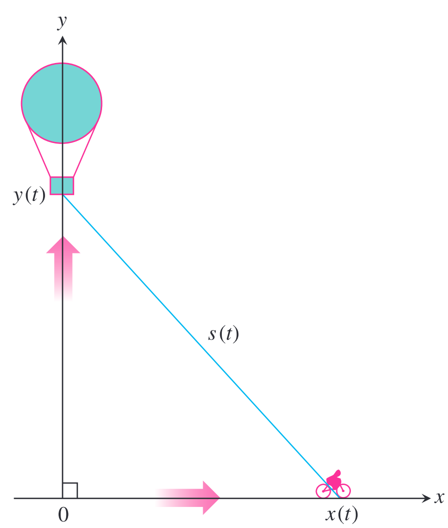
21. ***Hauling in a Dinghy*** A dinghy is pulled toward a dock by a rope from the bow through a ring on the dock 6 ft above the bow as shown in the figure. The rope is hauled in at the rate of 2 ft/sec.

(a) How fast is the boat approaching the dock when 10 ft of rope are out?

(b) At what rate is angle θ changing at that moment?



22. ***Rising Balloon*** A balloon is rising vertically above a level, straight road at a constant rate of 1 ft/sec. Just when the balloon is 65 ft above the ground, a bicycle moving at a constant rate of 17 ft/sec passes under it. How fast is the distance between the bicycle and balloon increasing 3 sec later (see figure)?



In Exercises 23 and 24, a particle is moving along the curve $y = f(x)$.

23. Let $y = f(x) = \dfrac{10}{1 + x^2}$.

If $dx/dt = 3$ cm/sec, find dy/dt at the point where

(a) $x = -2$. **(b)** $x = 0$. **(c)** $x = 20$.

24. Let $y = f(x) = x^3 - 4x$.

If $dx/dt = -2$ cm/sec, find dy/dt at the point where

(a) $x = -3$. **(b)** $x = 1$. **(c)** $x = 4$.

25. ***Particle Motion*** A particle moves along the parabola $y = x^2$ in the first quadrant in such a way that its x-coordinate (in meters) increases at a constant rate of 10 m/sec. How fast is the angle of inclination θ of theline joining the particle to the origin changing when $x = 3$?

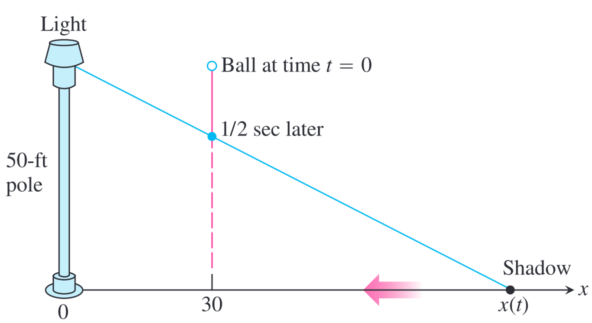
26. ***Particle Motion*** A particle moves from right to left along the parabolic curve $y = \sqrt{-x}$ in such a way that its x-coordinate (in meters) decreases at the rate of 8 m/sec. How fast is the angle of inclination θ of the line joining the particle to the origin changing when $x = -4$?

27. ***Melting Ice*** A spherical iron ball is coated with a layer of ice of uniform thickness. If the ice melts at the rate of 8 mL/min, how fast is the outer surface area of ice decreasing when the outer diameter (ball plus ice) is 20 cm?

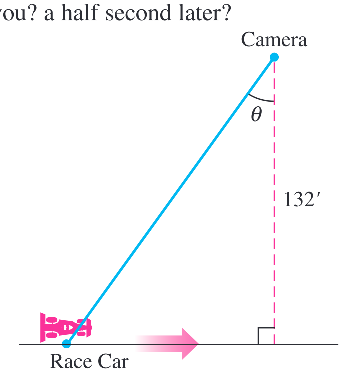
28. ***Particle Motion*** A particle $P(x, y)$ is moving in the coordinate plane in such a way that $dx/dt = -1$ m/sec and $dy/dt = -5$ m/sec. How fast is the particle's distance from the origin changing as it passes through the point (5, 12)?

29. ***Moving Shadow*** A man 6 ft tall walks at the rate of 5 ft/sec toward a streetlight that is 16 ft above the ground. At what rate is the length of his shadow changing when he is 10 ft from the base of the light?

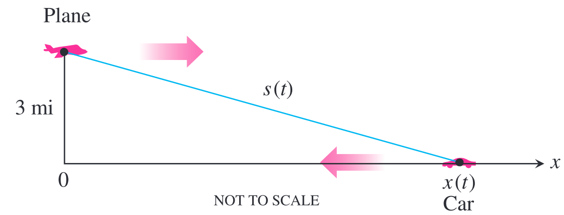
30. ***Moving Shadow*** A light shines from the top of a pole 50 ft high. A ball is dropped from the same height from a point 30 ft away from the light as shown below. How fast is the ball's shadow moving along the ground 1/2 sec later? (Assume the ball falls a distance $s = 16t^2$ in t sec.)



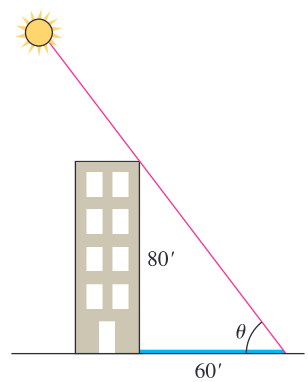
31. ***Moving Race Car*** You are videotaping a race from a stand 132 ft from the track, following a car that is moving at 180 mph (264 ft/sec) as shown in the figure. About how fast will your camera angle θ be changing when the car is right in front of you? a half second later?



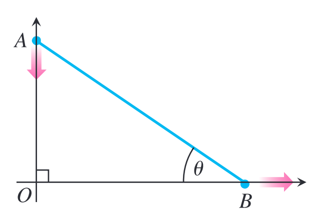
32. ***Speed Trap*** A highway patrol airplane flies 3 mi above a level, straight road at a constant rate of 120 mph. The pilot sees an oncoming car and with radar determines that at the instant the line-of-sight distance from plane to car is 5 mi the line-of-sight distance is decreasing at the rate of 160 mph. Find the car's speed along the highway.



33. ***Building's Shadow*** On a morning of a day when the sun will pass directly overhead, the shadow of an 80-ft building on level ground is 60 ft long as shown in the figure. At the moment in question, the angle θ the sun makes with the ground is increasing at the rate of 0.27°/min. At what rate is the shadow length decreasing? Express your answer in in./min, to the nearest tenth. (Remember to use radians.)



34. ***Walkers*** A and B are walking on straight streets that meet at right angles. A approaches the intersection at 2 m/sec and B moves away from the intersection at 1 m/sec as shown in the figure. At what rate is the angle θ changing when A is 10 m from the intersection and B is 20 m from the intersection? Express your answer in degrees per second to the nearest degree.



35. ***Moving Ships*** Two ships are steaming away from a point O along routes that make a 120° angle. Ship A moves at 14 knots (nautical miles per hour; a nautical mile is 2000 yards). Ship B moves at 21 knots. How fast are the ships moving apart when $OA = 5$ and $OB = 3$ nautical miles?

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

36. True or False If the radius of a circle is expanding at a constant rate, then its circumference is increasing at a constant rate. Justify your answer.

37. True or False If the radius of a circle is expanding at a constant rate, then its area is increasing at a constant rate. Justify your answer.

38. Multiple Choice If the volume of a cube is increasing at 24 in^3/min and each edge of the cube is increasing at 2 in./min, what is the length of each edge of the cube?

(A) 2 in. **(B)** $2\sqrt{2}$ in. **(C)** $\sqrt[3]{12}$ in. **(D)** 4 in. **(E)** 8 in.

39. Multiple Choice If the volume of a cube is increasing at 24 in^3/min and the surface area of the cube is increasing at 12 in^2/min, what is the length of each edge of the cube?

(A) 2 in. **(B)** $2\sqrt{2}$ in. **(C)** $\sqrt[3]{12}$ in. **(D)** 4 in. **(E)** 8 in.

40. Multiple Choice A particle is moving around the unit circle (the circle of radius 1 centered at the origin). At the point (0.6, 0.8) the particle has horizontal velocity $dx/dt = 3$. What is its vertical velocity dy/dt at that point?

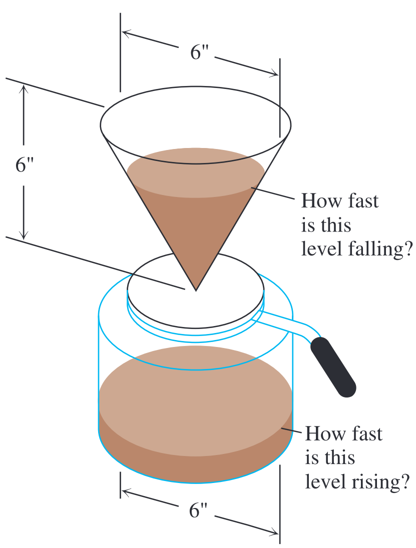
(A) -3.875 **(B)** -3.75 **(C)** -2.25 **(D)** 3.75 **(E)** 3.875

41. Multiple Choice A cylindrical rubber cord is stretched at a constant rate of 2 cm per second. Assuming its volume does not change, how fast is its radius shrinking when its length is 100 cm and its radius is 1 cm?

(A) 0 cm/sec **(B)** 0.01 cm/sec **(C)** 0.02 cm/sec
(D) 2 cm/sec **(E)** 3.979 cm/sec

Explorations

42. *Making Coffee* Coffee is draining from a conical filter into a cylindrical coffeepot at the rate of 10 in^3/min.



(a) How fast is the level in the pot rising when the coffee in the cone is 5 in. deep?

(b) How fast is the level in the cone falling at that moment?

43. *Cost, Revenue, and Profit* A company can manufacture x items at a cost of $c(x)$ dollars, a sales revenue of $r(x)$ dollars, and a profit of $p(x) = r(x) - c(x)$ dollars (all amounts in thousands). Find dc/dt, dr/dt, and dp/dt for the following values of x and dx/dt.

(a) $r(x) = 9x$, $c(x) = x^3 - 6x^2 + 15x$, and $dx/dt = 0.1$ when $x = 2$.

(b) $r(x) = 70x$, $c(x) = x^3 - 6x^2 + 45/x$, and $dx/dt = 0.05$ when $x = 1.5$.

44. Group Activity *Cardiac Output* In the late 1860s, Adolf Fick, a professor of physiology in the Faculty of Medicine in Würtzberg, Germany, developed one of the methods we use today for measuring how much blood your heart pumps in a minute. Your cardiac output as you read this sentence is probably about 7 liters a minute. At rest it is likely to be a bit under 6 L/min. If you are a trained marathon runner running a marathon, your cardiac output can be as high as 30 L/min.

Your cardiac output can be calculated with the formula

$$y = \frac{Q}{D},$$

where Q is the number of milliliters of CO_2 you exhale in a minute and D is the difference between the CO_2 concentration (mL/L) in the blood pumped to the lungs and the CO_2 concentration in the blood returning from the lungs. With $Q = 233$ mL/min and $D = 97 - 56 = 41$ mL/L,

$$y = \frac{233 \text{ mL/min}}{41 \text{ mL/L}} \approx 5.68 \text{ L/min},$$

fairly close to the 6 L/min that most people have at basal (resting) conditions. (Data courtesy of J. Kenneth Herd, M.D., Quillan College of Medicine, East Tennessee State University.)

Suppose that when $Q = 233$ and $D = 41$, we also know that D is decreasing at the rate of 2 units a minute but that Q remains unchanged. What is happening to the cardiac output?

Extending the Ideas

45. *Motion along a Circle* A wheel of radius 2 ft makes 8 revolutions about its center every second.

(a) Explain how the parametric equations

$$x = 2\cos\theta, \quad y = 2\sin\theta$$

can be used to represent the motion of the wheel.

(b) Express θ as a function of time t.

(c) Find the rate of horizontal movement and the rate of vertical movement of a point on the edge of the wheel when it is at the position given by $\theta = \pi/4$, $\pi/2$, and π.

46. *Ferris Wheel* A Ferris wheel with radius 30 ft makes one revolution every 10 sec.

(a) Assume that the center of the Ferris wheel is located at the point (0, 40), and write parametric equations to model its motion. [*Hint:* See Exercise 45.]

(b) At $t = 0$ the point P on the Ferris wheel is located at (30, 40). Find the rate of horizontal movement, and the rate of vertical movement of the point P when $t = 5$ sec and $t = 8$ sec.

47. ***Industrial Production*** **(a)** Economists often use the expression "rate of growth" in relative rather than absolute terms. For example, let $u = f(t)$ be the number of people in the labor force at time t in a given industry. (We treat this function as though it were differentiable even though it is an integer-valued step function.)

Let $v = g(t)$ be the average production per person in the labor force at time t. The total production is then $y = uv$.
If the labor force is growing at the rate of 4% per year ($du/dt = 0.04u$) and the production per worker is growing at the rate of 5% per year ($dv/dt = 0.05v$), find the rate of growth of the total production, y.

(b) Suppose that the labor force in part (a) is decreasing at the rate of 2% per year while the production per person is increasing at the rate of 3% per year. Is the total production increasing, or is it decreasing, and at what rate?

Quick Quiz for AP* Preparation: Sections 4.4–4.6

You may use a graphing calculator to solve the following problems.

1. Multiple Choice If Newton's method is used to approximate the real root of $x^3 + 2x - 1 = 0$, what would the third approximation, x_3, be if the first approximation is $x_1 = 1$?

(A) 0.453 **(B)** 0.465 **(C)** 0.495 **(D)** 0.600 **(E)** 1.977

2. Multiple Choice The sides of a right triangle with legs x and y and hypotenuse z increase in such a way that $dz/dt = 1$ and $dx/dt = 3\,dy/dt$. At the instant when $x = 4$ and $y = 3$, what is dx/dt?

(A) $\frac{1}{3}$ **(B)** 1 **(C)** 2 **(D)** $\sqrt{5}$ **(E)** 5

3. Multiple Choice An observer 70 meters south of a railroad crossing watches an eastbound train traveling at 60 meters per second. At how many meters per second is the train moving away from the observer 4 seconds after it passes through the intersection?

(A) 57.60 **(B)** 57.88 **(C)** 59.20 **(D)** 60.00 **(E)** 67.40

4. Free Response **(a)** Approximate $\sqrt{26}$ by using the linearization of $y = \sqrt{x}$ at the point (25, 5). Show the computation that leads to your conclusion.

(b) Approximate $\sqrt{26}$ by using a first guess of 5 and one iteration of Newton's method to approximate the zero of $x^2 - 26$. Show the computation that leads to your conclusion.

(c) Approximate $\sqrt[3]{26}$ by using an appropriate linearization. Show the computation that leads to your conclusion.

Chapter 4 Key Terms

absolute change (p. 240)
absolute maximum value (p. 187)
absolute minimum value (p. 187)
antiderivative (p. 200)
antidifferentiation (p. 200)
arithmetic mean (p. 204)
average cost (p. 224)
center of linear approximation (p. 233)
concave down (p. 207)
concave up (p. 207)
concavity test (p. 208)
critical point (p. 190)
decreasing function (p. 198)
differential (p. 237)
differential estimate of change (p. 239)
differential of a function (p. 239)
extrema (p. 187)
Extreme Value Theorem (p. 188)
first derivative test (p. 205)
first derivative test for local extrema (p. 205)
geometric mean (p. 204)
global maximum value (p. 177)
global minimum value (p. 177)
increasing function (p. 198)
linear approximation (p. 233)
linearization (p. 233)
local linearity (p. 233)
local maximum value (p. 189)
local minimum value (p. 189)
logistic curve (p. 210)
logistic regression (p. 211)
marginal analysis (p. 223)
marginal cost and revenue (p. 223)
Mean Value Theorem (p. 196)
monotonic function (p. 198)
Newton's method (p. 235)
optimization (p. 219)
percentage change (p. 240)
point of inflection (p. 208)
profit (p. 223)
quadratic approximation (p. 245)
related rates (p. 246)
relative change (p. 240)
relative extrema (p. 189)
Rolle's Theorem (p. 196)
second derivative test for local extrema (p. 211)
standard linear approximation (p. 233)

Chapter 4 Review Exercises

The collection of exercises marked in red could be used as a chapter test.

In Exercises 1 and 2, use analytic methods to find the global extreme values of the function on the interval and state where they occur.

1. $y = x\sqrt{2-x}, \quad -2 \le x \le 2$

2. $y = x^3 - 9x^2 - 21x - 11, \quad -\infty < x < \infty$

In Exercises 3 and 4, use analytic methods. Find the intervals on which the function is

(a) increasing, **(b)** decreasing,

(c) concave up, **(d)** concave down.

Then find any

(e) local extreme values, **(f)** inflection points.

3. $y = x^2 e^{1/x^2}$ **4.** $y = x\sqrt{4-x^2}$

In Exercises 5–16, find the intervals on which the function is

(a) increasing, **(b)** decreasing,

(c) concave up, **(d)** concave down.

Then find any

(e) local extreme values, **(f)** inflection points.

5. $y = 1 + x - x^2 - x^4$ **6.** $y = e^{x-1} - x$

7. $y = \dfrac{1}{\sqrt[4]{1-x^2}}$ **8.** $y = \dfrac{x}{x^3 - 1}$

9. $y = \cos^{-1} x$ **10.** $y = \dfrac{x}{x^2 + 2x + 3}$

11. $y = \ln|x|, \quad -2 \le x \le 2, \quad x \ne 0$

12. $y = \sin 3x + \cos 4x, \quad 0 \le x \le 2\pi$

13. $y = \begin{cases} e^{-x}, & x \le 0 \\ 4x - x^3, & x > 0 \end{cases}$

14. $y = -x^5 + \dfrac{7}{3}x^3 + 5x^2 + 4x + 2$

15. $y = x^{4/5}(2 - x)$

16. $y = \dfrac{5 - 4x + 4x^2 - x^3}{x - 2}$

In Exercises 17 and 18, use the derivative of the function $y = f(x)$ to find the points at which f has a

(a) local maximum, **(b)** local minimum, or

(c) point of inflection.

17. $y' = 6(x+1)(x-2)^2$ **18.** $y' = 6(x+1)(x-2)$

In Exercises 19–22, find all possible functions with the given derivative.

19. $f'(x) = x^{-5} + e^{-x}$ **20.** $f'(x) = \sec x \tan x$

21. $f'(x) = \dfrac{2}{x} + x^2 + 1, \; x > 0$ **22.** $f'(x) = \sqrt{x} + \dfrac{1}{\sqrt{x}}$

In Exercises 23 and 24, find the function with the given derivative whose graph passes through the point P.

23. $f'(x) = \sin x + \cos x, \quad P(\pi, 3)$

24. $f'(x) = x^{1/3} + x^2 + x + 1, \quad P(1, 0)$

In Exercises 25 and 26, the velocity v or acceleration a of a particle is given. Find the particle's position s at time t.

25. $v = 9.8t + 5, \quad s = 10 \quad$ when $\quad t = 0$

26. $a = 32, \quad v = 20 \quad$ and $\quad s = 5 \quad$ when $\quad t = 0$

In Exercises 27–30, find the linearization $L(x)$ of $f(x)$ at $x = a$.

27. $f(x) = \tan x, \quad a = -\pi/4$ **28.** $f(x) = \sec x, \quad a = \pi/4$

29. $f(x) = \dfrac{1}{1 + \tan x}, \quad a = 0$ **30.** $f(x) = e^x + \sin x, \quad a = 0$

In Exercises 31–34, use the graph to answer the questions.

31. Identify any global extreme values of f and the values of x at which they occur.

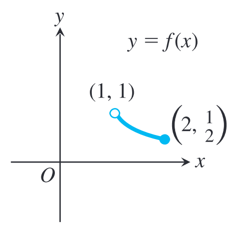


Figure for Exercise 31

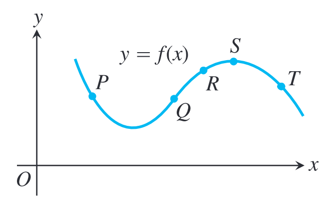


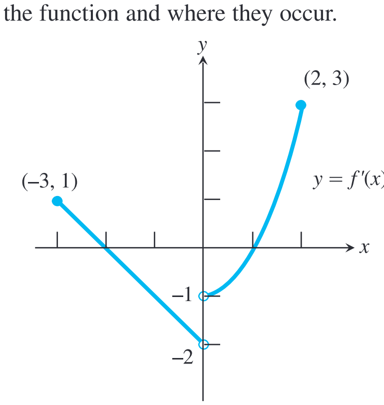
Figure for Exercise 32

32. At which of the five points on the graph of $y = f(x)$ shown here

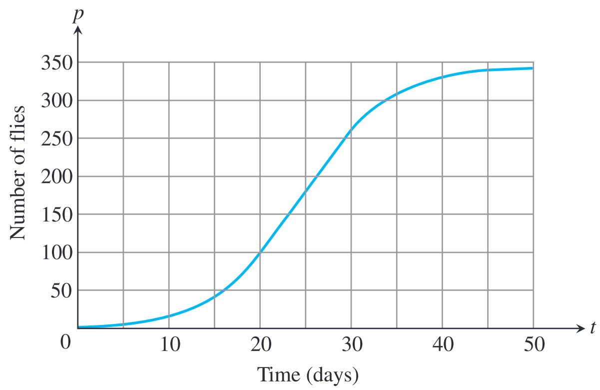
(a) are y' and y'' both negative?

(b) is y' negative and y'' positive?

33. Estimate the intervals on which the function $y = f(x)$ is **(a)** increasing; **(b)** decreasing. **(c)** Estimate any local extreme values of the function and where they occur.



34. Here is the graph of the fruit fly population from Section 2.4, Example 2. On approximately what day did the population's growth rate change from increasing to decreasing?



35. *Connecting f and f′* The graph of f' is shown in Exercise 33. Sketch a possible graph of f given that it is continuous with domain $[-3, 2]$ and $f(-3) = 0$.

36. *Connecting f, f′, and f″* The function f is continuous on $[0, 3]$ and satisfies the following.

x	0	1	2	3
f	0	-2	0	3
f'	-3	0	does not exist	4
f''	0	1	does not exist	0

x	$0 < x < 1$	$1 < x < 2$	$2 < x < 3$
f	$-$	$-$	$+$
f'	$-$	$+$	$+$
f''	$+$	$+$	$+$

(a) Find the absolute extrema of f and where they occur.

(b) Find any points of inflection.

(c) Sketch a possible graph of f.

37. *Mean Value Theorem* Let $f(x) = x \ln x$.

(a) Writing to Learn Show that f satisfies the hypotheses of the Mean Value Theorem on the interval $[a, b] = [0.5, 3]$.

(b) Find the value(s) of c in (a, b) for which

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

(c) Write an equation for the secant line AB where $A = (a, f(a))$ and $B = (b, f(b))$.

(d) Write an equation for the tangent line that is parallel to the secant line AB.

38. *Motion along a Line* A particle is moving along a line with position function $s(t) = 3 + 4t - 3t^2 - t^3$. Find the **(a)** velocity and **(b)** acceleration, and **(c)** describe the motion of the particle for $t \geq 0$.

39. *Approximating Functions* Let f be a function with $f'(x) = \sin x^2$ and $f(0) = -1$.

(a) Find the linearization of f at $x = 0$.

(b) Approximate the value of f at $x = 0.1$.

(c) Writing to Learn Is the actual value of f at $x = 0.1$ greater than or less than the approximation in (b)?

40. *Differentials* Let $y = x^2 e^{-x}$. Find **(a)** dy and **(b)** evaluate dy for $x = 1$ and $dx = 0.01$.

41. Table 4.5 shows the growth of the population of Tennessee from the 1850 census to the 1910 census. The table gives the population growth beyond the baseline number from the 1840 census, which was 829,210.

Table 4.5 Population Growth of Tennessee

Years since 1840	Growth Beyond 1840 Population
10	173,507
20	280,591
30	429,310
40	713,149
50	938,308
60	1,191,406
70	1,355,579

Source: Bureau of the Census, U.S. Chamber of Commerce

(a) Find the logistic regression for the data.

(b) Graph the data in a scatter plot and superimpose the regression curve.

(c) Use the regression equation to predict the Tennessee population in the 1920 census. Be sure to add the baseline 1840 number. (The actual 1920 census value was 2,337,885.)

(d) In what year during the period was the Tennessee population growing the fastest? What significant behavior does the graph of the regression equation exhibit at that point?

(e) What does the regression equation indicate about the population of Tennessee in the long run?

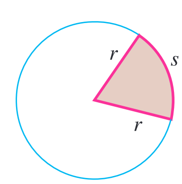
(f) Writing to Learn In fact, the population of Tennessee had already passed the long-run value predicted by this regression curve by 1930. By 2000 it had surpassed the prediction by more than 3 million people! What historical circumstances could have made the early regression so unreliable?

42. *Newton's Method* Use Newton's method to estimate all real solutions to $2 \cos x - \sqrt{1 + x} = 0$. State your answers accurate to 6 decimal places.

43. *Rocket Launch* A rocket lifts off the surface of Earth with a constant acceleration of 20 m/sec^2. How fast will the rocket be going 1 min later?

44. *Launching on Mars* The acceleration of gravity near the surface of Mars is 3.72 m/sec^2. If a rock is blasted straight up from the surface with an initial velocity of 93 m/sec (about 208 mph), how high does it go?

45. *Area of Sector* If the perimeter of the circular sector shown here is fixed at 100 ft, what values of r and s will give the sector the greatest area?

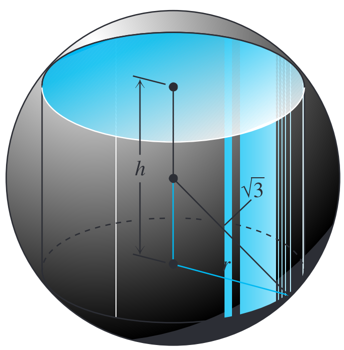


46. *Area of Triangle* An isosceles triangle has its vertex at the origin and its base parallel to the x-axis with the vertices above the axis on the curve $y = 27 - x^2$. Find the largest area the triangle can have.

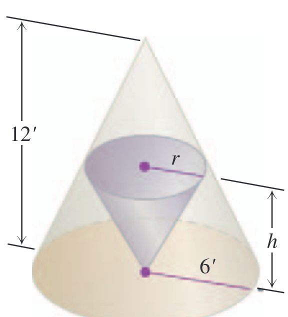
47. *Storage Bin* Find the dimensions of the largest open-top storage bin with a square base and vertical sides that can be made from 108 ft^2 of sheet steel. (Neglect the thickness of the steel and assume that there is no waste.)

48. *Designing a Vat* You are to design an open-top rectangular stainless-steel vat. It is to have a square base and a volume of 32 ft^3; to be welded from quarter-inch plate, and weigh no more than necessary. What dimensions do you recommend?

49. *Inscribing a Cylinder* Find the height and radius of the largest right circular cylinder that can be put into a sphere of radius $\sqrt{3}$ as described in the figure.



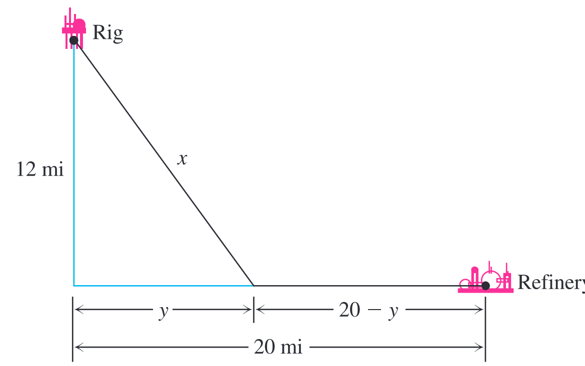
50. *Cone in a Cone* The figure shows two right circular cones, one upside down inside the other. The two bases are parallel, and the vertex of the smaller cone lies at the center of the larger cone's base. What values of r and h will give the smaller cone the largest possible volume?



51. *Box with Lid* Repeat Exercise 18 of Section 4.4 but this time remove the two equal squares from the corners of a 15-in. side.

52. *Inscribing a Rectangle* A rectangle is inscribed under one arch of $y = 8 \cos (0.3x)$ with its base on the x-axis and its upper two vertices on the curve symmetric about the y-axis. What is the largest area the rectangle can have?

53. *Oil Refinery* A drilling rig 12 mi offshore is to be connected by a pipe to a refinery onshore, 20 mi down the coast from the rig as shown in the figure. If underwater pipe costs \$40,000 per mile and land-based pipe costs \$30,000 per mile, what values of x and y give the least expensive connection?



54. *Designing an Athletic Field* An athletic field is to be built in the shape of a rectangle x units long capped by semicircular regions of radius r at the two ends. The field is to be bounded by a 400-m running track. What values of x and r will give the rectangle the largest possible area?

55. *Manufacturing Tires* Your company can manufacture x hundred grade A tires and y hundred grade B tires a day, where $0 \le x \le 4$ and

$$y = \frac{40 - 10x}{5 - x}.$$

Your profit on a grade A tire is twice your profit on a grade B tire. What is the most profitable number of each kind to make?

56. *Particle Motion* The positions of two particles on the s-axis are $s_1 = \cos t$ and $s_2 = \cos (t + \pi/4)$.

(a) What is the farthest apart the particles ever get?

(b) When do the particles collide?

57. *Open-top Box* An open-top rectangular box is constructed from a 10- by 16-in. piece of cardboard by cutting squares of equal side length from the corners and folding up the sides. Find analytically the dimensions of the box of largest volume and the maximum volume. Support your answers graphically.

58. *Changing Area* The radius of a circle is changing at the rate of $-2/\pi$ m/sec. At what rate is the circle's area changing when $r = 10$ m?

59. ***Particle Motion*** The coordinates of a particle moving in the plane are differentiable functions of time t with $dx/dt = -1$ m/sec and $dy/dt = -5$ m/sec. How fast is the particle approaching the origin as it passes through the point (5, 12)?

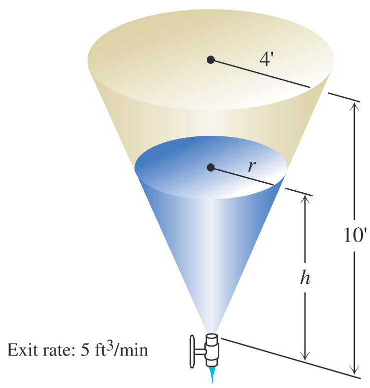
60. ***Changing Cube*** The volume of a cube is increasing at the rate of 1200 cm^3/min at the instant its edges are 20 cm long. At what rate are the edges changing at that instant?

61. ***Particle Motion*** A point moves smoothly along the curve $y = x^{3/2}$ in the first quadrant in such a way that its distance from the origin increases at the constant rate of 11 units per second. Find dx/dt when $x = 3$.

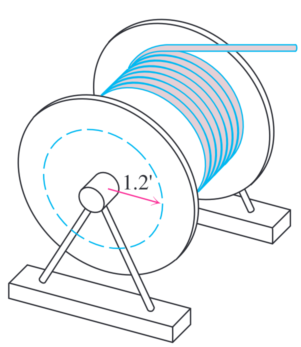
62. ***Draining Water*** Water drains from the conical tank shown in the figure at the rate of 5 ft^3/min.

(a) What is the relation between the variables h and r?

(b) How fast is the water level dropping when $h = 6$ ft?

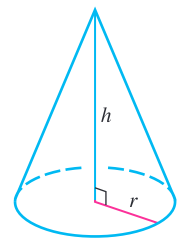


63. ***Stringing Telephone Cable*** As telephone cable is pulled from a large spool to be strung from the telephone poles along a street, it unwinds from the spool in layers of constant radius as suggested in the figure. If the truck pulling the cable moves at a constant rate of 6 ft/sec, use the equation $s = r\theta$ to find how fast (in rad/sec) the spool is turning when the layer of radius 1.2 ft is being unwound.



64. ***Throwing Dirt*** You sling a shovelful of dirt up from the bottom of a 17-ft hole with an initial velocity of 32 ft/sec. Is that enough speed to get the dirt out of the hole, or had you better duck?

65. ***Estimating Change*** Write a formula that estimates the change that occurs in the volume of a right circular cone (see figure) when the radius changes from a to $a + dr$ and the height does not change.



$V = \frac{1}{3}\pi r^2 h$

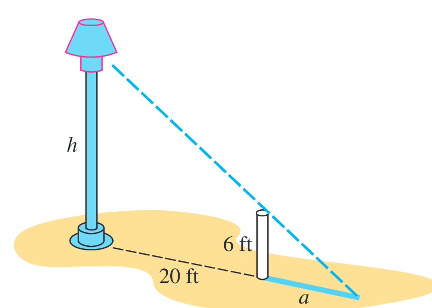
66. ***Controlling Error***

(a) How accurately should you measure the edge of a cube to be reasonably sure of calculating the cube's surface area with an error of no more than 2%?

(b) Suppose the edge is measured with the accuracy required in part (a). About how accurately can the cube's volume be calculated from the edge measurement? To find out, estimate the percentage error in the volume calculation that might result from using the edge measurement.

67. ***Compounding Error*** The circumference of the equator of a sphere is measured as 10 cm with a possible error of 0.4 cm. This measurement is then used to calculate the radius. The radius is then used to calculate the surface area and volume of the sphere. Estimate the percentage errors in the calculated values of **(a)** the radius, **(b)** the surface area, and **(c)** the volume.

68. ***Finding Height*** To find the height of a lamppost (see figure), you stand a 6-ft pole 20 ft from the lamp and measure the length a of its shadow, finding it to be 15 ft, give or take an inch. Calculate the height of the lamppost using the value $a = 15$, and estimate the possible error in the result.



69. ***Decreasing Function*** Show that the function $y = \sin^2 x - 3x$ decreases on every interval in its domain.

AP Examination Preparation

You should solve the following problems without using a calculator.

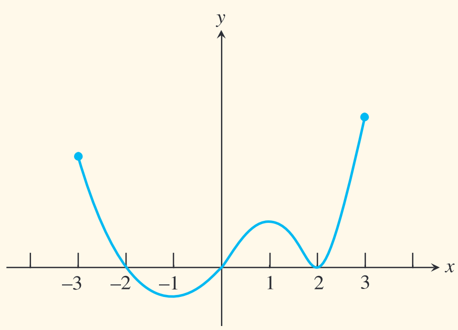
70. The accompanying figure shows the graph of the derivative of a function f. The domain of f is the closed interval $[-3, 3]$.

(a) For what values of x in the open interval $(-3, 3)$ does f have a relative maximum? Justify your answer.

(b) For what values of x in the open interval $(-3, 3)$ does f have a relative minimum? Justify your answer.

(c) For what values of x is the graph of f concave up? Justify your answer.

(d) Suppose $f(-3) = 0$. Sketch a possible graph of f on the domain $[-3, 3]$.



71. The volume V of a cone $(V = \frac{1}{3}\pi r^2 h)$ is increasing at the rate of 4π cubic inches per second. At the instant when the radius of the cone is 2 inches, its volume is 8π cubic inches and the radius is increasing at 1/3 inch per second.

(a) At the instant when the radius of the cone is 2 inches, what is the rate of change of the area of its base?

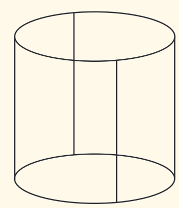
(b) At the instant when the radius of the cone is 2 inches, what is the rate of change of its height h?

(c) At the instant when the radius of the cone is 2 inches, what is the instantaneous rate of change of the area of its base with respect to its height h?

72. A piece of wire 60 inches long is cut into six sections, two of length a and four of length b. Each of the two sections of length a is bent into the form of a circle and the circles are then joined by the four sections of length b to make a frame for a model of a right circular cylinder, as shown in the accompanying figure.

(a) Find the values of a and b that will make the cylinder of maximum volume.

(b) Use differential calculus to justify your answer in part (a).



Chapter 5 The Definite Integral

The 1995 Reader's Digest Sweepstakes grand prize winner is being paid a total of \$5,010,000 over 30 years. If invested, the winnings plus the interest earned generate an amount defined by:

$$A = e^{rT} \int_0^T mPe^{-rt}\, dt$$

(r = interest rate, P = size of payment, T = term in years, m = number of payments per year.)

Would the prize have a different value if it were paid in 15 annual installments of \$334,000 instead of 30 annual installments of \$167,000? Section 5.4 can help you compare the total amounts.

Chapter 5 Overview

The need to calculate instantaneous rates of change led the discoverers of calculus to an investigation of the slopes of tangent lines and, ultimately, to the derivative—to what we call *differential* calculus. But derivatives revealed only half the story. In addition to a calculation method (a "calculus") to describe how functions change at any given instant, they needed a method to describe how those instantaneous changes could accumulate over an interval to produce the function. That is why they also investigated *areas under curves,* which ultimately led to the second main branch of calculus, called *integral* calculus.

Once Newton and Leibniz had the calculus for finding slopes of tangent lines and the calculus for finding areas under curves—two geometric operations that would seem to have nothing at all to do with each other—the challenge for them was to prove the connection that they knew intuitively had to be there. The discovery of this connection (called the Fundamental Theorem of Calculus) brought differential and integral calculus together to become the single most powerful insight mathematicians had ever acquired for understanding how the universe worked.

5.1 Estimating with Finite Sums

What you'll learn about

- Distance Traveled
- Rectangular Approximation Method (RAM)
- Volume of a Sphere
- Cardiac Output

. . . and why

Learning about estimating with finite sums sets the foundation for understanding integral calculus.

Distance Traveled

We know why a mathematician pondering motion problems might have been led to consider slopes of curves, but what do those same motion problems have to do with areas under curves? Consider the following problem from a typical elementary school textbook:

> A train moves along a track at a steady rate of 75 miles per hour from 7:00 A.M. to 9:00 A.M. What is the total distance traveled by the train?

Applying the well-known formula *distance* = *rate* × *time,* we find that the answer is 150 miles. Simple. Now suppose that you are Isaac Newton trying to make a connection between this formula and the graph of the velocity function.

You might notice that the distance traveled by the train (150 miles) is exactly the *area* of the rectangle whose base is the time interval $[7, 9]$ and whose height at each point is the value of the constant velocity function $v = 75$ (Figure 5.1). This is no accident, either, since *the distance traveled* and *the area* in this case are both found by multiplying the rate (75) by the change in time (2).

This same connection between distance traveled and rectangle area could be made no matter how fast the train was going or how long or short the time interval was. But what if the train had a velocity v that *varied* as a function of time? The graph (Figure 5.2) would no longer be a horizontal line, so the region under the graph would no longer be rectangular.

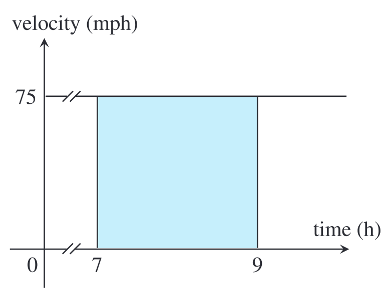


Figure 5.1 The distance traveled by a 75 mph train in 2 hours is 150 miles, which corresponds to the area of the shaded rectangle.

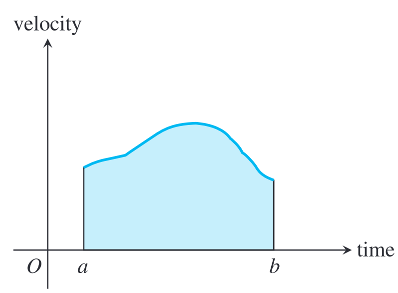


Figure 5.2 If the velocity varies over the time interval $[a, b]$, does the shaded region give the distance traveled?

Would the area of this irregular region still give the total distance traveled over the time interval? Newton and Leibniz (and, actually, many others who had considered this question) thought that it obviously would, and that is why they were interested in a calculus for finding areas under curves. They imagined the time interval being partitioned into many tiny subintervals, each one so small that the velocity over it would essentially be constant. Geometrically, this was equivalent to slicing the irregular region into narrow strips, each of which would be nearly indistinguishable from a narrow rectangle (Figure 5.3).

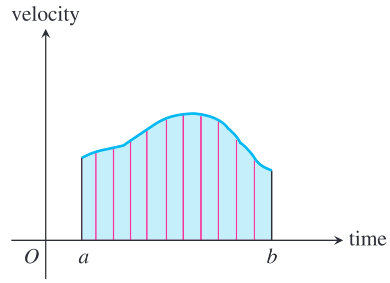


Figure 5.3 The region is partitioned into vertical strips. If the strips are narrow enough, they are almost indistinguishable from rectangles. The sum of the areas of these "rectangles" will give the total area and can be interpreted as distance traveled.

They argued that, just as the total area could be found by summing the areas of the (essentially rectangular) strips, the total distance traveled could be found by summing the small distances traveled over the tiny time intervals.

EXAMPLE 1 Finding Distance Traveled when Velocity Varies

A particle starts at $x = 0$ and moves along the x-axis with velocity $v(t) = t^2$ for time $t \geq 0$. Where is the particle at $t = 3$?

SOLUTION

We graph v and partition the time interval $[0, 3]$ into subintervals of length Δt. (Figure 5.4 shows twelve subintervals of length 3/12 each.)

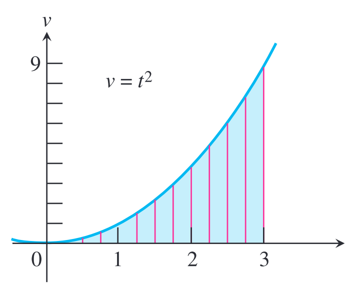


Figure 5.4 The region under the parabola $v = t^2$ from $t = 0$ to $t = 3$ is partitioned into 12 thin strips, each with base $\Delta t = 1/4$. The strips have curved tops. (Example 1)

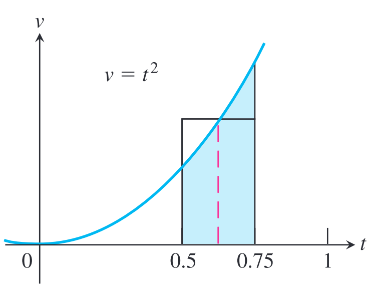


Figure 5.5 The area of the shaded region is approximated by the area of the rectangle whose height is the function value at the midpoint of the interval. (Example 1)

Notice that the region under the curve is partitioned into thin strips with bases of length 1/4 and *curved* tops that slope upward from left to right. You might not know how to find the area of such a strip, but you can get a good approximation of it by finding the area of a suitable rectangle. In Figure 5.5, we use the rectangle whose height is the y-coordinate of the function at the midpoint of its base.

continued

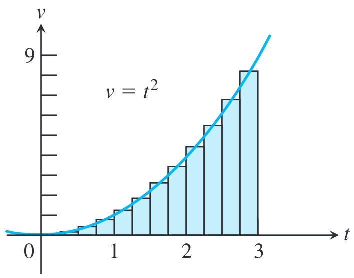


Figure 5.6 These rectangles have approximately the same areas as the strips in Figure 5.4. Each rectangle has height m_i^2, where m_i is the midpoint of its base. (Example 1)

The area of this narrow rectangle approximates the distance traveled over the time subinterval. Adding all the areas (distances) gives an approximation of the total area under the curve (total distance traveled) from $t = 0$ to $t = 3$ (Figure 5.6).

Computing this sum of areas is straightforward. Each rectangle has a base of length $\Delta t = 1/4$, while the height of each rectangle can be found by evaluating the function at the midpoint of the subinterval. Table 5.1 shows the computations for the first four rectangles.

Table 5.1

Subinterval	$\left[0, \frac{1}{4}\right]$	$\left[\frac{1}{4}, \frac{1}{2}\right]$	$\left[\frac{1}{2}, \frac{3}{4}\right]$	$\left[\frac{3}{4}, 1\right]$
Midpoint m_i	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{5}{8}$	$\frac{7}{8}$
Height $= (m_i)^2$	$\frac{1}{64}$	$\frac{9}{64}$	$\frac{25}{64}$	$\frac{49}{64}$
Area $= (1/4)(m_i)^2$	$\frac{1}{256}$	$\frac{9}{256}$	$\frac{25}{256}$	$\frac{49}{256}$

Continuing in this manner, we derive the area $(1/4)(m_i)^2$ for each of the twelve subintervals and add them:

$$\frac{1}{256} + \frac{9}{256} + \frac{25}{256} + \frac{49}{256} + \frac{81}{256} + \frac{121}{256} + \frac{169}{256} + \frac{225}{256}$$

$$+ \frac{289}{256} + \frac{361}{256} + \frac{441}{256} + \frac{529}{256} = \frac{2300}{256} \approx 8.98.$$

Since this number approximates the area and hence the total distance traveled by the particle, we conclude that the particle has moved approximately 9 units in 3 seconds. If it starts at $x = 0$, then it is very close to $x = 9$ when $t = 3$. ***Now try Exercise 3.***

Approximation by Rectangles

Approximating irregularly-shaped regions by regularly-shaped regions for the purpose of computing areas is not new. Archimedes used the idea more than 2200 years ago to find the area of a circle, demonstrating in the process that π was located between 3.140845 and 3.142857. He also used approximations to find the area under a parabolic arch, anticipating the answer to an important seventeenth-century question nearly 2000 years before anyone thought to ask it. The fact that we still measure the area of anything—even a circle—in "square units" is obvious testimony to the historical effectiveness of using rectangles for approximating areas.

To make it easier to talk about approximations with rectangles, we now introduce some new terminology.

Rectangular Approximation Method (RAM)

In Example 1 we used the *Midpoint Rectangular Approximation Method* (MRAM) to approximate the area under the curve. The name suggests the choice we made when determining the heights of the approximating rectangles: We evaluated the function at the midpoint of each subinterval. If instead we had evaluated the function at the left-hand endpoint we would have obtained the *LRAM* approximation, and if we had used the right-hand endpoints we would have obtained the *RRAM* approximation. Figure 5.7 shows what the three approximations look like graphically when we approximate the area under the curve $y = x^2$ from $x = 0$ to $x = 3$ with six subintervals.

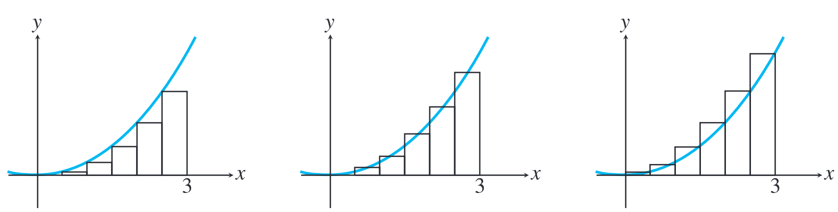


Figure 5.7 LRAM, MRAM, and RRAM approximations to the area under the graph of $y = x^2$ from $x = 0$ to $x = 3$.

No matter which RAM approximation we compute, we are adding products of the form $f(x_i) \cdot \Delta x$, or, in this case, $(x_i)^2 \cdot (3/6)$.

LRAM:

$$\left(0\right)^2\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)+\left(1\right)^2\left(\frac{1}{2}\right)+\left(\frac{3}{2}\right)^2\left(\frac{1}{2}\right)+\left(2\right)^2\left(\frac{1}{2}\right)+\left(\frac{5}{2}\right)^2\left(\frac{1}{2}\right)=6.875$$

MRAM:

$$\left(\frac{1}{4}\right)^2\left(\frac{1}{2}\right)+\left(\frac{3}{4}\right)^2\left(\frac{1}{2}\right)+\left(\frac{5}{4}\right)^2\left(\frac{1}{2}\right)+\left(\frac{7}{4}\right)^2\left(\frac{1}{2}\right)+\left(\frac{9}{4}\right)^2\left(\frac{1}{2}\right)+\left(\frac{11}{4}\right)^2\left(\frac{1}{2}\right)=8.9375$$

RRAM:

$$\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)+\left(1\right)^2\left(\frac{1}{2}\right)+\left(\frac{3}{2}\right)^2\left(\frac{1}{2}\right)+\left(2\right)^2\left(\frac{1}{2}\right)+\left(\frac{5}{2}\right)^2\left(\frac{1}{2}\right)+\left(3\right)^2\left(\frac{1}{2}\right)=11.375$$

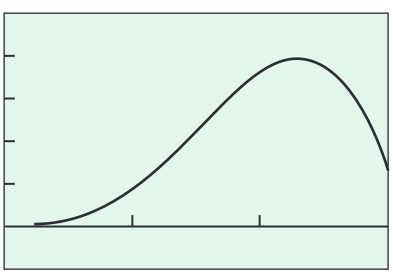
As we can see from Figure 5.7, LRAM is smaller than the true area and RRAM is larger. MRAM appears to be the closest of the three approximations. However, observe what happens as the number n of subintervals increases:

n	LRAM_n	MRAM_n	RRAM_n
6	6.875	8.9375	11.375
12	7.90625	8.984375	10.15625
24	8.4453125	8.99609375	9.5703125
48	8.720703125	8.999023438	9.283203125
100	8.86545	8.999775	9.13545
1000	8.9865045	8.99999775	9.0135045

We computed the numbers in this table using a graphing calculator and a summing program called RAM. A version of this program for most graphing calculators can be found in the *Technology Resource Manual* that accompanies this textbook. *All three sums* approach the same number (in this case, 9).

EXAMPLE 2 Estimating Area Under the Graph of a Nonnegative Function

Figure 5.8 shows the graph of $f(x) = x^2 \sin x$ on the interval $[0, 3]$. Estimate the area under the curve from $x = 0$ to $x = 3$.



[0, 3] by [–1, 5]

Figure 5.8 The graph of $y = x^2 \sin x$ over the interval $[0, 3]$. (Example 2)

SOLUTION

We apply our RAM program to get the numbers in this table.

n	LRAM_n	MRAM_n	RRAM_n
5	5.15480	5.89668	5.91685
10	5.52574	5.80685	5.90677
25	5.69079	5.78150	5.84320
50	5.73615	5.77788	5.81235
100	5.75701	5.77697	5.79511
1000	5.77476	5.77667	5.77857

It is not necessary to compute all three sums each time just to approximate the area, but we wanted to show again how all three sums approach the same number. With 1000 subintervals, all three agree in the first three digits. (The *exact* area is $-7 \cos 3 + 6 \sin 3 - 2$, which is 5.77666752456 to twelve digits.)

Now try Exercise 7.

EXPLORATION 1 Which RAM is the Biggest?

You might think from the previous two RAM tables that LRAM is always a little low and RRAM a little high, with MRAM somewhere in between. That, however, depends on n and on the shape of the curve.

1. Graph $y = 5 - 4\sin(x/2)$ in the window $[0, 3]$ by $[0, 5]$. Copy the graph on paper and sketch the rectangles for the LRAM, MRAM, and RRAM sums with $n = 3$. Order the three approximations from greatest to smallest.
2. Graph $y = 2\sin(5x) + 3$ in the same window. Copy the graph on paper and sketch the rectangles for the LRAM, MRAM, and RRAM sums with $n = 3$. Order the three approximations from greatest to smallest.
3. If a positive, continuous function is increasing on an interval, what can we say about the relative sizes of LRAM, MRAM, and RRAM? Explain.
4. If a positive, continuous function is decreasing on an interval, what can we say about the relative sizes of LRAM, MRAM, and RRAM? Explain.

Volume of a Sphere

Although the visual representation of RAM approximation focuses on area, remember that our original motivation for looking at sums of this type was to find distance traveled by an object moving with a nonconstant velocity. The connection between Examples 1 and 2 is that in each case, we have a function f defined on a closed interval and estimate what we want to know with a sum of function values multiplied by interval lengths. Many other physical quantities can be estimated this way.

EXAMPLE 3 Estimating the Volume of a Sphere

Estimate the volume of a solid sphere of radius 4.

SOLUTION

We picture the sphere as if its surface were generated by revolving the graph of the function $f(x) = \sqrt{16 - x^2}$ about the x-axis (Figure 5.9a). We partition the interval $-4 \le x \le 4$ into n subintervals of equal length $\Delta x = 8/n$. We then slice the sphere with planes perpendicular to the x-axis at the partition points, cutting it like a round loaf of bread into n parallel slices of width Δx. When n is large, each slice can be approximated by a cylinder, a familiar geometric shape of known volume, $\pi r^2 h$. In our case, the cylinders lie on their sides and h is Δx while r varies according to where we are on the x-axis (Figure 5.9b). A logical radius to choose for each cylinder is $f(m_i) = \sqrt{16 - m_i^2}$, where m_i is the midpoint of the interval where the i^{th} slice intersects the x-axis (Figure 5.9c).

We can now approximate the volume of the sphere by using MRAM to sum the cylinder volumes,

$$\pi r^2 h = \pi\left(\sqrt{16 - m_i^2}\right)^2 \Delta x.$$

The function we use in the RAM program is $\pi\left(\sqrt{16 - x^2}\right)^2 = \pi(16 - x^2)$. The interval is $[-4, 4]$.

Number of Slices (n)	MRAM_n
10	269.42299
25	268.29704
50	268.13619
100	268.09598
1000	268.08271

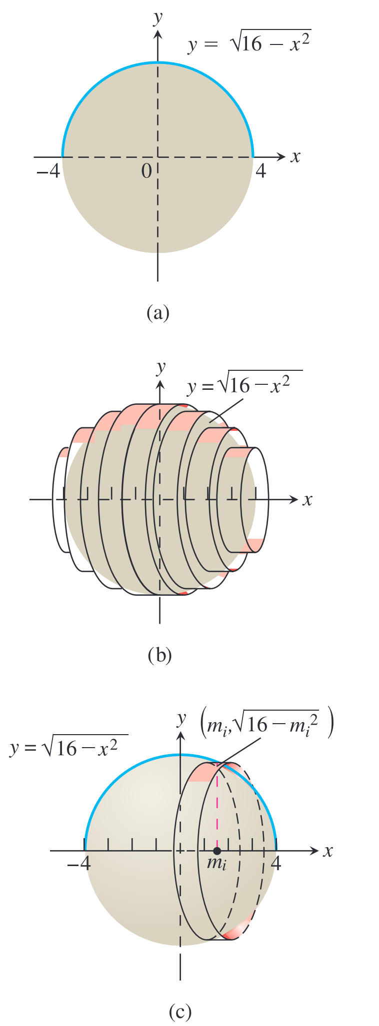


Figure 5.9 (a) The semicircle $y = \sqrt{16 - x^2}$ revolved about the x-axis to generate a sphere. (b) Slices of the solid sphere approximated with cylinders (drawn for $n = 8$). (c) The typical approximating cylinder has radius $f(m_i) = \sqrt{16 - m_i^2}$. (Example 3)

continued

Keeping Track of Units

Notice in Example 3 that we are summing products of the form $\pi(16 - x^2)$ (a cross section area, measured in square units) times Δx (a length, measured in units). The products are therefore measured in cubic units, which are the correct units for volume.

The value for $n = 1000$ compares *very* favorably with the true volume,

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(4)^3 = \frac{256\pi}{3} \approx 268.0825731.$$

Even for $n = 10$ the difference between the MRAM approximation and the true volume is a small percentage of V:

$$\frac{|\text{MRAM}_{10} - V|}{V} = \frac{\text{MRAM}_{10} - 256\pi/3}{256\pi/3} \leq 0.005.$$

That is, the error percentage is about one half of one percent! ***Now try Exercise 13.***

Cardiac Output

So far we have seen applications of the RAM process to finding distance traveled and volume. These applications hint at the usefulness of this technique. To suggest its versatility we will present an application from human physiology.

The number of liters of blood your heart pumps in a fixed time interval is called your *cardiac output*. For a person at rest, the rate might be 5 or 6 liters per minute. During strenuous exercise the rate might be as high as 30 liters per minute. It might also be altered significantly by disease. How can a physician measure a patient's cardiac output without interrupting the flow of blood?

One technique is to inject a dye into a main vein near the heart. The dye is drawn into the right side of the heart and pumped through the lungs and out the left side of the heart into the aorta, where its concentration can be measured every few seconds as the blood flows past. The data in Table 5.2 and the plot in Figure 5.10 (obtained from the data) show the response of a healthy, resting patient to an injection of 5.6 mg of dye.

Table 5.2 Dye Concentration Data

Seconds after Injection t	Dye Concentration (adjusted for recirculation) c
5	0
7	3.8
9	8.0
11	6.1
13	3.6
15	2.3
17	1.45
19	0.91
21	0.57
23	0.36
25	0.23
27	0.14
29	0.09
31	0

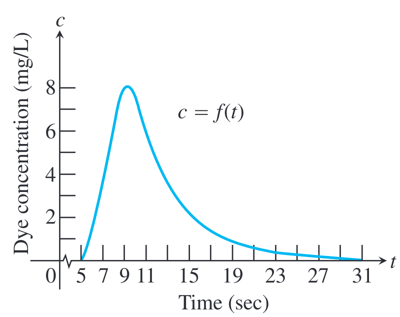


Figure 5.10 The dye concentration data from Table 5.2, plotted and fitted with a smooth curve. Time is measured with $t = 0$ at the time of injection. The dye concentration is zero at the beginning while the dye passes through the lungs. It then rises to a maximum at about $t = 9$ sec and tapers to zero by $t = 31$ sec.

The graph shows dye concentration (measured in milligrams of dye per liter of blood) as a function of time (in seconds). How can we use this graph to obtain the cardiac output (measured in liters of blood per second)? The trick is to divide the *number of mg of dye* by the *area under the dye concentration curve.* You can see why this works if you consider what happens to the units:

$$\frac{\text{mg of dye}}{\text{units of area under curve}} = \frac{\text{mg of dye}}{\dfrac{\text{mg of dye}}{\text{L of blood}} \cdot \text{sec}} = \frac{\text{mg of dye}}{\text{sec}} \cdot \frac{\text{L of blood}}{\text{mg of dye}} = \frac{\text{L of blood}}{\text{sec}}.$$

So you are now ready to compute like a cardiologist.

Charles Richard Drew

(1904–1950)

Millions of people are alive today because of Charles Drew's pioneering work on blood plasma and the preservation of human blood for transfusion. After directing the Red Cross program that collected plasma for the Armed Forces in World War II, Dr. Drew went on to become Head of Surgery at Howard University and Chief of Staff at Freedmen's Hospital in Washington, D.C.

EXAMPLE 4 Computing Cardiac Output from Dye Concentration

Estimate the cardiac output of the patient whose data appear in Table 5.2 and Figure 5.10. Give the estimate in liters per minute.

SOLUTION

We have seen that we can obtain the cardiac output by dividing the amount of dye (5.6 mg for our patient) by the area under the curve in Figure 5.10. Now we need to find the area. Our geometry formulas do not apply to this irregularly shaped region, and the RAM program is useless without a formula for the function. Nonetheless, we can draw the MRAM rectangles ourselves and estimate their heights from the graph. In Figure 5.11 each rectangle has a base 2 units long and a height $f(m_i)$ equal to the height of the curve above the midpoint of the base.

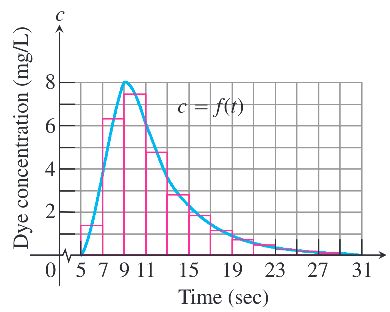


Figure 5.11 The region under the concentration curve of Figure 5.10 is approximated with rectangles. We ignore the portion from $t = 29$ to $t = 31$; its concentration is negligible. (Example 4)

The area of each rectangle, then, is $f(m_i)$ times 2, and the sum of the rectangular areas is the MRAM estimate for the area under the curve:

$$\begin{aligned}\text{Area} &\approx f(6)\cdot 2 + f(8)\cdot 2 + f(10)\cdot 2 + \cdots + f(28)\cdot 2\\ &\approx 2\cdot(1.4 + 6.3 + 7.5 + 4.8 + 2.8 + 1.9 + 1.1\\ &\qquad + 0.7 + 0.5 + 0.3 + 0.2 + 0.1)\\ &= 2\cdot(27.6) = 55.2\ (\text{mg/L})\cdot\text{sec}.\end{aligned}$$

Dividing 5.6 mg by this figure gives an estimate for cardiac output in liters per second. Multiplying by 60 converts the estimate to liters per minute:

$$\frac{5.6\text{ mg}}{55.2\text{ mg}\cdot\text{sec/L}}\cdot\frac{60\text{ sec}}{1\text{ min}} \approx 6.09\text{ L/min}.$$

Now try Exercise 15.

Quick Review 5.1

As you answer the questions in Exercises 1–10, try to associate the answers with area, as in Figure 5.1.

1. A train travels at 80 mph for 5 hours. How far does it travel?
2. A truck travels at an average speed of 48 mph for 3 hours. How far does it travel?
3. Beginning at a standstill, a car maintains a constant acceleration of 10 ft/sec^2 for 10 seconds. What is its velocity after 10 seconds? Give your answer in ft/sec and then convert it to mi/h.
4. In a vacuum, light travels at a speed of 300,000 kilometers per second. How many kilometers does it travel in a year? (This distance equals one *light-year.*)
5. A long distance runner ran a race in 5 hours, averaging 6 mph for the first 3 hours and 5 mph for the last 2 hours. How far did she run?
6. A pump working at 20 gallons/minute pumps for an hour. How many gallons are pumped?

7. At 8:00 P.M. the temperature began dropping at a rate of 1 degree Celsius per hour. Twelve hours later it began rising at a rate of 1.5 degrees per hour for six hours. What was the net change in temperature over the 18-hour period?

8. Water flows over a spillway at a steady rate of 300 cubic feet per second. How many cubic feet of water pass over the spillway in one day?

9. A city has a population density of 350 people per square mile in an area of 50 square miles. What is the population of the city?

10. A hummingbird in flight beats its wings at a rate of 70 times per second. How many times does it beat its wings in an hour if it is in flight 70% of the time?

Section 5.1 Exercises

1. A particle starts at $x = 0$ and moves along the x-axis with velocity $v(t) = 5$ for time $t \geq 0$. Where is the particle at $t = 4$?

2. A particle starts at $x = 0$ and moves along the x-axis with velocity $v(t) = 2t + 1$ for time $t \geq 0$. Where is the particle at $t = 4$?

3. A particle starts at $x = 0$ and moves along the x-axis with velocity $v(t) = t^2 + 1$ for time $t \geq 0$. Where is the particle at $t = 4$? Approximate the area under the curve using four rectangles of equal width and heights determined by the midpoints of the intervals, as in Example 1.

4. A particle starts at $x = 0$ and moves along the x-axis with velocity $v(t) = t^2 + 1$ for time $t \geq 0$. Where is the particle at $t = 5$? Approximate the area under the curve using five rectangles of equal width and heights determined by the midpoints of the intervals, as in Example 1.

Exercises 5–8 refer to the region R enclosed between the graph of the function $y = 2x - x^2$ and the x-axis for $0 \leq x \leq 2$.

5. **(a)** Sketch the region R.

(b) Partition $[0, 2]$ into 4 subintervals and show the four rectangles that LRAM uses to approximate the area of R. Compute the LRAM sum without a calculator.

6. Repeat Exercise 1(b) for RRAM and MRAM.

7. Using a calculator program, find the RAM sums that complete the following table.

n	LRAM_n	MRAM_n	RRAM_n
10			
50			
100			
500			

8. Make a conjecture about the area of the region R.

In Exercises 9–12, use RAM to estimate the area of the region enclosed between the graph of f and the x-axis for $a \leq x \leq b$.

9. $f(x) = x^2 - x + 3, \quad a = 0, \quad b = 3$

10. $f(x) = \dfrac{1}{x}, \quad a = 1, \quad b = 3$

11. $f(x) = e^{-x^2}, \quad a = 0, \quad b = 2$

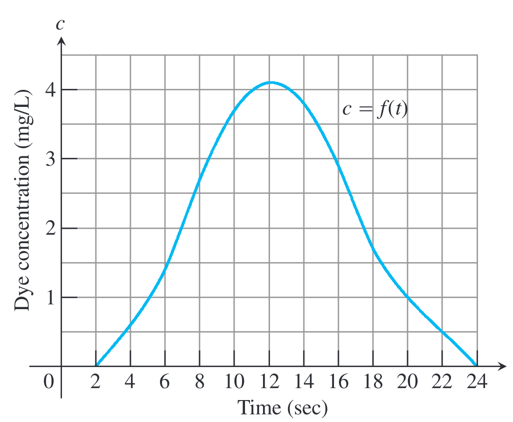
12. $f(x) = \sin x, \quad a = 0, \quad b = \pi$

13. *(Continuation of Example 3)* Use the slicing technique of Example 3 to find the MRAM sums that approximate the volume of a sphere of radius 5. Use $n = 10, 20, 40, 80$, and 160.

14. *(Continuation of Exercise 13)* Use a geometry formula to find the volume V of the sphere in Exercise 13 and find **(a)** the error and **(b)** the percentage error in the MRAM approximation for each value of n given.

15. ***Cardiac Output*** The following table gives dye concentrations for a dye-concentration cardiac-output determination like the one in Example 4. The amount of dye injected in this patient was 5 mg instead of 5.6 mg. Use rectangles to estimate the area under the dye concentration curve and then go on to estimate the patient's cardiac output.

Seconds after Injection t	Dye Concentration (adjusted for recirculation) c
2	0
4	0.6
6	1.4
8	2.7
10	3.7
12	4.1
14	3.8
16	2.9
18	1.7
20	1.0
22	0.5
24	0



16. ***Distance Traveled*** The table below shows the velocity of a model train engine moving along a track for 10 sec. Estimate the distance traveled by the engine, using 10 subintervals of length 1 with **(a)** left-endpoint values (LRAM) and **(b)** right-endpoint values (RRAM).

Time (sec)	Velocity (in./sec)	Time (sec)	Velocity (in./sec)
0	0	6	11
1	12	7	6
2	22	8	2
3	10	9	6
4	5	10	0
5	13		

17. ***Distance Traveled Upstream*** You are walking along the bank of a tidal river watching the incoming tide carry a bottle upstream. You record the velocity of the flow every 5 minutes for an hour, with the results shown in the table below. About how far upstream does the bottle travel during that hour? Find the **(a)** LRAM and **(b)** RRAM estimates using 12 subintervals of length 5.

Time (min)	Velocity (m/sec)	Time (min)	Velocity (m/sec)
0	1	35	1.2
5	1.2	40	1.0
10	1.7	45	1.8
15	2.0	50	1.5
20	1.8	55	1.2
25	1.6	60	0
30	1.4		

18. ***Length of a Road*** You and a companion are driving along a twisty stretch of dirt road in a car whose speedometer works but whose odometer (mileage counter) is broken. To find out how long this particular stretch of road is, you record the car's velocity at 10-sec intervals, with the results shown in the table below. (The velocity was converted from mi/h to ft/sec using 30 mi/h = 44 ft/sec.) Estimate the length of the road by averaging the LRAM and RRAM sums.

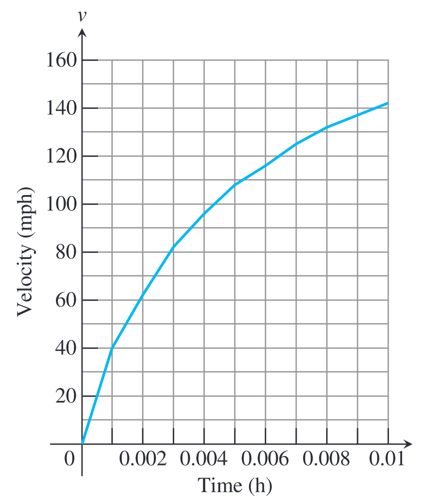
Time (sec)	Velocity (ft/sec)	Time (sec)	Velocity (ft/sec)
0	0	70	15
10	44	80	22
20	15	90	35
30	35	100	44
40	30	110	30
50	44	120	35
60	35		

19. ***Distance from Velocity Data*** The table below gives data for the velocity of a vintage sports car accelerating from 0 to 142 mi/h in 36 sec (10 thousandths of an hour.)

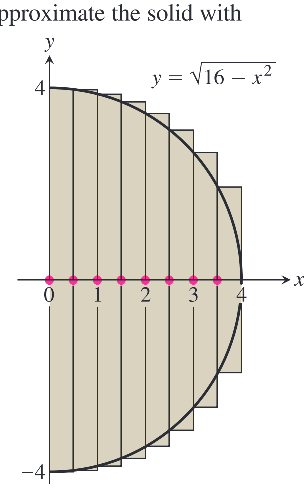
Time (h)	Velocity (mi/h)	Time (h)	Velocity (mi/h)
0.0	0	0.006	116
0.001	40	0.007	125
0.002	62	0.008	132
0.003	82	0.009	137
0.004	96	0.010	142
0.005	108		

(a) Use rectangles to estimate how far the car traveled during the 36 sec it took to reach 142 mi/h.

(b) Roughly how many seconds did it take the car to reach the halfway point? About how fast was the car going then?



20. ***Volume of a Solid Hemisphere*** To estimate the volume of a solid hemisphere of radius 4, imagine its axis of symmetry to be the interval [0, 4] on the x-axis. Partition [0, 4] into eight subintervals of equal length and approximate the solid with cylinders based on the circular cross sections of the hemisphere perpendicular to the x-axis at the subintervals' left endpoints. (See the accompanying profile view.)



(a) **Writing to Learn** Find the sum S_8 of the volumes of the cylinders. Do you expect S_8 to overestimate V? Give reasons for your answer.

(b) Express $|V - S_8|$ as a percentage of V to the nearest percent.

21. Repeat Exercise 20 using cylinders based on cross sections at the right endpoints of the subintervals.

22. ***Volume of Water in a Reservoir*** A reservoir shaped like a hemispherical bowl of radius 8 m is filled with water to a depth of 4 m.

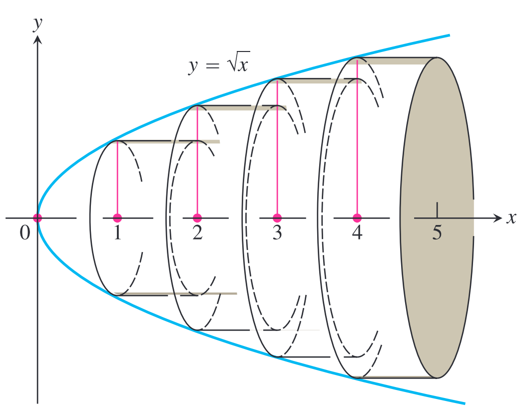
(a) Find an estimate S of the water's volume by approximating the water with eight circumscribed solid cylinders.

(b) It can be shown that the water's volume is $V = (320\pi)/3 \text{ m}^3$. Find the error $|V - S|$ as a percentage of V to the nearest percent.

23. ***Volume of Water in a Swimming Pool*** A rectangular swimming pool is 30 ft wide and 50 ft long. The table below shows the depth $h(x)$ of the water at 5-ft intervals from one end of the pool to the other. Estimate the volume of water in the pool using **(a)** left-endpoint values and **(b)** right-endpoint values.

Position (ft) x	Depth (ft) $h(x)$	Position (ft) x	Depth (ft) $h(x)$
0	6.0	30	11.5
5	8.2	35	11.9
10	9.1	40	12.3
15	9.9	45	12.7
20	10.5	50	13.0
25	11.0		

24. ***Volume of a Nose Cone*** The nose "cone" of a rocket is a *paraboloid* obtained by revolving the curve $y = \sqrt{x}, 0 \le x \le 5$ about the x-axis, where x is measured in feet. Estimate the volume V of the nose cone by partitioning $[0, 5]$ into five subintervals of equal length, slicing the cone with planes perpendicular to the x-axis at the subintervals' left endpoints, constructing cylinders of height 1 based on cross sections at these points, and finding the volumes of these cylinders. (See the accompanying figure.)



25. ***Volume of a Nose Cone*** Repeat Exercise 24 using cylinders based on cross sections at the *midpoints* of the subintervals.

26. ***Free Fall with Air Resistance*** An object is dropped straight down from a helicopter. The object falls faster and faster but its acceleration (rate of change of its velocity) decreases over time because of air resistance. The acceleration is measured in ft/sec^2 and recorded every second after the drop for 5 sec, as shown in the table below.

t	0	1	2	3	4	5
a	32.00	19.41	11.77	7.14	4.33	2.63

(a) Use LRAM_5 to find an upper estimate for the speed when $t = 5$.

(b) Use RRAM_5 to find a lower estimate for the speed when $t = 5$.

(c) Use upper estimates for the speed during the first second, second second, and third second to find an upper estimate for the distance fallen when $t = 3$.

27. ***Distance Traveled by a Projectile*** An object is shot straight upward from sea level with an initial velocity of 400 ft/sec.

(a) Assuming gravity is the only force acting on the object, give an upper estimate for its velocity after 5 sec have elapsed. Use $g = 32 \text{ ft/sec}^2$ for the gravitational constant.

(b) Find a lower estimate for the height attained after 5 sec.

28. ***Water Pollution*** Oil is leaking out of a tanker damaged at sea. The damage to the tanker is worsening as evidenced by the increased leakage each hour, recorded in the table below.

Time (h)	0	1	2	3	4
Leakage (gal/h)	50	70	97	136	190

Time (h)	5	6	7	8
Leakage (gal/h)	265	369	516	720

(a) Give an upper and lower estimate of the total quantity of oil that has escaped after 5 hours.

(b) Repeat part (a) for the quantity of oil that has escaped after 8 hours.

(c) The tanker continues to leak 720 gal/h after the first 8 hours. If the tanker originally contained 25,000 gal of oil, approximately how many more hours will elapse in the worst case before all of the oil has leaked? in the best case?

29. ***Air Pollution*** A power plant generates electricity by burning oil. Pollutants produced by the burning process are removed by scrubbers in the smokestacks. Over time the scrubbers become less efficient and eventually must be replaced when the amount of pollutants released exceeds government standards. Measurements taken at the end of each month determine the rate at which pollutants are released into the atmosphere as recorded in the table below.

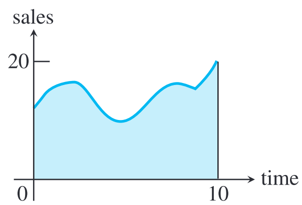
Month	Jan	Feb	Mar	Apr	May	Jun
Pollutant Release Rate (tons/day)	0.20	0.25	0.27	0.34	0.45	0.52

Month	Jul	Aug	Sep	Oct	Nov	Dec
Pollutant Release Rate (tons/day)	0.63	0.70	0.81	0.85	0.89	0.95

(a) Assuming a 30-day month and that new scrubbers allow only 0.05 ton/day released, give an upper estimate of the total tonnage of pollutants released by the end of June. What is a lower estimate?

(b) In the best case, approximately when will a total of 125 tons of pollutants have been released into the atmosphere?

30. Writing to Learn The graph shows the sales record for a company over a 10-year period. If sales are measured in millions of units per year, explain what information can be obtained from the area of the region, and why.



Standardized Test Questions

You should solve the following problems without using a graphing calculator.

31. True or False If f is a positive, continuous, increasing function on $[a, b]$, then LRAM gives an area estimate that is less than the true area under the curve. Justify your answer.

32. True or False For a given number of rectangles, MRAM always gives a more accurate approximation to the true area under the curve than RRAM or LRAM. Justify your answer.

33. Multiple Choice If an MRAM sum with four rectangles of equal width is used to approximate the area enclosed between the x-axis and the graph of $y = 4x - x^2$, the approximation is

(A) 10 **(B)** 10.5 **(C)** $10.\overline{6}$ **(D)** 10.75 **(E)** 11

34. Multiple Choice If f is a positive, continuous function on an interval $[a, b]$, which of the following rectangular approximation methods has a limit equal to the actual area under the curve from a to b as the number of rectangles approaches infinity?

I. LRAM

II. RRAM

III. MRAM

(A) I and II only

(B) III only

(C) I and III only

(D) I, II, and III

(E) None of these

35. Multiple Choice An LRAM sum with 4 equal subdivisions is used to approximate the area under the sine curve from $x = 0$ to $x = \pi$. What is the approximation?

(A) $\frac{\pi}{4}\left(0 + \frac{\pi}{4} + \frac{\pi}{2} + \frac{3\pi}{4}\right)$ **(B)** $\frac{\pi}{4}\left(0 + \frac{1}{2} + \frac{\sqrt{3}}{2} + 1\right)$

(C) $\frac{\pi}{4}\left(0 + \frac{\sqrt{2}}{2} + 1 + \frac{\sqrt{2}}{2}\right)$ **(D)** $\frac{\pi}{4}\left(0 + \frac{1}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\right)$

(E) $\frac{\pi}{4}\left(\frac{1}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} + 1\right)$

36. Multiple Choice A truck moves with positive velocity $v(t)$ from time $t = 3$ to time $t = 15$. The area under the graph of $y = v(t)$ between 3 and 15 gives

(A) the velocity of the truck at $t = 15$.

(B) the acceleration of the truck at $t = 15$.

(C) the position of the truck at $t = 15$.

(D) the distance traveled by the truck from $t = 3$ to $t = 15$.

(E) the average position of the truck in the interval from $t = 3$ to $t = 15$.

Exploration

37. Group Activity ***Area of a Circle*** Inscribe a regular n-sided polygon inside a circle of radius 1 and compute the area of the polygon for the following values of n.

(a) 4 (square) **(b)** 8 (octagon) **(c)** 16

(d) Compare the areas in parts (a), (b), and (c) with the area of the circle.

Extending the Ideas

38. *Rectangular Approximation Methods* Prove or disprove the following statement: MRAM_n is always the average of LRAM_n and RRAM_n.

39. *Rectangular Approximation Methods* Show that if f is a nonnegative function on the interval $[a, b]$ and the line $x = (a + b)/2$ is a line of symmetry of the graph of $y = f(x)$, then $\text{LRAM}_n\, f = \text{RRAM}_n\, f$ for every positive integer n.

40. *(Continuation of Exercise 37)*

(a) Inscribe a regular n-sided polygon inside a circle of radius 1 and compute the area of one of the n congruent triangles formed by drawing radii to the vertices of the polygon.

(b) Compute the limit of the area of the inscribed polygon as $n \to \infty$.

(c) Repeat the computations in parts (a) and (b) for a circle of radius r.

5.2 Definite Integrals

What you'll learn about

- Riemann Sums
- Terminology and Notation of Integration
- The Definite Integral
- Computing Definite Integrals on a Calculator
- Integrability

. . . and why

The definite integral is the basis of integral calculus, just as the derivative is the basis of differential calculus.

Riemann Sums

In the preceding section, we estimated distances, areas, and volumes with finite sums. The terms in the sums were obtained by multiplying selected function values by the lengths of intervals. In this section we move beyond finite sums to see what happens in the limit, as the terms become infinitely small and their number infinitely large.

Sigma notation enables us to express a large sum in compact form:

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n.$$

The Greek capital letter $\sum$ (sigma) stands for "sum." The index k tells us where to begin the sum (at the number below the $\sum$) and where to end (at the number above). If the symbol ∞ appears above the $\sum$, it indicates that the terms go on indefinitely.

The sums in which we will be interested are called *Riemann* ("*ree*-mahn") *sums,* after Georg Friedrich Bernhard Riemann (1826–1866). LRAM, MRAM, and RRAM in the previous section are all examples of Riemann sums—not because they estimated area, but because they were constructed in a particular way. We now describe that construction formally, in a more general context that does not confine us to nonnegative functions.

We begin with an arbitrary continuous function $f(x)$ defined on a closed interval $[a, b]$. Like the function graphed in Figure 5.12, it may have negative values as well as positive values.

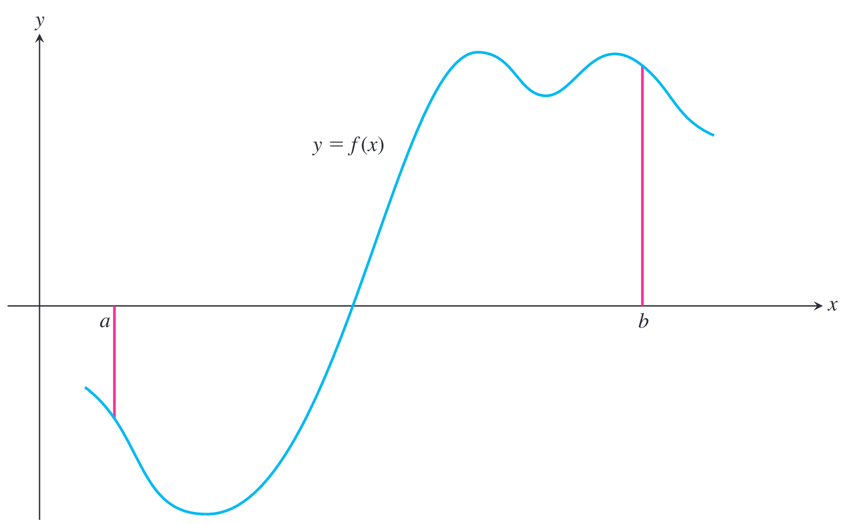


Figure 5.12 The graph of a typical function $y = f(x)$ over a closed interval $[a, b]$.

We then partition the interval $[a, b]$ into n subintervals by choosing $n - 1$ points, say x_1, $x_2, \cdots, x_{n-1}$, between a and b subject only to the condition that

$$a < x_1 < x_2 < \cdots < x_{n-1} < b.$$

To make the notation consistent, we denote a by x_0 and b by x_n. The set

$$P = \{x_0, x_1, x_2, \cdots, x_n\}$$

is called a **partition** of $[a, b]$.

The partition P determines n closed **subintervals,** as shown in Figure 5.13. The k^{th} subinterval is $[x_{k-1}, x_k]$, which has length $\Delta x_k = x_k - x_{k-1}$.

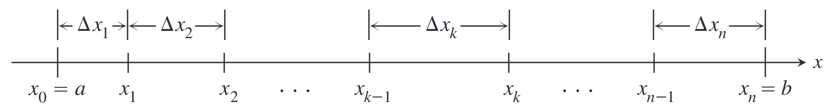


Figure 5.13 The partition $P = \{a = x_0, x_1, x_2, \cdots, x_n = b\}$ divides $[a, b]$ into n subintervals of lengths $\Delta x_1, \Delta x_2, \cdots, \Delta x_n$. The k^{th} subinterval has length Δx_k.

In each subinterval we select some number. Denote the number chosen from the k^{th} subinterval by c_k.

Then, on each subinterval we stand a vertical rectangle that reaches from the x-axis to touch the curve at $(c_k, f(c_k))$. These rectangles could lie either above or below the x-axis (Figure 5.14).

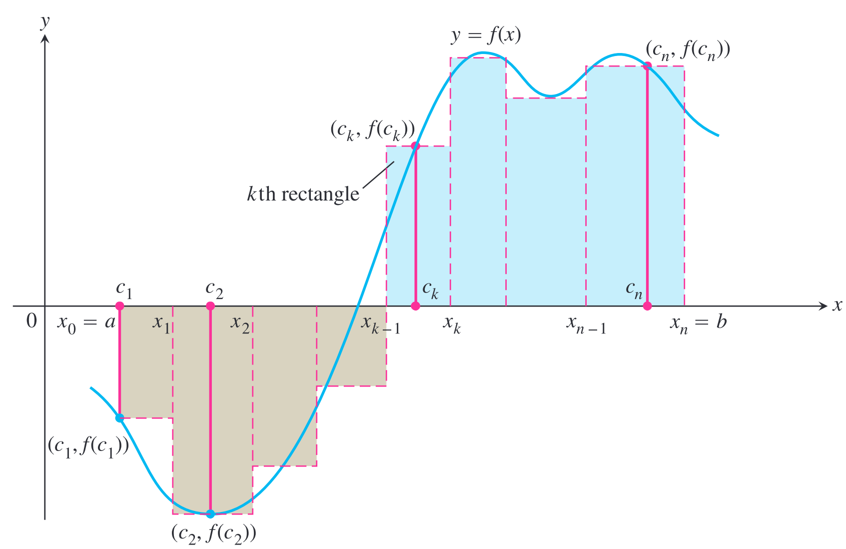


Figure 5.14 Rectangles extending from the x-axis to intersect the curve at the points $(c_k, f(c_k))$. The rectangles approximate the region between the x-axis and the graph of the function.

On each subinterval, we form the product $f(c_k) \cdot \Delta x_k$. This product can be positive, negative, or zero, depending on $f(c_k)$.

Finally, we take the sum of these products:

$$S_n = \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k.$$

This sum, which depends on the partition P and the choice of the numbers c_k, is a **Riemann sum for f on the interval $[a, b]$.**

As the partitions of $[a, b]$ become finer and finer, we would expect the rectangles defined by the partitions to approximate the region between the x-axis and the graph of f with increasing accuracy (Figure 5.15).

Just as LRAM, MRAM, and RRAM in our earlier examples converged to a common value in the limit, *all* Riemann sums for a given function on $[a, b]$ converge to a common value, as long as the lengths of the subintervals all tend to zero. This latter condition is assured by requiring the longest subinterval length (called the **norm** of the partition and denoted by $\|P\|$) to tend to zero.

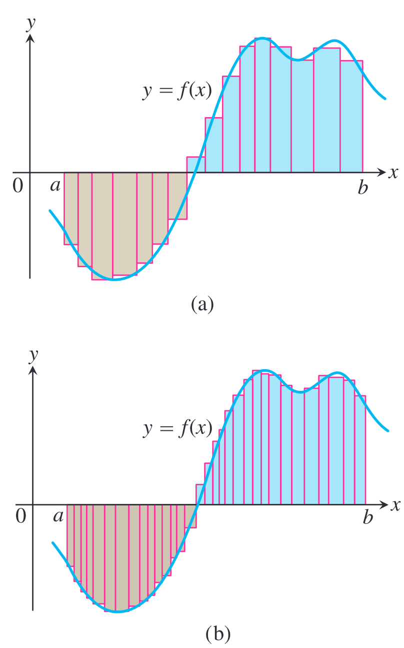


Figure 5.15 The curve of Figure 5.12 with rectangles from finer partitions of $[a, b]$. Finer partitions create more rectangles, with shorter bases.

DEFINITION The Definite Integral as a Limit of Riemann Sums

Let f be a function defined on a closed interval $[a, b]$. For any partition P of $[a, b]$, let the numbers c_k be chosen arbitrarily in the subintervals $[x_{k-1}, x_k]$.

If there exists a number I such that

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k = I$$

no matter how P and the c_k's are chosen, then f is **integrable** on $[a, b]$ and I is the **definite integral** of f over $[a, b]$.

Despite the potential for variety in the sums $\sum f(c_k)\Delta x_k$ as the partitions change and as the c_k's are chosen arbitrarily in the intervals of each partition, the sums always have the same limit as $\|P\|\to 0$ as long as f is *continuous* on $[a, b]$.

Georg Riemann (1826-1866)

The mathematicians of the 17th and 18th centuries blithely assumed the existence of limits of Riemann sums (as we admittedly did in our RAM explorations of the last section), but the existence was not established mathematically until Georg Riemann proved Theorem 1 in 1854. You can find a current version of Riemann's proof in most advanced calculus books.

THEOREM 1 The Existence of Definite Integrals

All continuous functions are integrable. That is, if a function f is continuous on an interval $[a, b]$, then its definite integral over $[a, b]$ exists.

Because of Theorem 1, we can get by with a simpler construction for definite integrals of continuous functions. Since we know for these functions that the Riemann sums tend to the same limit for *all* partitions in which $\|P\|\to 0$, we need only to consider the limit of the so-called **regular partitions,** in which all the subintervals have the same length.

The Definite Integral of a Continuous Function on $[a, b]$

Let f be continuous on $[a, b]$, and let $[a, b]$ be partitioned into n subintervals of equal length $\Delta x = (b - a)/n$. Then the definite integral of f over $[a, b]$ is given by

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x,$$

where each c_k is chosen arbitrarily in the k^{th} subinterval.

Terminology and Notation of Integration

Leibniz's clever choice of notation for the derivative, dy/dx, had the advantage of retaining an identity as a "fraction" even though both numerator and denominator had tended to zero. Although not really fractions, derivatives can *behave* like fractions, so the notation makes profound results like the Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

seem almost simple.

The notation that Leibniz introduced for the definite integral was equally inspired. In his derivative notation, the Greek letters ("Δ" for "difference") switch to Roman letters ("d" for "differential") in the limit,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}.$$

In his definite integral notation, the Greek letters again become Roman letters in the limit,

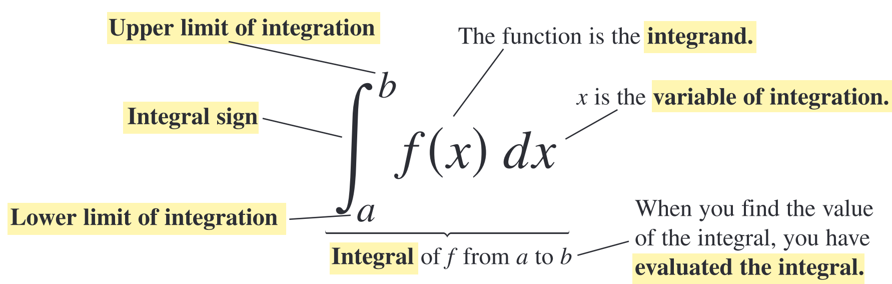
$$\lim_{n \to \infty} \sum_{k=1}^{n} f(c_k)\,\Delta x = \int_a^b f(x)\,dx.$$

Notice that the difference Δx has again tended to zero, becoming a differential dx. The Greek "$\sum$" has become an elongated Roman "S," so that the integral can retain its identity as a "sum." The c_k's have become so crowded together in the limit that we no longer think of a choppy selection of x values between a and b, but rather of a continuous, unbroken sampling of x values from a to b. It is as if we were summing *all* products of the form $f(x)\,dx$ as x goes from a to b, so we can abandon the k and the n used in the finite sum expression.

The symbol

$$\int_a^b f(x)\,dx$$

is read as "the integral from a to b of f of x dee x," or sometimes as "the integral from a to b of f of x with respect to x." The component parts also have names:



The value of the definite integral of a function over any particular interval depends on the function and not on the letter we choose to represent its independent variable. If we decide to use t or u instead of x, we simply write the integral as

$$\int_a^b f(t)\,dt \quad \text{or} \quad \int_a^b f(u)\,du \quad \text{instead of} \quad \int_a^b f(x)\,dx.$$

No matter how we represent the integral, it is the same *number,* defined as a limit of Riemann sums. Since it does not matter what letter we use to run from a to b, the variable of integration is called a **dummy variable.**

EXAMPLE 1 Using the Notation

The interval $[-1, 3]$ is partitioned into n subintervals of equal length $\Delta x = 4/n$. Let m_k denote the midpoint of the k^{th} subinterval. Express the limit

$$\lim_{n \to \infty} \sum_{k=1}^{n} (3(m_k)^2 - 2m_k + 5)\,\Delta x$$

as an integral.

continued

SOLUTION

Since the midpoints m_k have been chosen from the subintervals of the partition, this expression is indeed a limit of Riemann sums. (The points chosen did not have to be midpoints; they could have been chosen from the subintervals in any arbitrary fashion.) The function being integrated is $f(x) = 3x^2 - 2x + 5$ over the interval $[-1, 3]$. Therefore,

$$\lim_{n\to\infty} \sum_{k=1}^{n} (3(m_k)^2 - 2m_k + 5)\, \Delta x = \int_{-1}^{3} (3x^2 - 2x + 5)\, dx.$$

Now try Exercise 5.

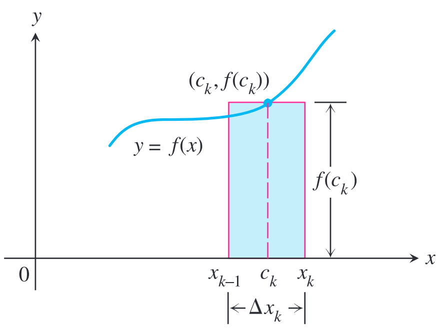


Figure 5.16 A term of a Riemann sum $\sum f(c_k)\Delta x_k$ for a nonnegative function f is either zero or the area of a rectangle such as the one shown.

Definite Integral and Area

If an integrable function $y = f(x)$ is nonnegative throughout an interval $[a, b]$, each nonzero term $f(c_k)\Delta x_k$ is the area of a rectangle reaching from the x-axis up to the curve $y = f(x)$. (See Figure 5.16.)

The Riemann sum

$$\sum f(c_k)\, \Delta x_k,$$

which is the sum of the areas of these rectangles, gives an estimate of the area of the region between the curve and the x-axis from a to b. Since the rectangles give an increasingly good approximation of the region as we use partitions with smaller and smaller norms, we call the limiting value the area under the curve.

DEFINITION Area Under a Curve (as a Definite Integral)

If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the **area under the curve $y = f(x)$ from a to b** is the integral of f from a to b,

$$A = \int_a^b f(x)\, dx.$$

This definition works both ways: We can use integrals to calculate areas *and* we can use areas to calculate integrals.

EXAMPLE 2 Revisiting Area Under a Curve

Evaluate the integral $\int_{-2}^{2} \sqrt{4 - x^2}\, dx$.

SOLUTION

We recognize $f(x) = \sqrt{4 - x^2}$ as a function whose graph is a semicircle of radius 2 centered at the origin (Figure 5.17).

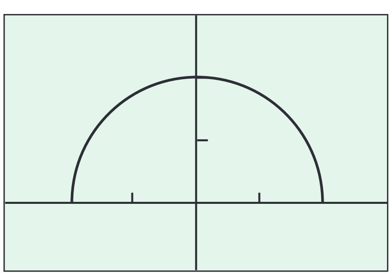
The area between the semicircle and the x-axis from -2 to 2 can be computed using the geometry formula

$$\text{Area} = \frac{1}{2} \cdot \pi r^2 = \frac{1}{2} \cdot \pi (2)^2 = 2\pi.$$

Because the area is also the value of the integral of f from -2 to 2,

$$\int_{-2}^{2} \sqrt{4 - x^2}\, dx = 2\pi.$$

Now try Exercise 15.



[–3, 3] by [–1, 3]

Figure 5.17 A square viewing window on $y = \sqrt{4 - x^2}$. The graph is a semicircle because $y = \sqrt{4 - x^2}$ is the same as $y^2 = 4 - x^2$, or $x^2 + y^2 = 4$, with $y \geq 0$. (Example 2)

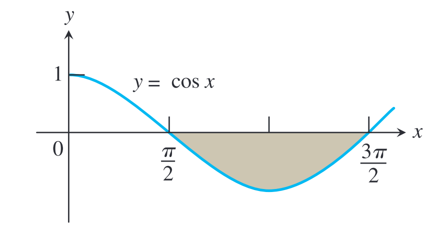


Figure 5.18 Because $f(x) = \cos x$ is nonpositive on $[\pi/2, 3\pi/2]$, the integral of f is a negative number. The area of the shaded region is the opposite of this integral,

$$\text{Area} = -\int_{\pi/2}^{3\pi/2} \cos x\, dx.$$

If an integrable function $y = f(x)$ is nonpositive, the nonzero terms $f(c_k)\Delta x_k$ in the Riemann sums for f over an interval $[a, b]$ are negatives of rectangle areas. The limit of the sums, the integral of f from a to b, is therefore the *negative* of the area of the region between the graph of f and the x-axis (Figure 5.18).

$$\int_a^b f(x)\, dx = -(\text{the area}) \quad \text{if} \quad f(x) \le 0.$$

Or, turning this around,

$$\text{Area} = -\int_a^b f(x)\, dx \quad \text{when} \quad f(x) \le 0.$$

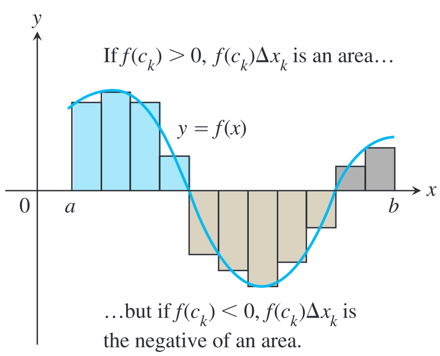


Figure 5.19 An integrable function f with negative as well as positive values.

If an integrable function $y = f(x)$ has both positive and negative values on an interval $[a, b]$, then the Riemann sums for f on $[a, b]$ add areas of rectangles that lie above the x-axis to the negatives of areas of rectangles that lie below the x-axis, as in Figure 5.19. The resulting cancellations mean that the limiting value is a number whose magnitude is less than the total area between the curve and the x-axis. The value of the integral is the area above the x-axis minus the area below.

For any integrable function,

$$\int_a^b f(x)\, dx = (\text{area above the } x\text{-axis}) - (\text{area below the } x\text{-axis}).$$

Net Area

Sometimes $\int_a^b f(x)\, dx$ is called the *net area* of the region determined by the curve $y = f(x)$ and the x-axis between $x = a$ and $x = b$.

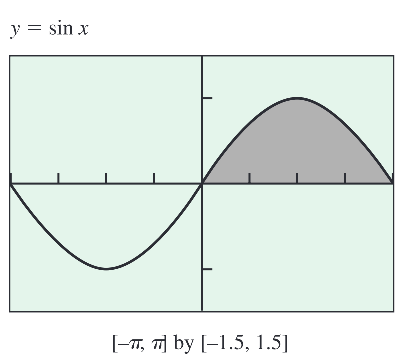


Figure 5.20

$$\int_0^{\pi} \sin x\, dx = 2. \text{ (Exploration 1)}$$

EXPLORATION 1 **Finding Integrals by Signed Areas**

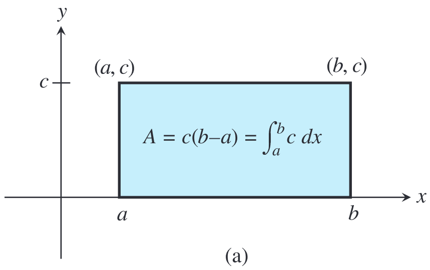
It is a fact (which we will revisit) that $\int_0^{\pi} \sin x\, dx = 2$ (Figure 5.20). With that information, what you know about integrals and areas, what you know about graphing curves, and sometimes a bit of intuition, determine the values of the following integrals. Give as convincing an argument as you can for each value, based on the graph of the function.

1. $\displaystyle\int_{\pi}^{2\pi} \sin x\, dx$ **2.** $\displaystyle\int_0^{2\pi} \sin x\, dx$ **3.** $\displaystyle\int_0^{\pi/2} \sin x\, dx$

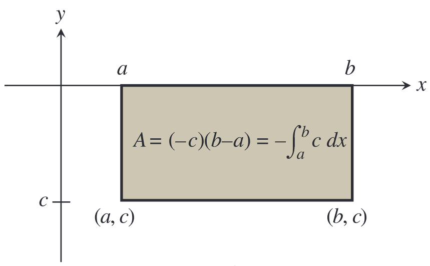
4. $\displaystyle\int_0^{\pi} (2 + \sin x)\, dx$ **5.** $\displaystyle\int_0^{\pi} 2 \sin x\, dx$ **6.** $\displaystyle\int_2^{\pi+2} \sin (x - 2)\, dx$

7. $\displaystyle\int_{-\pi}^{\pi} \sin u\, du$ **8.** $\displaystyle\int_0^{2\pi} \sin (x/2)\, dx$ **9.** $\displaystyle\int_0^{\pi} \cos x\, dx$

10. Suppose k is *any* positive number. Make a conjecture about $\int_{-k}^{k} \sin x\, dx$ and support your conjecture.



(a)



(b)

Figure 5.21 (a) If c is a positive constant, then $\int_a^b c\,dx$ is the area of the rectangle shown. (b) If c is negative, then $\int_a^b c\,dx$ is the opposite of the area of the rectangle.

Constant Functions

Integrals of constant functions are easy to evaluate. Over a closed interval, they are simply the constant times the length of the interval (Figure 5.21).

THEOREM 2 The Integral of a Constant

If $f(x) = c$, where c is a constant, on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = \int_a^b c\,dx = c(b-a).$$

Proof A constant function is continuous, so the integral exists, and we can evaluate it as a limit of Riemann sums with subintervals of equal length $(b-a)/n$. Any such sum looks like

$$\sum_{k=1}^{n} f(c_k)\cdot\Delta x, \quad \text{which is} \quad \sum_{k=1}^{n} c\cdot\frac{b-a}{n}.$$

Then

$$\begin{aligned}\sum_{k=1}^{n} c\cdot\frac{b-a}{n} &= c\cdot(b-a)\sum_{k=1}^{n}\frac{1}{n}\\ &= c(b-a)\cdot n\left(\frac{1}{n}\right)\\ &= c(b-a).\end{aligned}$$

Since the sum is *always* $c(b-a)$ for any value of n, it follows that the limit of the sums, the integral to which they converge, is also $c(b-a)$. ■

EXAMPLE 3 Revisiting the Train Problem

A train moves along a track at a steady 75 miles per hour from 7:00 A.M. to 9:00 A.M. Express its total distance traveled as an integral. Evaluate the integral using Theorem 2.

SOLUTION (See Figure 5.22.)

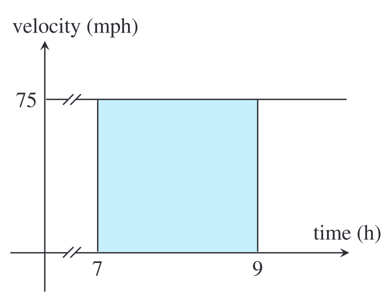


Figure 5.22 The area of the rectangle is a special case of Theorem 2. (Example 3)

$$\text{Distance traveled} = \int_7^9 75\,dt = 75\cdot(9-7) = 150$$

Since the 75 is measured in miles/hour and the $(9-7)$ is measured in hours, the 150 is measured in miles. The train traveled 150 miles. *Now try Exercise 29.*

Integrals on a Calculator

You do not have to know much about your calculator to realize that finding the limit of a Riemann sum is exactly the kind of thing that it does best. We have seen how effectively it can approximate areas using MRAM, but most modern calculators have sophisticated built-in programs that converge to integrals with much greater speed and precision than that. We will assume that your calculator has such a numerical integration capability, which we will denote as **NINT.** In particular, we will use NINT $(f(x), x, a, b)$ to denote a calculator (or computer) approximation of $\int_a^b f(x)\,dx$. When we write

$$\int_a^b f(x)\,dx = \text{NINT}\,(f(x), x, a, b),$$

we do so with the understanding that the right-hand side of the equation is an approximation of the left-hand side.

EXAMPLE 4 Using NINT

Evaluate the following integrals numerically.

(a) $\int_{-1}^{2} x \sin x\,dx$ **(b)** $\int_{0}^{1} \frac{4}{1+x^2}\,dx$ **(c)** $\int_{0}^{5} e^{-x^2}\,dx$

SOLUTION

(a) NINT $(x \sin x, x, -1, 2) \approx 2.04$

(b) NINT $(4/(1+x^2), x, 0, 1) \approx 3.14$

(c) NINT $(e^{-x^2}, x, 0, 5) \approx 0.89$ ***Now try Exercise 33.***

Bounded Functions

We say a function is *bounded* on a given domain if its range is confined between some minimum value m and some maximum value M. That is, given any x in the domain, $m \le f(x) \le M$. Equivalently, the graph of $y = f(x)$ lies between the horizontal lines $y = m$ and $y = M$.

We will eventually be able to confirm that the exact value for the integral in Example 4a is $-2\cos 2 + \sin 2 - \cos 1 + \sin 1$. You might want to conjecture for yourself what the exact answer to Example 4b might be. As for Example 4c, no explicit *exact* value has ever been found for this integral! The best we can do in this case (and in many like it) is to approximate the integral numerically. Here, technology is not only useful, it is essential.

Discontinuous Integrable Functions

Theorem 1 guarantees that all continuous functions are integrable. But some functions with discontinuities are also integrable. For example, a bounded function (see margin note) that has a finite number of points of discontinuity on an interval $[a, b]$ will still be integrable on the interval if it is continuous everywhere else.

EXAMPLE 5 Integrating a Discontinuous Function

Find $\int_{-1}^{2} \frac{|x|}{x}\,dx.$

SOLUTION

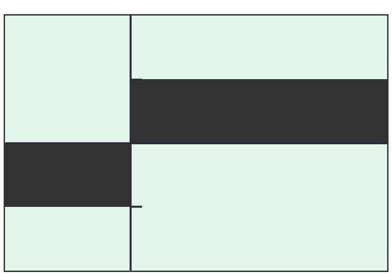
This function has a discontinuity at $x = 0$, where the graph jumps from $y = -1$ to $y = 1$. The graph, however, determines two rectangles, one below the x-axis and one above (Figure 5.23).

Using the idea of net area, we have

$$\int_{-1}^{2} \frac{|x|}{x}\,dx = -1 + 2 = 1.$$

Now try Exercise 37.

$y = |x|/x$



[–1, 2] by [–2, 2]

Figure 5.23 A discontinuous integrable function:

$$\int_{-1}^{2} \frac{|x|}{x}\,dx = -(\text{area below } x\text{-axis}) + (\text{area above } x\text{-axis}).$$

(Example 5)

A Nonintegrable Function

How "bad" does a function have to be before it is *not* integrable? One way to defeat integrability is to be unbounded (like $y = 1/x$ near $x = 0$), which can prevent the Riemann sums from tending to a finite limit. Another, more subtle, way is to be bounded but badly discontinuous, like the *characteristic function of the rationals:*

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

No matter what partition we take of the closed interval [0, 1], every subinterval contains both rational and irrational numbers. That means that we can always form a Riemann sum with all rational c_k's (a Riemann sum of 1) or all irrational c_k's (a Riemann sum of 0). The sums can therefore never tend toward a unique limit.

EXPLORATION 2 More Discontinuous Integrands

1. Explain why the function

$$f(x) = \frac{x^2 - 4}{x - 2}$$

is not continuous on [0, 3]. What kind of discontinuity occurs?

2. Use areas to show that

$$\int_0^3 \frac{x^2 - 4}{x - 2}\, dx = 10.5.$$

3. Use areas to show that

$$\int_0^5 \text{int}\,(x)\, dx = 10.$$

Quick Review 5.2

In Exercises 1–3, evaluate the sum.

1. $\sum_{n=1}^{5} n^2$

2. $\sum_{k=0}^{4} (3k - 2)$

3. $\sum_{j=0}^{4} 100\,(j + 1)^2$

In Exercises 4–6, write the sum in sigma notation.

4. $1 + 2 + 3 + \cdots + 98 + 99$

5. $0 + 2 + 4 + \cdots + 48 + 50$

6. $3(1)^2 + 3(2)^2 + \cdots + 3(500)^2$

In Exercises 7 and 8, write the expression as a single sum in sigma notation.

7. $2\sum_{x=1}^{50} x^2 + 3\sum_{x=1}^{50} x$

8. $\sum_{k=0}^{8} x^k + \sum_{k=9}^{20} x^k$

9. Find $\sum_{k=0}^{n} (-1)^k$ if n is odd.

10. Find $\sum_{k=0}^{n} (-1)^k$ if n is even.

Section 5.2 Exercises

In Exercises 1–6, each c_k is chosen from the kth subinterval of a regular partition of the indicated interval into n subintervals of length Δx. Express the limit as a definite integral.

1. $\lim_{n\to\infty} \sum_{k=1}^{n} c_k^2 \Delta x, \quad [0, 2]$

2. $\lim_{n\to\infty} \sum_{k=1}^{n} (c_k^2 - 3c_k)\,\Delta x, \quad [-7, 5]$

3. $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{c_k}\,\Delta x, \quad [1, 4]$

4. $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{1 - c_k}\,\Delta x, \quad [2, 3]$

5. $\lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{4 - c_k^2}\,\Delta x, \quad [0, 1]$

6. $\lim_{n\to\infty} \sum_{k=1}^{n} (\sin^3 c_k)\,\Delta x, \quad [-\pi, \pi]$

In Exercises 7–12, evaluate the integral.

7. $\int_{-2}^{1} 5\, dx$

8. $\int_{3}^{7} (-20)\, dx$

9. $\int_{0}^{3} (-160)\, dt$

10. $\int_{-4}^{-1} \frac{\pi}{2}\, d\theta$

11. $\int_{-2.1}^{3.4} 0.5\, ds$

12. $\int_{\sqrt{2}}^{\sqrt{18}} \sqrt{2}\, dr$

In Exercises 13–22, use the graph of the integrand and areas to evaluate the integral.

13. $\int_{-2}^{4}\left(\frac{x}{2}+3\right)dx$

14. $\int_{1/2}^{3/2}(-2x+4)\,dx$

15. $\int_{-3}^{3}\sqrt{9-x^2}\,dx$

16. $\int_{-4}^{0}\sqrt{16-x^2}\,dx$

17. $\int_{-2}^{1}|x|\,dx$

18. $\int_{-1}^{1}(1-|x|)\,dx$

19. $\int_{-1}^{1}(2-|x|)\,dx$

20. $\int_{-1}^{1}\left(1+\sqrt{1-x^2}\right)dx$

21. $\int_{\pi}^{2\pi}\theta\,d\theta$

22. $\int_{\sqrt{2}}^{5\sqrt{2}}r\,dr$

In Exercises 23–28, use areas to evaluate the integral.

23. $\int_{0}^{b}x\,dx,\quad b>0$

24. $\int_{0}^{b}4x\,dx,\quad b>0$

25. $\int_{a}^{b}2s\,ds,\quad 0<a<b$

26. $\int_{a}^{b}3t\,dt,\quad 0<a<b$

27. $\int_{a}^{2a}x\,dx,\quad a>0$

28. $\int_{a}^{\sqrt{3}a}x\,dx,\quad a>0$

In Exercises 29–32, express the desired quantity as a definite integral and evaluate the integral using Theorem 2.

29. Find the distance traveled by a train moving at 87 mph from 8:00 A.M. to 11:00 A.M.

30. Find the output from a pump producing 25 gallons per minute during the first hour of its operation.

31. Find the calories burned by a walker burning 300 calories per hour between 6:00 P.M. and 7:30 P.M.

32. Find the amount of water lost from a bucket leaking 0.4 liters per hour between 8:30 A.M. and 11:00 A.M.

In Exercises 33–36, use NINT to evaluate the expression.

33. $\int_{0}^{5}\frac{x}{x^2+4}\,dx$

34. $3+2\int_{0}^{\pi/3}\tan x\,dx$

35. Find the area enclosed between the x-axis and the graph of $y=4-x^2$ from $x=-2$ to $x=2$.

36. Find the area enclosed between the x-axis and the graph of $y=x^2e^{-x}$ from $x=-1$ to $x=3$.

In Exercises 37–40, **(a)** find the points of discontinuity of the integrand on the interval of integration, and **(b)** use area to evaluate the integral.

37. $\int_{-2}^{3}\frac{x}{|x|}\,dx$

38. $\int_{-6}^{5}2\,\text{int}\,(x-3)\,dx$

39. $\int_{-3}^{4}\frac{x^2-1}{x+1}\,dx$

40. $\int_{-5}^{6}\frac{9-x^2}{x-3}\,dx$

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

41. True or False If $\int_a^b f(x)\,dx>0$, then $f(x)$ is positive for all x in $[a, b]$. Justify your answer.

42. True or False If $f(x)$ is positive for all x in $[a, b]$, then $\int_a^b f(x)\,dx>0$. Justify your answer.

43. Multiple Choice If $\int_2^5 f(x)\,dx=18$, then $\int_2^5 (f(x)+4)\,dx=$

(A) 20 **(B)** 22 **(C)** 23 **(D)** 25 **(E)** 30

44. Multiple Choice $\int_{-4}^{4}(4-|x|)\,dx=$

(A) 0 **(B)** 4 **(C)** 8 **(D)** 16 **(E)** 32

45. Multiple Choice If the interval $[0, \pi]$ is divided into n subintervals of length π/n and c_k is chosen from the kth subinterval, which of the following is a Riemann sum?

(A) $\sum_{k=1}^{n}\sin(c_k)$ **(B)** $\sum_{k=1}^{\infty}\sin(c_k)$ **(C)** $\sum_{k=1}^{n}\sin(c_k)\left(\frac{\pi}{n}\right)$

(D) $\sum_{k=1}^{n}\sin\left(\frac{\pi}{n}\right)(c_k)$ **(E)** $\sum_{k=1}^{n}\sin(c_k)\left(\frac{\pi}{k}\right)$

46. Multiple Choice Which of the following quantities would *not* be represented by the definite integral $\int_0^8 70\,dt$?

(A) The distance traveled by a train moving at 70 mph for 8 minutes.

(B) The volume of ice cream produced by a machine making 70 gallons per hour for 8 hours.

(C) The length of a track left by a snail traveling at 70 cm per hour for 8 hours.

(D) The total sales of a company selling \$70 of merchandise per hour for 8 hours.

(E) The amount the tide has risen 8 minutes after low tide if it rises at a rate of 70 mm per minute during that period.

Explorations

In Exercises 47–56, use graphs, your knowledge of area, and the fact that

$$\int_0^1 x^3\,dx=\frac{1}{4}$$

to evaluate the integral.

47. $\int_{-1}^{1}x^3\,dx$

48. $\int_{0}^{1}(x^3+3)\,dx$

49. $\int_{2}^{3}(x-2)^3\,dx$

50. $\int_{-1}^{1}|x|^3\,dx$

51. $\int_{0}^{1}(1-x^3)\,dx$

52. $\int_{-1}^{2}(|x|-1)^3\,dx$

53. $\int_{0}^{2}\left(\frac{x}{2}\right)^3dx$

54. $\int_{-8}^{8}x^3\,dx$

55. $\int_{0}^{1}(x^3-1)\,dx$

56. $\int_{0}^{1}\sqrt[3]{x}\,dx$

Extending the Ideas

57. Writing to Learn The function

$$f(x) = \begin{cases} \dfrac{1}{x^2}, & 0 < x \le 1 \\ 0, & x = 0 \end{cases}$$

is defined on $[0, 1]$ and has a single point of discontinuity at $x = 0$.

(a) What happens to the graph of f as x approaches 0 from the right?

(b) The function f is not integrable on $[0, 1]$. Give a convincing argument based on Riemann sums to explain why it is not.

58. It can be shown by mathematical induction (see Appendix 2) that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Use this fact to give a formal proof that

$$\int_0^1 x^2\,dx = \frac{1}{3}$$

by following the steps given in the next column.

(a) Partition $[0, 1]$ into n subintervals of length $1/n$. Show that the RRAM Riemann sum for the integral is

$$\sum_{k=1}^{n} \left(\left(\frac{k}{n} \right)^2 \cdot \frac{1}{n} \right).$$

(b) Show that this sum can be written as

$$\frac{1}{n^3} \cdot \sum_{k=1}^{n} k^2.$$

(c) Show that the sum can therefore be written as

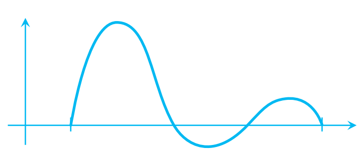
$$\frac{(n+1)(2n+1)}{6n^2}.$$

(d) Show that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\left(\frac{k}{n} \right)^2 \cdot \frac{1}{n} \right) = \frac{1}{3}.$$

(e) Explain why the equation in part (d) proves that

$$\int_0^1 x^2\,dx = \frac{1}{3}.$$



5.3 Definite Integrals and Antiderivatives

What you'll learn about

- Properties of Definite Integrals
- Average Value of a Function
- Mean Value Theorem for Definite Integrals
- Connecting Differential and Integral Calculus

. . . and why

Working with the properties of definite integrals helps us to understand better the definite integral. Connecting derivatives and definite integrals sets the stage for the Fundamental Theorem of Calculus.

Properties of Definite Integrals

In defining $\int_a^b f(x)$ as a limit of sums $\sum c_k \Delta x_k$, we moved from left to right across the interval $[a, b]$. What would happen if we integrated in the *opposite direction*? The integral would become $\int_b^a f(x)\,dx$—again a limit of sums of the form $\sum f(c_k)\Delta x_k$—but this time each of the Δx_k's would be negative as the x-values *decreased* from b to a. This would change the signs of all the terms in each Riemann sum, and ultimately the sign of the definite integral. This suggests the rule

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

Since the original definition did not apply to integrating backwards over an interval, we can treat this rule as a logical extension of the definition.

Although $[a, a]$ is technically not an interval, another logical extension of the definition is that $\int_a^a f(x)\,dx = 0$.

These are the first two rules in Table 5.3. The others are inherited from rules that hold for Riemann sums. However, the limit step required to *prove* that these rules hold in the limit (as the norms of the partitions tend to zero) places their mathematical verification beyond the scope of this course. They should make good sense nonetheless.

Table 5.3 Rules for Definite Integrals

1. *Order of Integration:* $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ — A definition

2. *Zero:* $\int_a^a f(x)\,dx = 0$ — Also a definition

3. *Constant Multiple:* $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ — Any number k

$\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$ — $k = -1$

4. *Sum and Difference:* $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5. *Additivity:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

6. *Max-Min Inequality:* If max f and min f are the maximum and minimum values of f on $[a, b]$, then

$$\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a).$$

7. *Domination:* $f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

$f(x) \ge 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$ — $g = 0$

EXAMPLE 1 Using the Rules for Definite Integrals

Suppose

$$\int_{-1}^{1} f(x)\,dx = 5, \quad \int_{1}^{4} f(x)\,dx = -2, \quad \text{and} \quad \int_{-1}^{1} h(x)\,dx = 7.$$

Find each of the following integrals, if possible.

(a) $\int_{4}^{1} f(x)\,dx$ **(b)** $\int_{-1}^{4} f(x)\,dx$ **(c)** $\int_{-1}^{1} [2f(x) + 3h(x)]\,dx$

(d) $\int_{0}^{1} f(x)\,dx$ **(e)** $\int_{-2}^{2} h(x)\,dx$ **(f)** $\int_{-1}^{4} [f(x) + h(x)]\,dx$

SOLUTION

(a) $\int_{4}^{1} f(x)\,dx = -\int_{1}^{4} f(x)\,dx = -(-2) = 2$

(b) $\int_{-1}^{4} f(x)\,dx = \int_{-1}^{1} f(x)\,dx + \int_{1}^{4} f(x)\,dx = 5 + (-2) = 3$

(c) $\int_{-1}^{1} [2f(x) + 3h(x)]\,dx = 2\int_{-1}^{1} f(x)\,dx + 3\int_{-1}^{1} h(x)\,dx = 2(5) + 3(7) = 31$

(d) Not enough information given. (We cannot assume, for example, that integrating over half the interval would give half the integral!)

(e) Not enough information given. (We have no information about the function h outside the interval $[-1, 1]$.)

(f) Not enough information given (same reason as in part (e)). *Now try Exercise 1.*

EXAMPLE 2 Finding Bounds for an Integral

Show that the value of $\int_0^1 \sqrt{1 + \cos x}\,dx$ is less than $3/2$.

SOLUTION

The Max-Min Inequality for definite integrals (Rule 6) says that $\min f \cdot (b - a)$ is a *lower bound* for the value of $\int_a^b f(x)\,dx$ and that $\max f \cdot (b - a)$ is an *upper bound*. The maximum value of $\sqrt{1 + \cos x}$ on $[0, 1]$ is $\sqrt{2}$, so

$$\int_0^1 \sqrt{1 + \cos x}\,dx \le \sqrt{2} \cdot (1 - 0) = \sqrt{2}.$$

Since $\int_0^1 \sqrt{1 + \cos x}\,dx$ is bounded above by $\sqrt{2}$ (which is 1.414...), it is less than $3/2$. *Now try Exercise 7.*

Average Value of a Function

The *average* of n numbers is the sum of the numbers divided by n. How would we define the average value of an arbitrary function f over a closed interval $[a, b]$? As there are infinitely many values to consider, adding them and then dividing by infinity is not an option.

Consider, then, what happens if we take a large *sample* of n numbers from regular subintervals of the interval $[a, b]$. One way would be to take some number c_k from each of the n subintervals of length

$$\Delta x = \frac{b-a}{n}.$$

The average of the n sampled values is

$$\begin{aligned}\frac{f(c_1) + f(c_2) + \cdots + f(c_n)}{n} &= \frac{1}{n} \cdot \sum_{k=1}^{n} f(c_k) \\ &= \frac{\Delta x}{b-a} \sum_{k=1}^{n} f(c_k) \qquad \frac{1}{n} = \frac{\Delta x}{b-a} \\ &= \frac{1}{b-a} \cdot \sum_{k=1}^{n} f(c_k)\Delta x.\end{aligned}$$

Does this last sum look familiar? It is $1/(b-a)$ times a Riemann sum for f on $[a, b]$. That means that when we consider this averaging process as $n \to \infty$, we find it *has a limit,* namely $1/(b-a)$ times the integral of f over $[a, b]$. We are led by this remarkable fact to the following definition.

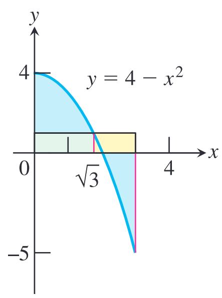


Figure 5.24 The rectangle with base $[0, 3]$ and with height equal to 1 (the average value of the function $f(x) = 4 - x^2$) has area equal to the net area between f and the x-axis from 0 to 3. (Example 3)

DEFINITION Average (Mean) Value

If f is integrable on $[a, b]$, its **average (mean) value** on $[a, b]$ is

$$av(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

EXAMPLE 3 Applying the Definition

Find the average value of $f(x) = 4 - x^2$ on $[0, 3]$. Does f actually take on this value at some point in the given interval?

SOLUTION

$$\begin{aligned} av(f) &= \frac{1}{b-a}\int_a^b f(x)\,dx \\ &= \frac{1}{3-0}\int_0^3 (4-x^2)\,dx \\ &= \frac{1}{3-0} \cdot 3 \\ &= 1 \end{aligned}$$

The average value of $f(x) = 4 - x^2$ over the interval $[0, 3]$ is 1. The function assumes this value when $4 - x^2 = 1$ or $x = \pm\sqrt{3}$. Since $x = \sqrt{3}$ lies in the interval $[0, 3]$, the function does assume its average value in the given interval (Figure 5.24).

Now try Exercise 11.

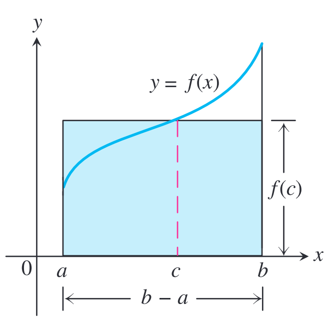


Figure 5.25 The value $f(c)$ in the Mean Value Theorem is, in a sense, the average (or *mean*) height of f on $[a, b]$. When $f \geq 0$, the area of the shaded rectangle

$$f(c)(b-a) = \int_a^b f(x)\,dx,$$

is the area under the graph of f from a to b.

Mean Value Theorem for Definite Integrals

It was no mere coincidence that the function in Example 3 took on its average value at some point in the interval. Look at the graph in Figure 5.25 and imagine rectangles with base $(b - a)$ and heights ranging from the minimum of f (a rectangle too small to give the integral) to the maximum of f (a rectangle too large). Somewhere in between there is a "just right" rectangle, and its topside will intersect the graph of f if f is continuous. The statement that a continuous function on a closed interval *always* assumes its average value at least once in the interval is known as the Mean Value Theorem for Definite Integrals.

THEOREM 3 The Mean Value Theorem for Definite Integrals

If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

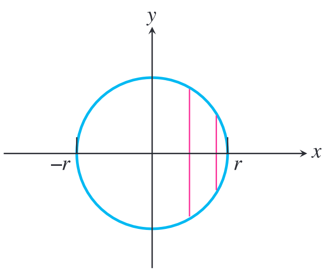


Figure 5.26 Chords perpendicular to the diameter $[-r, r]$ in a circle of radius r centered at the origin. (Exploration 1)

EXPLORATION 1 How Long is the Average Chord of a Circle?

Suppose we have a circle of radius r centered at the origin. We want to know the average length of the chords perpendicular to the diameter $[-r, r]$ on the x-axis.

1. Show that the length of the chord at x is $2\sqrt{r^2 - x^2}$ (Figure 5.26).
2. Set up an integral expression for the average value of $2\sqrt{r^2 - x^2}$ over the interval $[-r, r]$.
3. Evaluate the integral by identifying its value as an area.
4. So, what is the average length of a chord of a circle of radius r?
5. Explain how we can use the Mean Value Theorem for Definite Integrals (Theorem 3) to show that the function assumes the value in step 4.

Connecting Differential and Integral Calculus

Before we move on to the next section, let us pause for a moment of historical perspective that can help you to appreciate the power of the theorem that you are about to encounter. In Example 3 we used NINT to find the integral, and in Section 5.2, Example 2 we were fortunate that we could use our knowledge of the area of a circle. The area of a circle has been around for a long time, but NINT has not; so how did people evaluate definite integrals when they could not apply some known area formula? For example, in Exploration 1 of the previous section we used the fact that

$$\int_0^{\pi} \sin x\,dx = 2.$$

Would Newton and Leibniz have known this fact? How?

They did know that *quotients of infinitely small quantities,* as they put it, could be used to get velocity functions from position functions, and that *sums of infinitely thin "rectangle areas"* could be used to get position functions from velocity functions. In some way, then, there had to be a connection between these two seemingly different processes. Newton and Leibniz were able to picture that connection, and it led them to the Fundamental Theorem of Calculus. Can you picture it? Try Exploration 2.

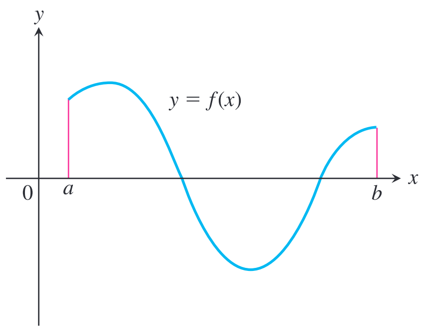


Figure 5.27 The graph of the function in Exploration 2.

EXPLORATION 2 Finding the Derivative of an Integral

Group Activity Suppose we are given the graph of a continuous function f, as in Figure 5.27.

1. Copy the graph of f onto your own paper. Choose any x greater than a in the interval $[a, b]$ and mark it on the x-axis.
2. Using only *vertical line segments,* shade in the region between the graph of f and the x-axis from a to x. (Some shading might be below the x-axis.)
3. Your shaded region represents a definite integral. Explain why this integral can be written as $\int_a^x f(t)\,dt$. (Why don't we write it as $\int_a^x f(x)\,dx$?)
4. Compare your picture with others produced by your group. Notice how your integral (a real number) depends on which x you chose in the interval $[a, b]$. The integral is therefore a *function of x* on $[a, b]$. Call it F.
5. Recall that $F'(x)$ is the limit of $\Delta F/\Delta x$ as Δx gets smaller and smaller. Represent ΔF in your picture by drawing *one more vertical shading segment* to the right of the last one you drew in step 2. ΔF is the (signed) *area* of your vertical segment.
6. Represent Δx in your picture by moving x to beneath your newly-drawn segment. That small change in Δx is the *thickness* of your vertical segment.
7. What is now the *height* of your vertical segment?
8. Can you see why Newton and Leibniz concluded that $F'(x) = f(x)$?

If all went well in Exploration 2, you concluded that the derivative with respect to x of the integral of f from a to x is simply f. Specifically,

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

This means that the integral is an *antiderivative* of f, a fact we can exploit in the following way.

If F is any antiderivative of f, then

$$\int_a^x f(t)\,dt = F(x) + C$$

for some constant C. Setting x in this equation equal to a gives

$$\int_a^a f(t)\,dt = F(a) + C$$

$$0 = F(a) + C$$

$$C = -F(a).$$

Putting it all together,

$$\int_a^x f(t)\,dt = F(x) - F(a).$$

The implications of the previous last equation were enormous for the discoverers of calculus. It meant that they could evaluate the definite integral of f from a to any number x simply by computing $F(x) - F(a)$, where F *is any antiderivative of* f.

EXAMPLE 4 Finding an Integral Using Antiderivatives

Find $\int_0^\pi \sin x\, dx$ using the formula $\int_a^x f(t)\, dt = F(x) - F(a)$.

SOLUTION

Since $F(x) = -\cos x$ is an antiderivative of $\sin x$, we have

$$\begin{aligned}\int_0^\pi \sin x\, dx &= -\cos(\pi) - (-\cos(0)) \\ &= -(-1) - (-1) \\ &= 2.\end{aligned}$$

This explains how we obtained the value for Exploration 1 of the previous section.

Now try Exercise 21.

Quick Review 5.3 *(For help, go to Sections 3.6, 3.8, and 3.9.)*

In Exercises 1–10, find dy/dx.

1. $y = -\cos x$

2. $y = \sin x$

3. $y = \ln(\sec x)$

4. $y = \ln(\sin x)$

5. $y = \ln(\sec x + \tan x)$

6. $y = x \ln x - x$

7. $y = \dfrac{x^{n+1}}{n+1} \quad (n \neq -1)$

8. $y = \dfrac{1}{2^x + 1}$

9. $y = xe^x$

10. $y = \tan^{-1} x$

Section 5.3 Exercises

The exercises in this section are designed to reinforce your understanding of the definite integral from the algebraic and geometric points of view. For this reason, you should not use the numerical integration capability of your calculator (NINT) except perhaps to support an answer.

1. Suppose that f and g are continuous functions and that

$$\int_1^2 f(x)\, dx = -4, \quad \int_1^5 f(x)\, dx = 6, \quad \int_1^5 g(x)\, dx = 8.$$

Use the rules in Table 5.3 to find each integral.

(a) $\int_2^2 g(x)\, dx$ **(b)** $\int_5^1 g(x)\, dx$

(c) $\int_1^2 3f(x)\, dx$ **(d)** $\int_2^5 f(x)\, dx$

(e) $\int_1^5 [f(x) - g(x)]\, dx$ **(f)** $\int_1^5 [4f(x) - g(x)]\, dx$

2. Suppose that f and h are continuous functions and that

$$\int_1^9 f(x)\, dx = -1, \quad \int_7^9 f(x)\, dx = 5, \quad \int_7^9 h(x)\, dx = 4.$$

Use the rules in Table 5.3 to find each integral.

(a) $\int_1^9 -2f(x)\, dx$ **(b)** $\int_7^9 [f(x) + h(x)]\, dx$

(c) $\int_7^9 [2f(x) - 3h(x)]\, dx$ **(d)** $\int_9^1 f(x)\, dx$

(e) $\int_1^7 f(x)\, dx$ **(f)** $\int_9^7 [h(x) - f(x)]\, dx$

3. Suppose that $\int_1^2 f(x)\, dx = 5$. Find each integral.

(a) $\int_1^2 f(u)\, du$ **(b)** $\int_1^2 \sqrt{3} f(z)\, dz$

(c) $\int_2^1 f(t)\, dt$ **(d)** $\int_1^2 [-f(x)]\, dx$

4. Suppose that $\int_{-3}^0 g(t)\, dt = \sqrt{2}$. Find each integral.

(a) $\int_0^{-3} g(t)\, dt$ **(b)** $\int_{-3}^0 g(u)\, du$

(c) $\int_{-3}^0 [-g(x)]\, dx$ **(d)** $\int_{-3}^0 \dfrac{g(r)}{\sqrt{2}}\, dr$

5. Suppose that f is continuous and that

$$\int_0^3 f(z)\,dz = 3 \quad \text{and} \quad \int_0^4 f(z)\,dz = 7.$$

Find each integral.

(a) $\int_3^4 f(z)\,dz$ **(b)** $\int_4^3 f(t)\,dt$

6. Suppose that h is continuous and that

$$\int_{-1}^1 h(r)\,dr = 0 \quad \text{and} \quad \int_{-1}^3 h(r)\,dr = 6.$$

Find each integral.

(a) $\int_1^3 h(r)\,dr$ **(b)** $-\int_3^1 h(u)\,du$

7. Show that the value of $\int_0^1 \sin(x^2)\,dx$ cannot possibly be 2.

8. Show that the value of $\int_0^1 \sqrt{x+8}\,dx$ lies between $2\sqrt{2} \approx 2.8$ and 3.

9. ***Integrals of Nonnegative Functions*** Use the Max-Min Inequality to show that if f is integrable then

$$f(x) \geq 0 \text{ on } [a, b] \quad \Rightarrow \quad \int_a^b f(x)\,dx \geq 0.$$

10. ***Integrals of Nonpositive Functions*** Show that if f is integrable then

$$f(x) \leq 0 \text{ on } [a, b] \quad \Rightarrow \quad \int_a^b f(x)\,dx \leq 0.$$

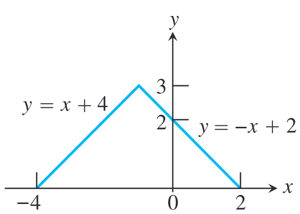
In Exercises 11–14, use NINT to find the average value of the function on the interval. At what point(s) in the interval does the function assume its average value?

11. $y = x^2 - 1, \quad [0, \sqrt{3}]$ **12.** $y = -\frac{x^2}{2}, \quad [0, 3]$

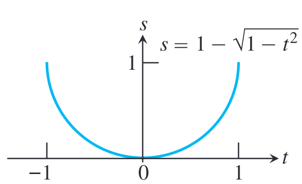
13. $y = -3x^2 - 1, \quad [0, 1]$ **14.** $y = (x-1)^2, \quad [0, 3]$

In Exercises 15–18, find the average value of the function on the interval without integrating, by appealing to the geometry of the region between the graph and the x-axis.

15. $f(x) = \begin{cases} x + 4, & -4 \leq x \leq -1, \\ -x + 2, & -1 < x \leq 2, \end{cases}$ on $[-4, 2]$



16. $f(t) = 1 - \sqrt{1 - t^2}, \quad [-1, 1]$



17. $f(t) = \sin t, \quad [0, 2\pi]$

18. $f(\theta) = \tan\theta, \quad \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

In Exercises 19–30, evaluate the integral using antiderivatives, as in Example 4.

19. $\int_\pi^{2\pi} \sin x\,dx$ **20.** $\int_0^{\pi/2} \cos x\,dx$

21. $\int_0^1 e^x\,dx$ **22.** $\int_0^{\pi/4} \sec^2 x\,dx$

23. $\int_1^4 2x\,dx$ **24.** $\int_{-1}^2 3x^2\,dx$

25. $\int_{-2}^6 5\,dx$ **26.** $\int_3^7 8\,dx$

27. $\int_{-1}^1 \frac{1}{1+x^2}\,dx$ **28.** $\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx$

29. $\int_1^e \frac{1}{x}\,dx$ **30.** $\int_1^4 -x^{-2}\,dx$

In Exercises 31–36, find the average value of the function on the interval, using antiderivatives to compute the integral.

31. $y = \sin x, \quad [0, \pi]$ **32.** $y = \frac{1}{x}, \quad [e, 2e]$

33. $y = \sec^2 x, \quad \left[0, \frac{\pi}{4}\right]$ **34.** $y = \frac{1}{1+x^2}, \quad [0, 1]$

35. $y = 3x^2 + 2x, \quad [-1, 2]$ **36.** $y = \sec x \tan x, \quad \left[0, \frac{\pi}{3}\right]$

37. **Group Activity** Use the Max-Min Inequality to find upper and lower bounds for the value of

$$\int_0^1 \frac{1}{1+x^4}\,dx.$$

38. **Group Activity** *(Continuation of Exercise 37)* Use the Max-Min Inequality to find upper and lower bounds for the values of

$$\int_0^{0.5} \frac{1}{1+x^4}\,dx \quad \text{and} \quad \int_{0.5}^1 \frac{1}{1+x^4}\,dx.$$

Add these to arrive at an improved estimate for

$$\int_0^1 \frac{1}{1+x^4}\,dx.$$

39. **Writing to Learn** If $av(f)$ really is a typical value of the integrable function $f(x)$ on $[a, b]$, then the number $av(f)$ should have the same integral over $[a, b]$ that f does. Does it? That is, does

$$\int_a^b av(f)\,dx = \int_a^b f(x)\,dx?$$

Give reasons for your answer.

40. **Writing to Learn** A driver averaged 30 mph on a 150-mile trip and then returned over the same 150 miles at the rate of 50 mph. He figured that his average speed was 40 mph for the entire trip.

(a) What was his total distance traveled?

(b) What was his total time spent for the trip?

(c) What was his average speed for the trip?

(d) Explain the error in the driver's reasoning.

41. **Writing to Learn** A dam released 1000 m^3 of water at 10 m^3/min and then released another 1000 m^3 at 20 m^3/min. What was the average rate at which the water was released? Give reasons for your answer.

42. Use the inequality $\sin x \le x$, which holds for $x \ge 0$, to find an upper bound for the value of $\int_0^1 \sin x\, dx$.

43. The inequality $\sec x \ge 1 + (x^2/2)$ holds on $(-\pi/2, \pi/2)$. Use it to find a lower bound for the value of $\int_0^1 \sec x\, dx$.

44. Show that the average value of a linear function $L(x)$ on $[a, b]$ is

$$\frac{L(a) + L(b)}{2}.$$

[*Caution*: This simple formula for average value does *not* work for functions in general!]

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

45. **True or False** The average value of a function f on $[a, b]$ always lies between $f(a)$ and $f(b)$. Justify your answer.

46. **True or False** If $\int_a^b f(x)\, dx = 0$, then $f(a) = f(b)$. Justify your answer.

47. **Multiple Choice** If $\int_3^7 f(x)\, dx = 5$ and $\int_3^7 g(x)\, dx = 3$, then all of the following must be true *except*

(A) $\displaystyle\int_3^7 f(x)g(x)dx = 15$

(B) $\displaystyle\int_3^7 [f(x) + g(x)]dx = 8$

(C) $\displaystyle\int_3^7 2f(x)\, dx = 10$

(D) $\displaystyle\int_3^7 [f(x) - g(x)]dx = 2$

(E) $\displaystyle\int_7^3 [g(x) - f(x)]dx = 2$

48. **Multiple Choice** If $\int_2^5 f(x)\, dx = 12$ and $\int_5^8 f(x)\, dx = 4$, then all of the following must be true *except*

(A) $\displaystyle\int_2^8 f(x)\, dx = 16$

(B) $\displaystyle\int_2^5 f(x)\, dx - \int_5^8 3f(x)\, dx = 0$

(C) $\displaystyle\int_5^2 f(x)\, dx = -12$

(D) $\displaystyle\int_{-5}^{-8} f(x)\, dx = -4$

(E) $\displaystyle\int_2^6 f(x)\, dx + \int_6^8 f(x)\, dx = 16$

49. **Multiple Choice** What is the average value of the cosine function on the interval [1, 5]?

(A) -0.990 **(B)** -0.450 **(C)** -0.128

(D) 0.412 **(E)** 0.998

50. **Multiple Choice** If the average value of the function f on the interval $[a, b]$ is 10, then $\int_a^b f(x)\, dx =$

(A) $\dfrac{10}{b-a}$ **(B)** $\dfrac{f(a)+f(b)}{10}$ **(C)** $10b - 10a$

(D) $\dfrac{b-a}{10}$ **(E)** $\dfrac{f(b)+f(a)}{20}$

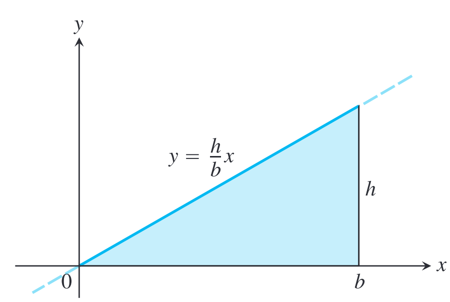
Exploration

51. ***Comparing Area Formulas*** Consider the region in the first quadrant under the curve $y = (h/b)x$ from $x = 0$ to $x = b$ (see figure).

(a) Use a geometry formula to calculate the area of the region.

(b) Find all antiderivatives of y.

(c) Use an antiderivative of y to evaluate $\int_0^b y(x)\, dx$.



Extending the Ideas

52. ***Graphing Calculator Challenge*** If $k > 1$, and if the average value of x^k on $[0, k]$ is k, what is k? Check your result with a CAS if you have one available.

53. Show that if $F'(x) = G'(x)$ on $[a, b]$, then

$$F(b) - F(a) = G(b) - G(a).$$

Quick Quiz for AP* Preparation: Sections 5.1–5.3

You should solve the following problems without using a calculator.

1. Multiple Choice If $\int_a^b f(x)\,dx = a + 2b$, then $\int_a^b (f(x) + 3)\,dx =$

(A) $a + 2b + 3$ **(B)** $3b - 3a$

(C) $4a - b$ **(D)** $5b - 2a$

(E) $5b - 3a$

2. Multiple Choice The expression

$$\frac{1}{20}\left(\sqrt{\frac{1}{20}} + \sqrt{\frac{2}{20}} + \sqrt{\frac{3}{20}} + \cdots + \sqrt{\frac{20}{20}}\right)$$

is a Riemann sum approximation for

(A) $\int_0^1 \sqrt{\frac{x}{20}}\,dx$ **(B)** $\int_0^1 \sqrt{x}\,dx$

(C) $\frac{1}{20}\int_0^1 \sqrt{\frac{x}{20}}\,dx$ **(D)** $\frac{1}{20}\int_0^1 \sqrt{x}\,dx$

(E) $\frac{1}{20}\int_0^{20} \sqrt{x}\,dx$

3. Multiple Choice What are all values of k for which $\int_2^k x^2\,dx = 0$?

(A) -2 **(B)** 0 **(C)** 2

(D) -2 and 2 **(E)** -2, 0, and 2

4. Free Response Let f be a function such that $f''(x) = 6x + 12$.

(a) Find $f(x)$ if the graph of f is tangent to the line $4x - y = 5$ at the point $(0, -5)$.

(b) Find the average value of $f(x)$ on the closed interval $[-1, 1]$.

5.4 Fundamental Theorem of Calculus

What you'll learn about

- Fundamental Theorem, Part 1
- Graphing the Function $\int_a^x f(t)\,dt$
- Fundamental Theorem, Part 2
- Area Connection
- Analyzing Antiderivatives Graphically

. . . and why

The Fundamental Theorem of Calculus is a triumph of mathematical discovery and the key to solving many problems.

Sir Isaac Newton
(1642–1727)

Sir Isaac Newton is considered to be one of the most influential mathematicians of all time. Moreover, by the age of 25, he had also made revolutionary advances in optics, physics, and astronomy.

Fundamental Theorem, Part 1

This section presents the discovery by Newton and Leibniz of the astonishing connection between integration and differentiation. This connection started the mathematical development that fueled the scientific revolution for the next 200 years, and is still regarded as the most important computational discovery in the history of mathematics: The Fundamental Theorem of Calculus.

The Fundamental Theorem comes in two parts, both of which were previewed in Exploration 2 of the previous section. The first part says that the definite integral of a continuous function is a differentiable function of its upper limit of integration. Moreover, it tells us what that derivative is. The second part says that the definite integral of a continuous function from a to b can be found from any one of the function's antiderivatives F as the number $F(b) - F(a)$.

THEOREM 4 The Fundamental Theorem of Calculus, Part 1

If f is continuous on $[a, b]$, then the function

$$F(x) = \int_a^x f(t)\,dt$$

has a derivative at every point x in $[a, b]$, and

$$\frac{dF}{dx} = \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Proof The geometric exploration at the end of the previous section contained the idea of the proof, but it glossed over the necessary limit arguments. Here we will be more precise.

Apply the definition of the derivative directly to the function F. That is,

$$\begin{aligned}
\frac{dF}{dx} &= \lim_{h\to 0}\frac{F(x+h)-F(x)}{h}\\
&= \lim_{h\to 0}\frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h}\\
&= \lim_{h\to 0}\frac{\int_x^{x+h} f(t)\,dt}{h}\\
&= \lim_{h\to 0}\left[\frac{1}{h}\int_x^{x+h} f(t)\,dt\right].
\end{aligned}$$

The expression in brackets in the last line is the average value of f from x to $x + h$. We know from the Mean Value Theorem for Definite Integrals (Theorem 3, Section 5.3) that f, being continuous, takes on its average value at least once in the interval; that is,

$$\frac{1}{h}\int_x^{x+h} f(t)\,dt = f(c) \qquad \text{for some } c \text{ between } x \text{ and } x + h.$$

We can therefore continue our proof, letting $(1/h)\int_x^{x+h} f(t)\,dt = f(c)$,

$$\frac{dF}{dx} = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt$$

$$= \lim_{h\to 0} f(c), \qquad \text{where } c \text{ lies between } x \text{ and } x + h.$$

What happens to c as h goes to zero? As $x + h$ gets closer to x, it carries c along with it like a bead on a wire, forcing c to approach x. Since f is continuous, this means that $f(c)$ approaches $f(x)$:

$$\lim_{h\to 0} f(c) = f(x).$$

Putting it all together,

$$\frac{dF}{dx} = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h\to 0} \frac{\int_x^{x+h} f(t)\,dt}{h}$$

$$= \lim_{h\to 0} f(c) \qquad \text{for some } c \text{ between } x \text{ and } x + h.$$

$$= f(x).$$

This concludes the proof. ■

It is difficult to overestimate the power of the equation

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x). \tag{1}$$

It says that every continuous function f is the derivative of some other function, namely $\int_a^x f(t)\,dt$. It says that every continuous function has an antiderivative. And it says that the processes of integration and differentiation are inverses of one another. If any equation deserves to be called the Fundamental Theorem of Calculus, this equation is surely the one.

EXAMPLE 1 Applying the Fundamental Theorem

Find

$$\frac{d}{dx}\int_{-\pi}^x \cos t\,dt \quad \text{and} \quad \frac{d}{dx}\int_0^x \frac{1}{1+t^2}\,dt$$

by using the Fundamental Theorem.

SOLUTION

$$\frac{d}{dx}\int_{-\pi}^x \cos t\,dt = \cos x$$

$$\frac{d}{dx}\int_0^x \frac{1}{1+t^2}\,dt = \frac{1}{1+x^2}.$$

Now try Exercise 3.

EXAMPLE 2 The Fundamental Theorem with the Chain Rule

Find dy/dx if $y = \int_1^{x^2} \cos t\, dt$.

SOLUTION

The upper limit of integration is not x but x^2. This makes y a composite of

$$y = \int_1^u \cos t\, dt \quad \text{and} \quad u = x^2.$$

We must therefore apply the Chain Rule when finding dy/dx.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\
&= \left(\frac{d}{du}\int_1^u \cos t\, dt\right) \cdot \frac{du}{dx} \\
&= \cos u \cdot \frac{du}{dx} \\
&= \cos(x^2) \cdot 2x \\
&= 2x \cos x^2
\end{aligned}$$

Now try Exercise 9.

EXAMPLE 3 Variable Lower Limits of Integration

Find dy/dx.

(a) $y = \displaystyle\int_x^5 3t \sin t\, dt$ **(b)** $y = \displaystyle\int_{2x}^{x^2} \frac{1}{2 + e^t}\, dt$

SOLUTION

The rules for integrals set these up for the Fundamental Theorem.

(a)
$$\begin{aligned}
\frac{d}{dx}\int_x^5 3t \sin t\, dt &= \frac{d}{dx}\left(-\int_5^x 3t \sin t\, dt\right) \\
&= -\frac{d}{dx}\int_5^x 3t \sin t\, dt \\
&= -3x \sin x
\end{aligned}$$

(b)
$$\begin{aligned}
\frac{d}{dx}\int_{2x}^{x^2} \frac{1}{2+e^t}\, dt &= \frac{d}{dx}\left(\int_0^{x^2} \frac{1}{2+e^t}\, dt - \int_0^{2x} \frac{1}{2+e^t}\, dt\right) \\
&= \frac{1}{2+e^{x^2}}\frac{d}{dx}(x^2) - \frac{1}{2+e^{2x}}\frac{d}{dx}(2x) \\
&= \frac{1}{2+e^{x^2}} \cdot 2x - \frac{1}{2+e^{2x}} \cdot 2 \\
&= \frac{2x}{2+e^{x^2}} - \frac{2}{2+e^{2x}}
\end{aligned}$$

Now try Exercise 19.

EXAMPLE 4 Constructing a Function with a Given Derivative and Value

Find a function $y = f(x)$ with derivative

$$\frac{dy}{dx} = \tan x$$

that satisfies the condition $f(3) = 5$.

SOLUTION

The Fundamental Theorem makes it easy to construct a function with derivative tan *x:*

$$y = \int_3^x \tan t\, dt.$$

Since $y(3) = 0$, we have only to add 5 to this function to construct one with derivative tan *x* whose value at $x = 3$ is 5:

$$f(x) = \int_3^x \tan t\, dt + 5.$$

Now try Exercise 25.

Although the solution to the problem in Example 4 satisfies the two required conditions, you might question whether it is in a useful form. Not many years ago, this form might have posed a computation problem. Indeed, for such problems much effort has been expended over the centuries trying to find solutions that do not involve integrals. We will see some in Chapter 6, where we will learn (for example) how to write the solution in Example 4 as

$$y = \ln\left|\frac{\cos 3}{\cos x}\right| + 5.$$

However, now that computers and calculators are capable of evaluating integrals, the form given in Example 4 is not only useful, but in some ways preferable. It is certainly easier to find and is always available.

Graphing the Function $\int_a^x f(t)\, dt$

Consider for a moment the two forms of the function we have just been discussing,

$$F(x) = \int_3^x \tan t\, dt + 5 \quad \text{and} \quad F(x) = \ln\left|\frac{\cos 3}{\cos x}\right| + 5.$$

With which expression is it easier to evaluate, say, $F(4)$? From the time of Newton almost to the present, there has been no contest: the expression on the right. At least it provides something to compute, and there have always been tables or slide rules or calculators to facilitate that computation. The expression on the left involved at best a tedious summing process and almost certainly an increased opportunity for error.

Today we can find $F(4)$ from either expression on the same machine. The choice is between NINT (tan *x*, *x*, 3, 4) + 5 and ln (abs(cos(3)/cos(4))) + 5. Both calculations give 5.415135083 in approximately the same amount of time.

We can even use NINT to graph the function. This modest technology feat would have absolutely dazzled the mathematicians of the 18th and 19th centuries, who knew how the solutions of differential equations, such as $dy/dx = \tan x$, could be written as integrals, but for whom integrals were of no practical use computationally unless they could be written in exact form. Since so few integrals could, in fact, be written in exact form, NINT would have spared generations of scientists much frustration.

Nevertheless, one must not proceed blindly into the world of calculator computation. Exploration 1 will demonstrate the need for caution.

EXPLORATION 1 Graphing NINT *f*

Let us use NINT to attempt to graph the function we just discussed,

$$F(x) = \int_3^x \tan t \, dt + 5.$$

1. Graph the function $y = F(x)$ in the window $[-10, 10]$ by $[-10, 10]$. You will probably wait a long time and see no graph. Break out of the graphing program if necessary.
2. Recall that the graph of the function $y = \tan x$ has vertical asymptotes. Where do they occur on the interval $[-10, 10]$?
3. When attempting to graph the function $F(x) = \int_3^x \tan t \, dt + 5$ on the interval $[-10, 10]$, your grapher begins by trying to find $F(-10)$. Explain why this might cause a problem for your calculator.
4. Set your viewing window so that your calculator graphs only over the domain of the continuous branch of the tangent function that contains the point $(3, \tan 3)$.
5. What is the domain in step 4? Is it an open interval or a closed interval?
6. What is the domain of $F(x)$? Is it an open interval or a closed interval?
7. Your calculator graphs over the closed interval $[x_{\min}, x_{\max}]$. Find a viewing window that will give you a good look at the graph of F and produce the graph on your calculator.
8. Describe the graph of F.

Graphing NINT *f*

Some graphers can graph the numerical integral $y = \text{NINT}\,(f(x), x, a, x)$ directly as a function of x. Others will require a toolbox program such as the one called NINTGRAF provided in the *Technology Resource Manual.*

You have probably noticed that your grapher moves slowly when graphing NINT. This is because it must compute each value as a limit of sums—comparatively slow work even for a microprocessor. Here are some ways to speed up the process:

1. Change the *tolerance* on your grapher. The smaller the tolerance, the more accurate the calculator will try to be when finding the limiting value of each sum (and the longer it will take to do so). The default value is usually quite small (like 0.00001), but a value as large as 1 can be used for graphing in a typical viewing window.
2. Change the *x-resolution.* The default resolution is 1, which means that the grapher will compute a function value for every vertical column of pixels. At resolution 2 it computes only every second value, and so on. With higher resolutions, some graph smoothness is sacrificed for speed.
3. Switch to parametric mode. To graph $y = \text{NINT}\,(f(x), x, a, x)$ in parametric mode, let $x(t) = t$ and let $y(t) = \text{NINT}\,(f(t), t, a, t)$. You can then control the speed of the grapher by changing the t-step. (Choosing a bigger t-step has the same effect as choosing a larger x-resolution.)

EXPLORATION 2 **The Effect of Changing *a* in $\int_a^x f(t)dt$**

The first part of the Fundamental Theorem of Calculus asserts that the derivative of $\int_a^x f(t)\,dt$ is $f(x)$, regardless of the value of a.

1. Graph NDER (NINT $(x^2, x, 0, x)$).
2. Graph NDER (NINT $(x^2, x, 5, x)$).
3. Without graphing, tell what the x-intercept of NINT $(x^2, x, 0, x)$ is. Explain.
4. Without graphing, tell what the x-intercept of NINT $(x^2, x, 5, x)$ is. Explain.
5. How does changing a affect the graph of $y = (d/dx)\int_a^x f(t)\,dt$?
6. How does changing a affect the graph of $y = \int_a^x f(t)\,dt$?

Fundamental Theorem, Part 2

The second part of the Fundamental Theorem of Calculus shows how to evaluate definite integrals directly from antiderivatives.

THEOREM 4 (continued) **The Fundamental Theorem of Calculus, Part 2**

If f is continuous at every point of $[a, b]$, and if F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

This part of the Fundamental Theorem is also called the **Integral Evaluation Theorem.**

Proof Part 1 of the Fundamental Theorem tells us that an antiderivative of f exists, namely

$$G(x) = \int_a^x f(t)\,dt.$$

Thus, if F is *any* antiderivative of f, then $F(x) = G(x) + C$ for some constant C (by Corollary 3 of the Mean Value Theorem for Derivatives, Section 4.2).

Evaluating $F(b) - F(a)$, we have

$$\begin{aligned} F(b) - F(a) &= [G(b) + C] - [G(a) + C] \\ &= G(b) - G(a) \\ &= \int_a^b f(t)\,dt - \int_a^a f(t)\,dt \\ &= \int_a^b f(t)\,dt - 0 \\ &= \int_a^b f(t)\,dt. \end{aligned}$$

■

At the risk of repeating ourselves: It is difficult to overestimate the power of the simple equation

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

It says that any definite integral of any continuous function f can be calculated without taking limits, without calculating Riemann sums, and often without effort—so long as an antiderivative of f can be found. If you can imagine what it was like before this theorem (and before computing machines), when approximations by tedious sums were the only alternative for solving many real-world problems, then you can imagine what a miracle calculus was thought to be. If any equation deserves to be called the Fundamental Theorem of Calculus, this equation is surely the (second) one.

Integral Evaluation Notation

The usual notation for $F(b) - F(a)$ is

$$F(x)\Big]_a^b \quad \text{or} \quad \Big[F(x)\Big]_a^b,$$

depending on whether F has one or more terms. This notation provides a compact "recipe" for the evaluation, allowing us to show the antiderivative in an intermediate step.

EXAMPLE 5 Evaluating an Integral

Evaluate $\int_{-1}^{3} (x^3 + 1)\,dx$ using an antiderivative.

SOLUTION

Solve Analytically A simple antiderivative of $x^3 + 1$ is $(x^4/4) + x$. Therefore,

$$\int_{-1}^{3} \left(x^3 + 1\right) dx = \left[\frac{x^4}{4} + x\right]_{-1}^{3}$$

$$= \left(\frac{81}{4} + 3\right) - \left(\frac{1}{4} - 1\right)$$

$$= 24.$$

Support Numerically NINT $(x^3 + 1, x, -1, 3) = 24.$ *Now try Exercise 29.*

Area Connection

In Section 5.2 we saw that the definite integral could be interpreted as the net area between the graph of a function and the x-axis. We can therefore compute areas using antiderivatives, but we must again be careful to distinguish net area (in which area below the x-axis is counted as negative) from total area. The unmodified word "area" will be taken to mean *total area.*

EXAMPLE 6 Finding Area Using Antiderivatives

Find the area of the region between the curve $y = 4 - x^2$, $0 \le x \le 3$, and the x-axis.

SOLUTION

The curve crosses the x-axis at $x = 2$, partitioning the interval $[0, 3]$ into two subintervals, on each of which $f(x) = 4 - x^2$ will not change sign.

We can see from the graph (Figure 5.28) that $f(x) > 0$ on $[0, 2)$ and $f(x) < 0$ on $(2, 3]$.

Over $[0, 2]$: $\displaystyle\int_0^2 (4 - x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_0^2 = \frac{16}{3}.$

Over $[2, 3]$: $\displaystyle\int_2^3 (4 - x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_2^3 = -\frac{7}{3}.$

The area of the region is $\left|\frac{16}{3}\right| + \left|-\frac{7}{3}\right| = \frac{23}{3}.$ *Now try Exercise 41.*

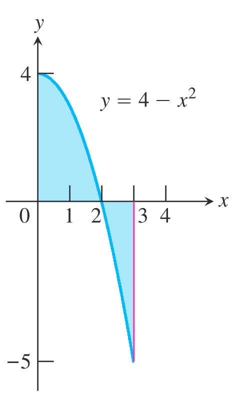
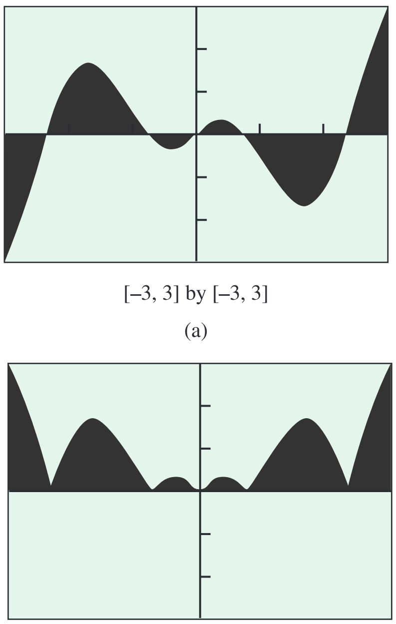


Figure 5.28 The function $f(x) = 4 - x^2$ changes sign only at $x = 2$ on the interval $[0, 3]$. (Example 6)

How to Find Total Area Analytically

To find the area between the graph of $y = f(x)$ and the x-axis over the interval $[a, b]$ analytically,

1. partition $[a, b]$ with the zeros of f,
2. integrate f over each subinterval,
3. add the absolute values of the integrals.


[–3, 3] by [–3, 3]

(a)

[–3, 3] by [–3, 3]

(b)

Figure 5.29 The graphs of (a) $y = x \cos 2x$ and (b) $y = |x \cos 2x|$ over $[-3, 3]$. The shaded regions have the same area.

We can find area numerically by using NINT to integrate the *absolute value* of the function over the given interval. There is no need to partition. By taking absolute values, we automatically reflect the negative portions of the graph across the x-axis to count all area as positive (Figure 5.29).

EXAMPLE 7 Finding Area Using NINT

Find the area of the region between the curve $y = x \cos 2x$ and the x-axis over the interval $-3 \le x \le 3$ (Figure 5.29).

SOLUTION

Rounded to two decimal places, we have

$$\text{NINT}\,(|x \cos 2x|, x, -3, 3) = 5.43.$$

Now try Exercise 51.

How to Find Total Area Numerically

To find the area between the graph of $y = f(x)$ and the x-axis over the interval $[a, b]$ numerically, evaluate

$$\text{NINT}\,(|f(x)|, x, a, b).$$

Analyzing Antiderivatives Graphically

A good way to put several calculus concepts together at this point is to start with the graph of a function f and consider a new function h defined as a definite integral of f. If $h(x) = \int_a^x f(t)\,dt$, for example, the Fundamental Theorem guarantees that $h'(x) = f(x)$, so the graph of f is also the graph of h'. We can therefore make conclusions about the behavior of h by considering the graphical behavior of its derivative f, just as we did in Section 4.3.

EXAMPLE 8 Using the Graph of f to Analyze $h(x) = \int_a^x f(t)\,dt$

The graph of a continuous function f with domain [0, 8] is shown in Figure 5.30. Let h be the function defined by $h(x) = \int_1^x f(t)\,dt$.

(a) Find $h(1)$.

(b) Is $h(0)$ positive or negative? Justify your answer.

(c) Find the value of x for which $h(x)$ is a maximum.

(d) Find the value of x for which $h(x)$ is a minimum.

(e) Find the x-coordinates of all points of inflections of the graph of $y = h(x)$.

continued

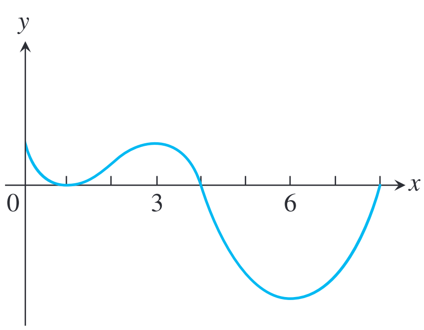


Figure 5.30 The graph of f in Example 8, in which questions are asked about the function $h(x) = \int_1^x f(t)\,dt$.

SOLUTION

First, we note that $h'(x) = f(x)$, so the graph of f is also the graph of the derivative of h. Also, h is continuous because it is differentiable.

(a) $h(1) = \int_1^1 f(t)\,dt = 0$.

(b) $h(0) = \int_1^0 f(t)\,dt < 0$, because we are integrating from right to left under a positive function.

(c) The derivative of h is positive on (0,1), positive on (1, 4), and negative on (4, 8), so the continuous function h is increasing on [0, 4] and decreasing on [4, 8]. Thus $f(4)$ is a maximum.

(d) The sign analysis of the derivative above shows that the minimum value occurs at an endpoint of the interval [0, 8]. We see by comparing areas that $h(0) = \int_1^0 f(t)\,dt \approx -0.5$, while $h(8) = \int_1^8 f(t)\,dt$ is a negative number considerably less than -1. Thus $f(8)$ is a minimum.

(e) The points of inflection occur where $h' = f$ changes direction, that is, at $x = 1$, $x = 3$, and $x = 6$. ***Now try Exercise 57.***

Quick Review 5.4 *(For help, go to Sections 3.6, 3.7, and 3.9.)*

In Exercises 1–10, find dy/dx.

1. $y = \sin(x^2)$

2. $y = (\sin x)^2$

3. $y = \sec^2 x - \tan^2 x$

4. $y = \ln(3x) - \ln(7x)$

5. $y = 2^x$

6. $y = \sqrt{x}$

7. $y = \dfrac{\cos x}{x}$

8. $y = \sin t$ and $x = \cos t$

9. $xy + x = y^2$

10. $dx/dy = 3x$

Section 5.4 Exercises

In Exercises 1–20, find dy/dx.

1. $y = \int_0^x (\sin^2 t)\,dt$

2. $y = \int_2^x (3t + \cos t^2)\,dt$

3. $y = \int_0^x (t^3 - t)^5\,dt$

4. $y = \int_{-2}^x \sqrt{1 + e^{5t}}\,dt$

5. $y = \int_2^x (\tan^3 u)du$

6. $y = \int_4^x e^u \sec u\,du$

7. $y = \int_7^x \dfrac{1+t}{1+t^2}dt$

8. $y = \int_{-\pi}^x \dfrac{2 - \sin t}{3 + \cos t}\,dt$

9. $y = \int_0^{x^2} e^{t^2}\,dt$

10. $y = \int_6^{x^2} \cot 3t\,dt$

11. $y = \int_2^{5x} \dfrac{\sqrt{1 + u^2}}{u}\,du$

12. $y = \int_\pi^{\pi - x} \dfrac{1 + \sin^2 u}{1 + \cos^2 u}\,du$

13. $y = \int_x^6 \ln(1 + t^2)\,dt$

14. $y = \int_x^7 \sqrt{2t^4 + t + 1}\,dt$

15. $y = \int_{x^3}^5 \dfrac{\cos t}{t^2 + 2}\,dt$

16. $y = \int_{5x^2}^{25} \dfrac{t^2 - 2t + 9}{t^3 + 6}\,dt$

17. $y = \int_{\sqrt{x}}^0 \sin(r^2)\,dr$

18. $y = \int_{3x^2}^{10} \ln(2 + p^2)\,dp$

19. $\int_{x^2}^{x^3} \cos(2t)\,dt$

20. $y = \int_{\sin x}^{\cos x} t^2\,dt$

In Exercises 21–26, construct a function of the form $y = \int_a^x f(t)\,dt + C$ that satisfies the given conditions.

21. $\dfrac{dy}{dx} = \sin^3 x$, and $y = 0$ when $x = 5$.

22. $\dfrac{dy}{dx} = e^x \tan x$, and $y = 0$ when $x = 8$.

23. $\dfrac{dy}{dx} = \ln(\sin x + 5)$, and $y = 3$ when $x = 2$.

24. $\dfrac{dy}{dx} = \sqrt{3 - \cos x}$, and $y = 4$ when $x = -3$.

25. $\frac{dy}{dx} = \cos^2 5x$, and $y = -2$ when $x = 7$.

26. $\frac{dy}{dx} = e^{\sqrt{x}}$, and $y = 1$ when $x = 0$.

In Exercises 27–40, evaluate each integral using Part 2 of the Fundamental Theorem. Support your answer with NINT if you are unsure.

27. $\int_{1/2}^{3} \left(2 - \frac{1}{x}\right) dx$

28. $\int_{2}^{-1} 3^x \, dx$

29. $\int_{0}^{1} (x^2 + \sqrt{x}) \, dx$

30. $\int_{0}^{5} x^{3/2} \, dx$

31. $\int_{1}^{32} x^{-6/5} \, dx$

32. $\int_{-2}^{-1} \frac{2}{x^2} \, dx$

33. $\int_{0}^{\pi} \sin x \, dx$

34. $\int_{0}^{\pi} (1 + \cos x) \, dx$

35. $\int_{0}^{\pi/3} 2 \sec^2 \theta \, d\theta$

36. $\int_{\pi/6}^{5\pi/6} \csc^2 \theta \, d\theta$

37. $\int_{\pi/4}^{3\pi/4} \csc x \cot x \, dx$

38. $\int_{0}^{\pi/3} 4 \sec x \tan x \, dx$

39. $\int_{-1}^{1} (r + 1)^2 \, dr$

40. $\int_{0}^{4} \frac{1 - \sqrt{u}}{\sqrt{u}} \, du$

In Exercises 41–44, find the total area of the region between the curve and the x-axis.

41. $y = 2 - x, \quad 0 \le x \le 3$

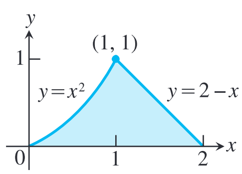
42. $y = 3x^2 - 3, \quad -2 \le x \le 2$

43. $y = x^3 - 3x^2 + 2x, \quad 0 \le x \le 2$

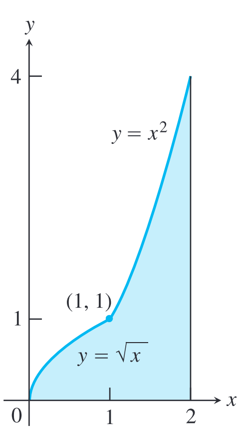
44. $y = x^3 - 4x, \quad -2 \le x \le 2$

In Exercises 45–48, find the area of the shaded region.

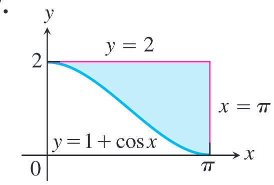
45.



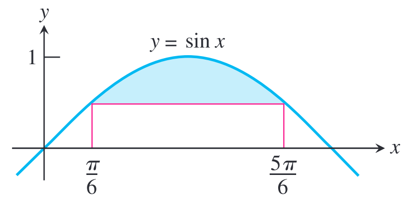
46.



47.



48.



In Exercises 49–54, use NINT to solve the problem.

49. Evaluate $\int_{0}^{10} \frac{1}{3 + 2 \sin x} \, dx$.

50. Evaluate $\int_{-0.8}^{0.8} \frac{2x^4 - 1}{x^4 - 1} \, dx$.

51. Find the area of the semielliptical region between the x-axis and the graph of $y = \sqrt{8 - 2x^2}$.

52. Find the average value of $\sqrt{\cos x}$ on the interval $[-1, 1]$.

53. For what value of x does $\int_0^x e^{-t^2} \, dt = 0.6$?

54. Find the area of the region in the first quadrant enclosed by the coordinate axes and the graph of $x^3 + y^3 = 1$.

In Exercises 55 and 56, find K so that

$$\int_a^x f(t) \, dt + K = \int_b^x f(t) \, dt.$$

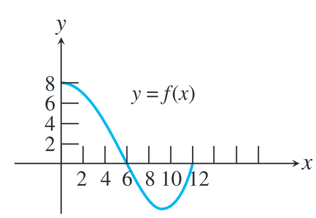
55. $f(x) = x^2 - 3x + 1; \quad a = -1; \quad b = 2$

56. $f(x) = \sin^2 x; \quad a = 0; \quad b = 2$

57. Let

$$H(x) = \int_0^x f(t) \, dt,$$

where f is the continuous function with domain $[0, 12]$ graphed here.



(a) Find $H(0)$.

(b) On what interval is H increasing? Explain.

(c) On what interval is the graph of H concave up? Explain.

(d) Is $H(12)$ positive or negative? Explain.

(e) Where does H achieve its maximum value? Explain.

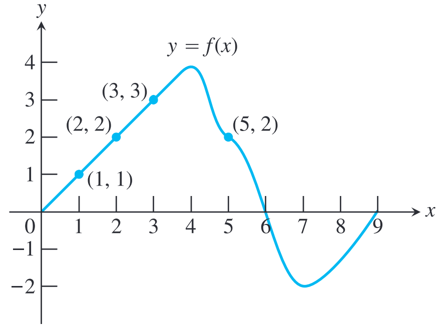
(f) Where does H achieve its minimum value? Explain.

In Exercises 58 and 59, f is the differentiable function whose graph is shown in the given figure. The position at time t (sec) of a particle moving along a coordinate axis is

$$s = \int_0^t f(x)\,dx$$

meters. Use the graph to answer the questions. Give reasons for your answers.

58.



(a) What is the particle's velocity at time $t = 5$?

(b) Is the acceleration of the particle at time $t = 5$ positive or negative?

(c) What is the particle's position at time $t = 3$?

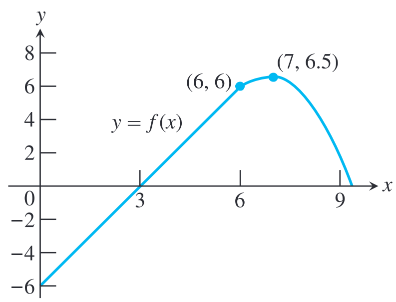
(d) At what time during the first 9 sec does s have its largest value?

(e) Approximately when is the acceleration zero?

(f) When is the particle moving toward the origin? away from the origin?

(g) On which side of the origin does the particle lie at time $t = 9$?

59.



(a) What is the particle's velocity at time $t = 3$?

(b) Is the acceleration of the particle at time $t = 3$ positive or negative?

(c) What is the particle's position at time $t = 3$?

(d) When does the particle pass through the origin?

(e) Approximately when is the acceleration zero?

(f) When is the particle moving toward the origin? away from the origin?

(g) On which side of the origin does the particle lie at time $t = 9$?

60. Suppose $\int_1^x f(t)\,dt = x^2 - 2x + 1$. Find $f(x)$.

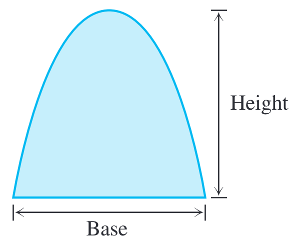
61. ***Linearization*** Find the linearization of

$$f(x) = 2 + \int_0^x \frac{10}{1+t}\,dt \quad \text{at} \quad x = 0.$$

62. Find $f(4)$ if $\int_0^x f(t)\,dt = x\cos \pi x$.

63. ***Finding Area*** Show that if k is a positive constant, then the area between the x-axis and one arch of the curve $y = \sin kx$ is always $2/k$.

64. ***Archimedes' Area Formula for Parabolas*** Archimedes (287–212 B.C.), inventor, military engineer, physicist, and the greatest mathematician of classical times, discovered that the area under a parabolic arch like the one shown here is always two-thirds the base times the height.



(a) Find the area under the parabolic arch

$$y = 6 - x - x^2, \quad -3 \le x \le 2.$$

(b) Find the height of the arch.

(c) Show that the area is two-thirds the base times the height.

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

65. True or False If f is continuous on an open interval I containing a, then F defined by $F(x) = \int_a^x f(t)\,dt$ is continuous on I. Justify your answer.

66. True or False If $b > a$, then $\frac{d}{dx}\int_a^b e^{x^2}\,dx$ is positive. Justify your answer.

67. Multiple Choice Let $f(x) = \int_a^x \ln(2 + \sin t)dt$. If $f(3) = 4$, then $f(5) =$

(A) 0.040 **(B)** 0.272 **(C)** 0.961 **(D)** 4.555 **(E)** 6.667

68. Multiple Choice What is $\lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt$?

(A) 0 **(B)** 1 **(C)** $f'(x)$ **(D)** $f(x)$ **(E)** nonexistent

69. Multiple Choice At $x = \pi$, the linearization of $f(x) = \int_\pi^x \cos^3 t\,dt$ is

(A) $y = -1$ **(B)** $y = -x$ **(C)** $y = \pi$
(D) $y = x - \pi$ **(E)** $y = \pi - x$

70. Multiple Choice The area of the region enclosed between the graph of $y = \sqrt{1 - x^4}$ and the x-axis is

(A) 0.886 **(B)** 1.253 **(C)** 1.414
(D) 1.571 **(E)** 1.748

Explorations

71. ***The Sine Integral Function*** The sine integral function

$$\text{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt$$

is one of the many useful functions in engineering that are defined as integrals. Although the notation does not show it, the function being integrated is

$$f(t) = \begin{cases} \dfrac{\sin t}{t}, & t \neq 0 \\ 1, & t = 0, \end{cases}$$

the continuous extension of $(\sin t)/t$ to the origin.

(a) Show that $\text{Si}(x)$ is an odd function of x.

(b) What is the value of $\text{Si}(0)$?

(c) Find the values of x at which $\text{Si}(x)$ has a local extreme value.

(d) Use NINT to graph $\text{Si}(x)$.

72. ***Cost from Marginal Cost*** The marginal cost of printing a poster when x posters have been printed is

$$\frac{dc}{dx} = \frac{1}{2\sqrt{x}}$$

dollars. Find

(a) $c(100) - c(1)$, the cost of printing posters 2 to 100.

(b) $c(400) - c(100)$, the cost of printing posters 101 to 400.

73. ***Revenue from Marginal Revenue*** Suppose that a company's marginal revenue from the manufacture and sale of eggbeaters is

$$\frac{dr}{dx} = 2 - \frac{2}{(x+1)^2},$$

where r is measured in thousands of dollars and x in thousands of units. How much money should the company expect from a production run of $x = 3$ thousand eggbeaters? To find out, integrate the marginal revenue from $x = 0$ to $x = 3$.

74. ***Average Daily Holding Cost*** Solon Container receives 450 drums of plastic pellets every 30 days. The inventory function (drums on hand as a function of days) is $I(x) = 450 - x^2/2$.

(a) Find the average daily inventory (that is, the average value of $I(x)$ for the 30-day period).

(b) If the holding cost for one drum is \$0.02 per day, find the average daily holding cost (that is, the per-drum holding cost times the average daily inventory).

75. Suppose that f has a negative derivative for all values of x and that $f(1) = 0$. Which of the following statements must be true of the function

$$h(x) = \int_0^x f(t)\,dt?$$

Give reasons for your answers.

(a) h is a twice-differentiable function of x.

(b) h and dh/dx are both continuous.

(c) The graph of h has a horizontal tangent at $x = 1$.

(d) h has a local maximum at $x = 1$.

(e) h has a local minimum at $x = 1$.

(f) The graph of h has an inflection point at $x = 1$.

(g) The graph of dh/dx crosses the x-axis at $x = 1$.

Extending the Ideas

76. **Writing to Learn** If f is an odd continuous function, give a graphical argument to explain why $\int_0^x f(t)\,dt$ is even.

77. **Writing to Learn** If f is an even continuous function, give a graphical argument to explain why $\int_0^x f(t)\,dt$ is odd.

78. **Writing to Learn** Explain why we can conclude from Exercises 76 and 77 that every even continuous function is the derivative of an odd continuous function and vice versa.

79. Give a convincing argument that the equation

$$\int_0^x \frac{\sin t}{t}\,dt = 1$$

has exactly one solution. Give its approximate value.

5.5 Trapezoidal Rule

What you'll learn about

- Trapezoidal Approximations
- Other Algorithms
- Error Analysis

. . . and why

Some definite integrals are best found by numerical approximations, and rectangles are not always the most efficient figures to use.

Trapezoidal Approximations

You probably noticed in Section 5.1 that MRAM was generally more efficient in approximating integrals than either LRAM or RRAM, even though all three RAM approximations approached the same limit. All three RAM approximations, however, depend on the areas of rectangles. Are there other geometric shapes with known areas that can do the job more efficiently? The answer is yes, and the most obvious one is the trapezoid.

As shown in Figure 5.31, if $[a, b]$ is partitioned into n subintervals of equal length $h = (b - a)/n$, the graph of f on $[a, b]$ can be approximated by a straight line segment over each subinterval.

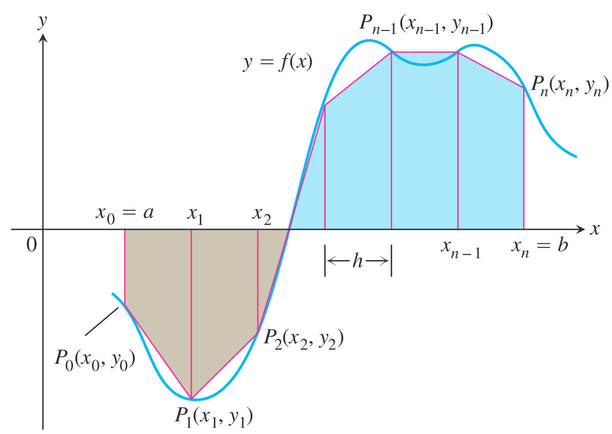


Figure 5.31 The trapezoidal rule approximates short stretches of the curve $y = f(x)$ with line segments. To approximate the integral of f from a to b, we add the "signed" areas of the trapezoids made by joining the ends of the segments to the x-axis.

The region between the curve and the x-axis is then approximated by the trapezoids, the area of each trapezoid being the length of its horizontal "altitude" times the average of its two vertical "bases." That is,

$$\int_a^b f(x)\,dx \approx h \cdot \frac{y_0 + y_1}{2} + h \cdot \frac{y_1 + y_2}{2} + \cdots + h \cdot \frac{y_{n-1} + y_n}{2}$$

$$= h\left(\frac{y_0}{2} + y_1 + y_2 + \cdots + y_{n-1} + \frac{y_n}{2}\right)$$

$$= \frac{h}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right),$$

where

$$y_0 = f(a), \quad y_1 = f(x_1), \quad \cdots, \quad y_{n-1} = f(x_{n-1}), \quad y_n = f(b).$$

This is algebraically equivalent to finding the numerical average of LRAM and RRAM; indeed, that is how some texts define the Trapezoidal Rule.

The Trapezoidal Rule

To approximate $\int_a^b f(x)\,dx$, use

$$T = \frac{h}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right),$$

where $[a, b]$ is partitioned into n subintervals of equal length $h = (b - a)/n$.
Equivalently,

$$T = \frac{\text{LRAM}_n + \text{RRAM}_n}{2},$$

where LRAM_n and RRAM_n are the Riemann sums using the left and right endpoints, respectively, for f for the partition.

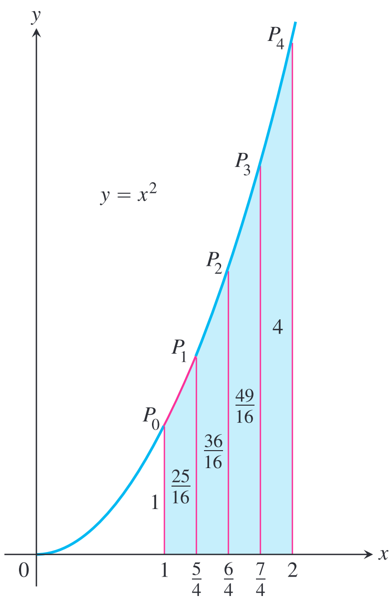


Figure 5.32 The trapezoidal approximation of the area under the graph of $y = x^2$ from $x = 1$ to $x = 2$ is a slight overestimate. (Example 1)

Table 5.4

x	$y = x^2$
1	1
$\frac{5}{4}$	$\frac{25}{16}$
$\frac{6}{4}$	$\frac{36}{16}$
$\frac{7}{4}$	$\frac{49}{16}$
2	4

EXAMPLE 1 Applying the Trapezoidal Rule

Use the Trapezoidal Rule with $n = 4$ to estimate $\int_1^2 x^2\,dx$. Compare the estimate with the value of NINT $(x^2, x, 1, 2)$ and with the exact value.

SOLUTION

Partition $[1, 2]$ into four subintervals of equal length (Figure 5.32). Then evaluate $y = x^2$ at each partition point (Table 5.4).

Using these y values, $n = 4$, and $h = (2 - 1)/4 = 1/4$ in the Trapezoidal Rule, we have

$$T = \frac{h}{2}\left(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4\right)$$

$$= \frac{1}{8}\left(1 + 2\left(\frac{25}{16}\right) + 2\left(\frac{36}{16}\right) + 2\left(\frac{49}{16}\right) + 4\right)$$

$$= \frac{75}{32} = 2.34375.$$

The value of NINT $(x^2, x, 1, 2)$ is 2.333333333.

The exact value of the integral is

$$\int_1^2 x^2\,dx = \frac{x^3}{3}\Bigg]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}.$$

The T approximation overestimates the integral by about half a percent of its true value of 7/3. The percentage error is $(2.34375 - 7/3)/(7/3) \approx 0.446\%$. ***Now try Exercise 3.***

We could have predicted that the Trapezoidal Rule would overestimate the integral in Example 1 by considering the geometry of the graph in Figure 5.32. Since the parabola is concave *up,* the approximating segments lie above the curve, giving each trapezoid slightly more area than the corresponding strip under the curve. In Figure 5.31 we see that the straight segments lie *under* the curve on those intervals where the curve is concave *down,* causing the Trapezoidal Rule to *underestimate* the integral on those intervals. The interpretation of "area" changes where the curve lies below the x-axis but it is still the case that the higher y-values give the greater signed area. So we can always say that T overestimates the integral where the graph is concave up and underestimates the integral where the graph is concave down.

EXAMPLE 2 Averaging Temperatures

An observer measures the outside temperature every hour from noon until midnight, recording the temperatures in the following table.

Time	N	1	2	3	4	5	6	7	8	9	10	11	M
Temp	63	65	66	68	70	69	68	68	65	64	62	58	55

What was the average temperature for the 12-hour period?

SOLUTION

We are looking for the average value of a continuous function (temperature) for which we know values at discrete times that are one unit apart. We need to find

$$av(f) = \frac{1}{b-a}\int_a^b f(x)\,dx,$$

without having a formula for $f(x)$. The integral, however, can be approximated by the Trapezoidal Rule, using the temperatures in the table as function values at the points of a 12-subinterval partition of the 12-hour interval (making $h = 1$).

$$T = \frac{h}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{11} + y_{12}\right)$$

$$= \frac{1}{2}\left(63 + 2 \cdot 65 + 2 \cdot 66 + \cdots + 2 \cdot 58 + 55\right)$$

$$= 782$$

Using T to approximate $\int_a^b f(x)\,dx$, we have

$$av(f) \approx \frac{1}{b-a} \cdot T = \frac{1}{12} \cdot 782 \approx 65.17.$$

Rounding to be consistent with the data given, we estimate the average temperature as 65 degrees. *Now try Exercise 7.*

Other Algorithms

LRAM, MRAM, RRAM, and the Trapezoidal Rule all give reasonable approximations to the integral of a continuous function over a closed interval. The Trapezoidal Rule is more efficient, giving a better approximation for small values of n, which makes it a faster algorithm for numerical integration.

Indeed, the only shortcoming of the Trapezoidal Rule seems to be that it depends on approximating curved arcs with straight segments. You might think that an algorithm that approximates the curve with *curved* pieces would be even more efficient (and hence faster for machines), and you would be right. All we need to do is find a geometric figure with a straight base, straight sides, and a curved top that has a known area. You might not know one, but the ancient Greeks did; it is one of the things they knew about parabolas.

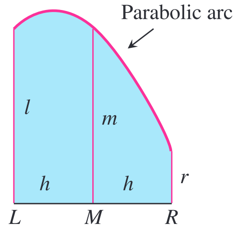


Figure 5.33 The area under the parabolic arc can be computed from the length of the base LR and the lengths of the altitudes constructed at L, R and midpoint M. (Exploration 1)

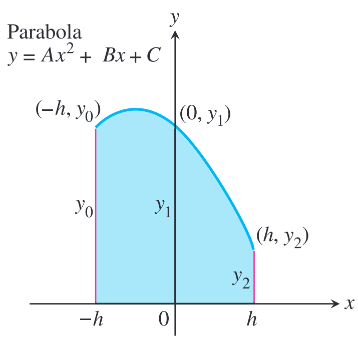


Figure 5.34 A convenient coordinatization of Figure 5.33. The parabola has equation $y = Ax^2 + Bx + C$, and the midpoint of the base is at the origin. (Exploration 1)

EXPLORATION 1 Area Under a Parabolic Arc

The area A_P of a figure having a horizontal base, vertical sides, and a parabolic top (Figure 5.33) can be computed by the formula

$$A_P = \frac{h}{3}\left(l + 4m + r\right),$$

where h is half the length of the base, l and r are the lengths of the left and right sides, and m is the altitude at the midpoint of the base. This formula, once a profound discovery of ancient geometers, is readily verified today with calculus.

1. Coordinatize Figure 5.33 by centering the base at the origin, as shown in Figure 5.34. Let $y = Ax^2 + Bx + C$ be the equation of the parabola. Using this equation, show that $y_0 = Ah^2 - Bh + C$, $y_1 = C$, and $y_2 = Ah^2 + Bh + C$.
2. Show that $y_0 + 4y_1 + y_2 = 2Ah^2 + 6C$.
3. Integrate to show that the area A_P is
$$\frac{h}{3}(2Ah^2 + 6C).$$
4. Combine these results to derive the formula
$$A_P = \frac{h}{3}\left(y_0 + 4y_1 + y_2\right).$$

This last formula leads to an efficient rule for approximating integrals numerically. Partition the interval of integration into an even number of subintervals, apply the formula for A_P to successive interval pairs, and add the results. This algorithm is known as Simpson's Rule.

Simpson's Rule

To approximate $\int_a^b f(x)\,dx$, use

$$S = \frac{h}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right),$$

where $[a, b]$ is partitioned into an *even* number n of subintervals of equal length $h = (b - a)/n$.

What's in a Name?

The formula that underlies Simpson's Rule (see Exploration 1) was discovered long before Thomas Simpson (1720–1761) was born. Just as Pythagoras did not discover the Pythagorean Theorem, Simpson did not discover Simpson's Rule. It is another of history's beautiful quirks that one of the ablest mathematicians of eighteenth-century England is remembered not for his successful textbooks and his contributions to mathematical analysis, but for a rule that was never his, that he never laid claim to, and that bears his name only because he happened to mention it in one of his books.

EXAMPLE 3 Applying Simpson's Rule

Use Simpson's Rule with $n = 4$ to approximate $\int_0^2 5x^4\,dx$.

SOLUTION

Partition $[0, 2]$ into four subintervals and evaluate $y = 5x^4$ at the partition points. (See Table 5.5 on the next page.)

continued

Table 5.5

x	$y = 5x^4$
0	0
$\frac{1}{2}$	$\frac{5}{16}$
1	5
$\frac{3}{2}$	$\frac{405}{16}$
2	80

Then apply Simpson's Rule with $n = 4$ and $h = 1/2$:

$$S = \frac{h}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right)$$

$$= \frac{1}{6}\left(0 + 4\left(\frac{5}{16}\right) + 2\left(5\right) + 4\left(\frac{405}{16}\right) + 80\right)$$

$$= \frac{385}{12}.$$

This estimate differs from the exact value (32) by only 1/12, a percentage error of less than three-tenths of one percent—and this was with just 4 subintervals.

Now try Exercise 17.

There are still other algorithms for approximating definite integrals, most of them involving fancy numerical analysis designed to make the calculations more efficient for high-speed computers. Some are kept secret by the companies that design the machines. In any case, we will not deal with them here.

Error Analysis

After finding that the trapezoidal approximation in Example 1 overestimated the integral, we pointed out that this could have been predicted from the concavity of the curve we were approximating.

Knowing something about the error in an approximation is more than just an interesting sidelight. Despite what your years of classroom experience might have suggested, exact answers are not always easy to find in mathematics. It is fortunate that for all *practical* purposes exact answers are also rarely necessary. (For example, a carpenter who computes the need for a board of length $\sqrt{34}$ feet will happily settle for an approximation when cutting the board.)

Suppose that an exact answer really can *not* be found, but that we know that an approximation within 0.001 unit is good enough. How can we tell that our approximation is within 0.001 if we do not know the exact answer? This is where knowing something about the error is critical.

Since the Trapezoidal Rule approximates curves with straight lines, it seems reasonable that the error depends on how "curvy" the graph is. This suggests that the error depends on the second derivative. It is also apparent that the error depends on the length h of the subintervals. It can be shown that if f'' is continuous the error in the trapezoidal approximation, denoted E_T, satisfies the inequality

$$|E_T| \le \frac{b-a}{12} h^2 M_{f''},$$

where $[a, b]$ is the interval of integration, h is the length of each subinterval, and $M_{f''}$ is the maximum value of $|f''|$ on $[a, b]$.

It can also be shown that the error E_S in Simpson's Rule depends on h and the *fourth* derivative. It satisfies the inequality

$$|E_S| \le \frac{b-a}{180} h^4 M_{f^{(4)}},$$

where $[a, b]$ is the interval of integration, h is the length of each subinterval, and $M_{f^{(4)}}$ is the maximum value of $|f^{(4)}|$ on $[a, b]$, provided that $f^{(4)}$ is continuous.

For comparison's sake, if all the assumptions hold, we have the following *error bounds.*

Error Bounds

If T and S represent the approximations to $\int_a^b f(x)\,dx$ given by the Trapezoidal Rule and Simpson's Rule, respectively, then the errors E_T and E_S satisfy

$$|E_T| \le \frac{b-a}{12} h^2 M_{f''} \quad \text{and} \quad |E_S| \le \frac{b-a}{180} h^4 M_{f^{(4)}}.$$

If we disregard possible differences in magnitude between $M_{f''}$ and $M_{f^{(4)}}$, we notice immediately that $(b-a)/180$ is one-fifteenth the size of $(b-a)/12$, giving S an obvious advantage over T as an approximation. That, however, is almost insignificant when compared to the fact that the trapezoid error varies as the *square* of h, while Simpson's error varies as the *fourth power* of h. (Remember that h is already a small number in most partitions.)

Table 5.6 shows T and S values for approximations of $\int_1^2 1/x\,dx$ using various values of n. Notice how Simpson's Rule dramatically improves over the Trapezoidal Rule. In particular, notice that when we double the value of n (thereby halving the value of h), the T error is divided by 2 *squared,* while the S error is divided by 2 *to the fourth.*

Table 5.6 **Trapezoidal Rule Approximations (T_n) and Simpson's Rule Approximations (S_n) of ln 2 = $\int_1^2 (1/x)\,dx$**

n	T_n	|Error| less than …	S_n	|Error| less than …
10	0.6937714032	0.0006242227	0.6931502307	0.0000030502
20	0.6933033818	0.0001562013	0.6931473747	0.0000001942
30	0.6932166154	0.0000694349	0.6931472190	0.0000000385
40	0.6931862400	0.0000390595	0.6931471927	0.0000000122
50	0.6931721793	0.0000249988	0.6931471856	0.0000000050
100	0.6931534305	0.0000062500	0.6931471809	0.0000000004

This has a dramatic effect as h gets very small. The Simpson approximation for $n = 50$ rounds accurately to seven places, and for $n = 100$ agrees to nine decimal places (billionths)!

We close by showing you the values (Table 5.7) we found for $\int_1^5 (\sin x)/x\,dx$ by six different calculator methods. The exact value of this integral to six decimal places is 0.603848, so both Simpson's method with 50 subintervals and NINT give results accurate to at least six places (millionths).

Table 5.7 **Approximations of $\int_1^5 (\sin x)/x\,dx$**

Method	Subintervals	Value
LRAM	50	0.6453898
RRAM	50	0.5627293
MRAM	50	0.6037425
TRAP	50	0.6040595
SIMP	50	0.6038481
NINT	Tol = 0.00001	0.6038482

Quick Review 5.5 *(For help, go to Sections 3.9 and 4.3.)*

In Exercises 1–10, tell whether the curve is concave up or concave down on the given interval.

1. $y = \cos x$ on $[-1, 1]$

2. $y = x^4 - 12x - 5$ on $[8, 17]$

3. $y = 4x^3 - 3x^2 + 6$ on $[-8, 0]$

4. $y = \sin(x/2)$ on $[48\pi, 50\pi]$

5. $y = e^{2x}$ on $[-5, 5]$

6. $y = \ln x$ on $[100, 200]$

7. $y = \dfrac{1}{x}$ on $[3, 6]$

8. $y = \csc x$ on $[0, \pi]$

9. $y = 10^{10} - 10x^{10}$ on $[10, 10^{10}]$

10. $y = \sin x - \cos x$ on $[1, 2]$

Section 5.5 Exercises

In Exercises 1–6, **(a)** use the Trapezoidal Rule with $n = 4$ to approximate the value of the integral. **(b)** Use the concavity of the function to predict whether the approximation is an overestimate or an underestimate. Finally, **(c)** find the integral's exact value to check your answer.

1. $\int_0^2 x\,dx$

2. $\int_0^2 x^2\,dx$

3. $\int_0^2 x^3\,dx$

4. $\int_1^2 \frac{1}{x}\,dx$

5. $\int_0^4 \sqrt{x}\,dx$

6. $\int_0^{\pi} \sin x\,dx$

7. Use the function values in the following table and the Trapezoidal Rule with $n = 6$ to approximate $\int_0^6 f(x)\,dx$.

x	0	1	2	3	4	5	6
$f(x)$	12	10	9	11	13	16	18

8. Use the function values in the following table and the Trapezoidal Rule with $n = 6$ to approximate $\int_2^8 f(x)\,dx$.

x	2	3	4	5	6	7	8
$f(x)$	16	19	17	14	13	16	20

9. ***Volume of Water in a Swimming Pool*** A rectangular swimming pool is 30 ft wide and 50 ft long. The table below shows the depth $h(x)$ of the water at 5-ft intervals from one end of the pool to the other. Estimate the volume of water in the pool using the Trapezoidal Rule with $n = 10$, applied to the integral

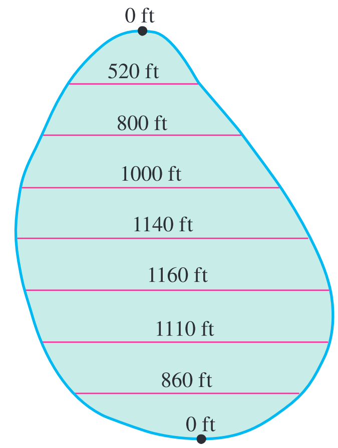
$$V = \int_0^{50} 30 \cdot h(x)\,dx.$$

Position (ft) x	Depth (ft) $h(x)$	Position (ft) x	Depth (ft) $h(x)$
0	6.0	30	11.5
5	8.2	35	11.9
10	9.1	40	12.3
15	9.9	45	12.7
20	10.5	50	13.0
25	11.0		

10. ***Stocking a Fish Pond*** As the fish and game warden of your township, you are responsible for stocking the town pond with fish before the fishing season. The average depth of the pond is 20 feet. Using a scaled map, you measure distances across the pond at 200-foot intervals, as shown in the diagram.

(a) Use the Trapezoidal Rule to estimate the volume of the pond.

(b) You plan to start the season with one fish per 1000 cubic feet. You intend to have at least 25% of the opening day's fish population left at the end of the season. What is the maximum number of licenses the town can sell if the average seasonal catch is 20 fish per license?



Vertical spacing = 200 ft

11. ***Audi S4 Quattro Cabriolet*** The accompanying table shows time-to-speed data for a 2004 Audi S4 Quattro Cabriolet accelerating from rest to 130 mph. How far had the Cabriolet traveled by the time it reached this speed? (Use trapezoids to estimate the area under the velocity curve, but be careful: the time intervals vary in length.)

Speed Change: Zero to	Time (sec)
30 mph	2.0
40 mph	3.2
50 mph	4.5
60 mph	5.8
70 mph	7.7
80 mph	9.5
90 mph	11.6
100 mph	14.9
110 mph	17.8
120 mph	21.7
130 mph	26.3

Source: Car and Driver, July 2004.

12. The table below records the velocity of a bobsled at 1-second intervals for the first eight seconds of its run. Use the Trapezoidal Rule to approximate the distance the bobsled travels during that 8-second interval. (Give your final answer in feet.)

Time (Seconds)	Speed (Miles/hr)
0	0
1	3
2	7
3	12
4	17
5	25
6	33
7	41
8	48

In Exercises 13–18, **(a)** use Simpson's Rule with $n = 4$ to approximate the value of the integral and **(b)** find the exact value of the integral to check your answer. (Note that these are the same integrals as Exercises 1–6, so you can also compare it with the Trapezoidal Rule approximation.)

13. $\int_0^2 x\,dx$ **14.** $\int_0^2 x^2\,dx$

15. $\int_0^2 x^3\,dx$ **16.** $\int_1^2 \frac{1}{x}\,dx$

17. $\int_0^4 \sqrt{x}\,dx$ **18.** $\int_0^{\pi} \sin x\,dx$

19. Consider the integral $\int_{-1}^{3} (x^3 - 2x)\,dx$.

(a) Use Simpson's Rule with $n = 4$ to approximate its value.

(b) Find the exact value of the integral. What is the error, $|E_S|$?

(c) Explain how you could have predicted what you found in (b) from knowing the error-bound formula.

(d) Writing to Learn Is it possible to make a general statement about using Simpson's Rule to approximate integrals of cubic polynomials? Explain.

20. Writing to Learn In Example 2 (before rounding) we found the average temperature to be 65.17 degrees when we used the integral approximation, yet the average of the 13 discrete temperatures is only 64.69 degrees. Considering the shape of the temperature curve, explain why you would expect the average of the 13 discrete temperatures to be less than the average value of the temperature function on the entire interval.

21. *(Continuation of Exercise 20)*

(a) In the Trapezoidal Rule, every function value in the sum is doubled except for the two endpoint values. Show that if you double the endpoint values, you get 70.08 for the average temperature.

(b) Explain why it makes more sense to not double the endpoint values if we are interested in the average temperature over the entire 12-hour period.

22. Group Activity For most functions, Simpson's Rule gives a better approximation to an integral than the Trapezoidal Rule for a given value of n. Sketch the graph of a function on a closed interval for which the Trapezoidal Rule obviously gives a better approximation than Simpson's Rule for $n = 4$.

In Exercises 23–26, use a calculator program to find the Simpson's Rule approximations with $n = 50$ and $n = 100$.

23. $\int_{-1}^{1} 2\sqrt{1 - x^2}\,dx$

24. $\int_0^1 \sqrt{1 + x^4}\,dx$

25. $\int_0^{\pi/2} \frac{\sin x}{x}\,dx$

26. $\int_0^{\pi/2} \sin(x^2)\,dx$

27. Consider the integral $\int_0^{\pi} \sin x\,dx$.

(a) Use a calculator program to find the Trapezoidal Rule approximations for $n = 10$, 100, and 1000.

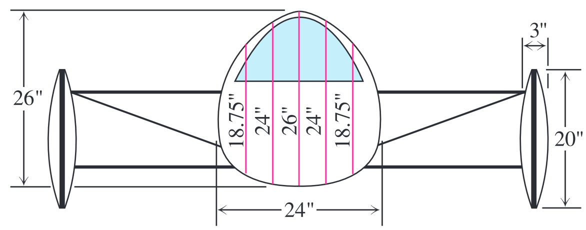
(b) Record the errors with as many decimal places of accuracy as you can.

(c) What pattern do you see?

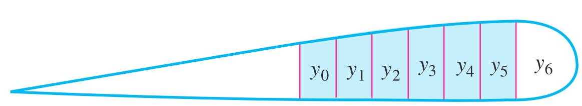
(d) Writing to Learn Explain how the error bound for E_T accounts for the pattern.

28. *(Continuation of Exercise 27)* Repeat Exercise 27 with Simpson's Rule and E_S.

29. *Aerodynamic Drag* A vehicle's aerodynamic drag is determined in part by its cross section area, so, all other things being equal, engineers try to make this area as small as possible. Use Simpson's Rule to estimate the cross section area of the body of James Worden's solar-powered Solectria® automobile at M.I.T. from the diagram below.



30. *Wing Design* The design of a new airplane requires a gasoline tank of constant cross section area in each wing. A scale drawing of a cross section is shown here. The tank must hold 5000 lb of gasoline, which has a density of 42 lb/ft^3. Estimate the length of the tank.



$y_0 = 1.5$ ft, $y_1 = 1.6$ ft, $y_2 = 1.8$ ft, $y_3 = 1.9$ ft,
$y_4 = 2.0$ ft, $y_5 = y_6 = 2.1$ ft Horizontal spacing = 1 ft

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

31. True or False The Trapezoidal Rule will underestimate $\int_a^b f(x)\,dx$ if the graph of f is concave up on $[a, b]$. Justify your answer.

32. True or False For a given value of n, the Trapezoidal Rule with n subdivisions will always give a more accurate estimate of $\int_a^b f(x)\,dx$ than a right Riemann sum with n subdivisions. Justify your answer.

33. Multiple Choice Using 8 equal subdivisions of the interval [2, 12], the LRAM approximation of $\int_2^{12} f(x)\,dx$ is 16.6 and the trapezoidal approximation is 16.4. What is the RRAM approximation?

(A) 16.2 **(B)** 16.5

(C) 16.6 **(D)** 16.8

(E) It cannot be determined from the given information.

34. Multiple Choice If three equal subdivisions of [−2, 4] are used, what is the trapezoidal approximation of $\int_{-2}^{4} \frac{e^x}{2}\,dx$?

(A) $e^4 + e^2 + e^0 + e^{-2}$

(B) $e^4 + 2e^2 + 2e^0 + e^{-2}$

(C) $\frac{1}{2}(e^4 + e^2 + e^0 + e^{-2})$

(D) $\frac{1}{2}(e^4 + 2e^2 + 2e^0 + e^{-2})$

(E) $\frac{1}{4}(e^4 + 2e^2 + 2e^0 + e^{-2})$

35. Multiple Choice The trapezoidal approximation of $\int_0^\pi \sin x\,dx$ using 4 equal subdivisions of the interval of integration is

(A) $\frac{\pi}{2}$

(B) π

(C) $\frac{\pi}{4}(1 + \sqrt{2})$

(D) $\frac{\pi}{2}(1 + \sqrt{2})$

(E) $\frac{\pi}{4}(2 + \sqrt{2})$

36. Multiple Choice Suppose f, f', and f'' are all positive on the interval $[a, b]$, and suppose we compute LRAM, RRAM, and trapezoidal approximations of $I = \int_a^b f(x)\,dx$ using the same number of equal subdivisions of $[a, b]$. If we denote the three approximations of I as L, R, and T respectively, which of the following is true?

(A) $R < T < I < L$ **(B)** $R < I < T < L$ **(C)** $L < I < T < R$

(D) $L < T < I < R$ **(E)** $L < I < R < T$

Explorations

37. Consider the integral $\int_{-1}^{1} \sin(x^2)\,dx$.

(a) Find f'' for $f(x) = \sin(x^2)$.

(b) Graph $y = f''(x)$ in the viewing window $[-1, 1]$ by $[-3, 3]$.

(c) Explain why the graph in part (b) suggests that $|f''(x)| \le 3$ for $-1 \le x \le 1$.

(d) Show that the error estimate for the Trapezoidal Rule in this case becomes

$$|E_T| \le \frac{h^2}{2}.$$

(e) Show that the Trapezoidal Rule error will be less than or equal to 0.01 if $h \le 0.1$.

(f) How large must n be for $h \le 0.1$?

38. Consider the integral $\int_{-1}^{1} \sin(x^2)\,dx$.

(a) Find $f^{(4)}$ for $f(x) = \sin(x^2)$. (You may want to check your work with a CAS if you have one available.)

(b) Graph $y = f^{(4)}(x)$ in the viewing window $[-1, 1]$ by $[-30, 10]$.

(c) Explain why the graph in part (b) suggests that $|f^{(4)}(x)| \le 30$ for $-1 \le x \le 1$.

(d) Show that the error estimate for Simpson's Rule in this case becomes

$$|E_S| \le \frac{h^4}{3}.$$

(e) Show that the Simpson's Rule error will be less than or equal to 0.01 if $h \le 0.4$.

(f) How large must n be for $h \le 0.4$?

Extending the Ideas

39. Using the definitions, prove that, in general,

$$T_n = \frac{\text{LRAM}_n + \text{RRAM}_n}{2}.$$

40. Using the definitions, prove that, in general,

$$S_{2n} = \frac{\text{MRAM}_n + 2T_{2n}}{3}.$$

Quick Quiz for AP* Preparation: Sections 5.4 and 5.5

You may use a graphing calculator to solve the following problems.

1. **Multiple Choice** The function f is continuous on the closed interval [1, 7] and has values that are given in the table below.

x	1	4	6	7
$f(x)$	10	30	40	20

Using the subintervals [1, 4], [4, 6], and [6, 7], what is the trapezoidal approximation of $\int_1^7 f(x)\,dx$?

(A) 110 **(B)** 130 **(C)** 160 **(D)** 190 **(E)** 210

2. **Multiple Choice** Let $F(x)$ be an antiderivative of $\sin^3 x$. If $F(1) = 0$, then $F(8) =$

(A) 0.00 **(B)** 0.021 **(C)** 0.373 **(D)** 0.632 **(E)** 0.968

3. **Multiple Choice** Let $f(x) = \int_{-2}^{x^2-3x} e^{t^2}dt$. At what value of x is $f(x)$ a minimum?

(A) For no value of x **(B)** $\frac{1}{2}$ **(C)** $\frac{3}{2}$ **(D)** 2 **(E)** 3

4. **Free Response** Let $F(x) = \int_0^x \sin(t^2)\,dt$ for $0 \le x \le 3$.

(a) Use the Trapezoidal Rule with four equal subdivisions of the closed interval [0, 2] to approximate $F(2)$.

(b) On what interval or intervals is F increasing? Justify your answer.

(c) If the average rate of change of F on the closed interval [0, 3] is k, find $\int_0^3 \sin(t^2)\,dt$ in terms of k.

Chapter 5 Key Terms

area under a curve (p. 263)
average value (p. 287)
bounded function (p. 281)
cardiac output (p. 268)
characteristic function of the rationals (p. 282)
definite integral (p. 276)
differential calculus (p. 263)
dummy variable (p. 277)
error bounds (p. 311)
Fundamental Theorem of Calculus (p. 294)
integrable function (p. 276)
integral calculus (p. 263)
Integral Evaluation Theorem (p. 299)
integral of f from a to b (p. 276)
integral sign (p. 277)
integrand (p. 277)
lower bound (p. 286)
lower limit of integration (p. 277)
LRAM (p. 265)
mean value (p. 287)
Mean Value Theorem for Definite Integrals (p. 288)
MRAM (p. 265)
net area (p. 279)
NINT (p. 281)
norm of a partition (p. 275)
partition (p. 274)
Rectangular Approximation Method (RAM) (p. 265)
regular partition (p. 276)
Riemann sum (p. 274)
RRAM (p. 265)
sigma notation (p. 274)
Simpson's Rule (p. 309)
subinterval (p. 275)
total area (p. 300)
Trapezoidal Rule (p. 307)
upper bound (p. 286)
upper limit of integration (p. 277)
variable of integration (p. 277)

Chapter 5 Review Exercises

The collection of exercises marked in red could be used as a chapter test.

Exercises 1–6 refer to the region R in the first quadrant enclosed by the x-axis and the graph of the function $y = 4x - x^3$.

1. Sketch R and partition it into four subregions, each with a base of length $\Delta x = 1/2$.

2. Sketch the rectangles and compute (by hand) the area for the $LRAM_4$ approximation.

3. Sketch the rectangles and compute (by hand) the area for the $MRAM_4$ approximation.

4. Sketch the rectangles and compute (by hand) the area for the $RRAM_4$ approximation.

5. Sketch the trapezoids and compute (by hand) the area for the T_4 approximation.

6. Find the exact area of R by using the Fundamental Theorem of Calculus.

7. Use a calculator program to compute the RAM approximations in the following table for the area under the graph of $y = 1/x$ from $x = 1$ to $x = 5$.

n	$LRAM_n$	$MRAM_n$	$RRAM_n$
10			
20			
30			
50			
100			
1000			

8. *(Continuation of Exercise 7)* Use the Fundamental Theorem of Calculus to determine the value to which the sums in the table are converging.

9. Suppose

$$\int_{-2}^{2} f(x)\,dx = 4, \quad \int_{2}^{5} f(x)\,dx = 3, \quad \int_{-2}^{5} g(x)\,dx = 2.$$

Which of the following statements are true, and which, if any, are false?

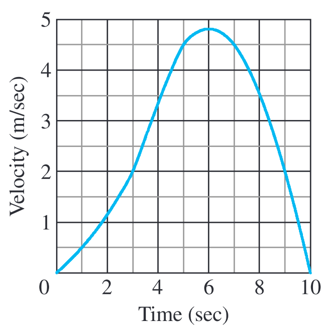
(a) $\int_{5}^{2} f(x)\,dx = -3$

(b) $\int_{-2}^{5} [f(x) + g(x)]\,dx = 9$

(c) $f(x) \le g(x)$ on the interval $-2 \le x \le 5$

10. The region under one arch of the curve $y = \sin x$ is revolved around the x-axis to form a solid. **(a)** Use the method of Example 3, Section 5.1, to set up a Riemann sum that approximates the volume of the solid. **(b)** Find the volume using NINT.

11. The accompanying graph shows the velocity (m/sec) of a body moving along the s-axis during the time interval from $t = 0$ to $t = 10$ sec. **(a)** About how far did the body travel during those 10 seconds?



(b) Sketch a graph of position (s) as a function of time (t) for $0 \le t \le 10$, assuming $s(0) = 0$.

12. The interval $[0, 10]$ is partitioned into n subintervals of length $\Delta x = 10/n$. We form the following Riemann sums, choosing each c_k in the k^{th} subinterval. Write the limit as $n \to \infty$ of each Riemann sum as a definite integral.

(a) $\sum_{k=1}^{n} (c_k)^3 \Delta x$

(b) $\sum_{k=1}^{n} c_k (\sin c_k) \Delta x$

(c) $\sum_{k=1}^{n} c_k (3c_k - 2)^2 \Delta x$

(d) $\sum_{k=1}^{n} (1 + c_k{}^2)^{-1} \Delta x$

(e) $\sum_{k=1}^{n} \pi (9 - \sin^2(\pi c_k/10)) \Delta x$

In Exercises 13 and 14, find the total area between the curve and the x-axis.

13. $y = 4 - x, \quad 0 \le x \le 6$

14. $y = \cos x, \quad 0 \le x \le \pi$

In Exercises 15–24, evaluate the integral analytically by using the Integral Evaluation Theorem (Part 2 of the Fundamental Theorem, Theorem 4).

15. $\int_{-2}^{2} 5\,dx$

16. $\int_{2}^{5} 4x\,dx$

17. $\int_{0}^{\pi/4} \cos x\,dx$

18. $\int_{-1}^{1} (3x^2 - 4x + 7)\,dx$

19. $\int_{0}^{1} (8s^3 - 12s^2 + 5)\,ds$

20. $\int_{1}^{2} \frac{4}{x^2}\,dx$

21. $\int_{1}^{27} y^{-4/3}\,dy$

22. $\int_{1}^{4} \frac{dt}{t\sqrt{t}}$

23. $\int_{0}^{\pi/3} \sec^2\theta\,d\theta$

24. $\int_{1}^{e} (1/x)\,dx$

In Exercises 25–29, evaluate the integral.

25. $\int_{0}^{1} \frac{36}{(2x+1)^3}\,dx$

26. $\int_{1}^{2} \left(x + \frac{1}{x^2}\right) dx$

27. $\int_{-\pi/3}^{0} \sec x \tan x\,dx$

28. $\int_{-1}^{1} 2x \sin(1 - x^2)\,dx$

29. $\int_{0}^{2} \frac{2}{y+1}\,dy$

In Exercises 30–32, evaluate the integral by interpreting it as area and using formulas from geometry.

30. $\int_{0}^{2} \sqrt{4 - x^2}\,dx$

31. $\int_{-4}^{8} |x|\,dx$

32. $\int_{-8}^{8} 2\sqrt{64 - x^2}\,dx$

33. *Oil Consumption on Pathfinder Island* A diesel generator runs continuously, consuming oil at a gradually increasing rate until it must be temporarily shut down to have the filters replaced.

Day	Oil Consumption Rate (liters/hour)
Sun	0.019
Mon	0.020
Tue	0.021
Wed	0.023
Thu	0.025
Fri	0.028
Sat	0.031
Sun	0.035

(a) Give an upper estimate and a lower estimate for the amount of oil consumed by the generator during that week.

(b) Use the Trapezoidal Rule to estimate the amount of oil consumed by the generator during that week.

34. *Rubber-Band–Powered Sled* A sled powered by a wound rubber band moves along a track until friction and the unwinding of the rubber band gradually slow it to a stop. A speedometer in the sled monitors its speed, which is recorded at 3-second intervals during the 27-second run.

Time (sec)	Speed (ft/sec)
0	5.30
3	5.25
6	5.04
9	4.71
12	4.25
15	3.66
18	2.94
21	2.09
24	1.11
27	0

(a) Give an upper estimate and a lower estimate for the distance traveled by the sled.

(b) Use the Trapezoidal Rule to estimate the distance traveled by the sled.

35. Writing to Learn Your friend knows how to compute integrals but never could understand what difference the "dx" makes, claiming that it is irrelevant. How would you explain to your friend why it is necessary?

36. The function

$$f(x) = \begin{cases} x^2, & x \geq 0 \\ x - 2, & x < 0 \end{cases}$$

is discontinuous at 0, but integrable on $[-4, 4]$. Find $\int_{-4}^{4} f(x)\,dx$.

37. Show that $0 \leq \int_0^1 \sqrt{1 + \sin^2 x}\,dx \leq \sqrt{2}$.

38. Find the average value of

(a) $y = \sqrt{x}$ over the interval $[0, 4]$.

(b) $y = a\sqrt{x}$ over the interval $[0, a]$.

In Exercises 39–42, find dy/dx.

39. $y = \displaystyle\int_2^x \sqrt{2 + \cos^3 t}\,dt$

40. $y = \displaystyle\int_2^{7x^2} \sqrt{2 + \cos^3 t}\,dt$

41. $y = \displaystyle\int_x^1 \frac{6}{3 + t^4}\,dt$

42. $y = \displaystyle\int_x^{2x} \frac{1}{t^2 + 1}\,dt$

43. *Printing Costs* Including start-up costs, it costs a printer \$50 to print 25 copies of a newsletter, after which the marginal cost at x copies is

$$\frac{dc}{dx} = \frac{2}{\sqrt{x}} \text{ dollars per copy.}$$

Find the total cost of printing 2500 newsletters.

44. *Average Daily Inventory* Rich Wholesale Foods, a manufacturer of cookies, stores its cases of cookies in an air-conditioned warehouse for shipment every 14 days. Rich tries to keep 600 cases on reserve to meet occasional peaks in demand, so a typical 14-day inventory function is $I(t) = 600 + 600t$, $0 \leq t \leq 14$. The holding cost for each case is 4¢ per day. Find Rich's average daily inventory and average daily holding cost (that is, the average of $I(x)$ for the 14-day period, and this average multiplied by the holding cost).

45. Solve for x: $\int_0^x (t^3 - 2t + 3)\,dt = 4$.

46. Suppose $f(x)$ has a positive derivative for all values of x and that $f(1) = 0$. Which of the following statements must be true of

$$g(x) = \int_0^x f(t)\,dt?$$

(a) g is a differentiable function of x.

(b) g is a continuous function of x.

(c) The graph of g has a horizontal tangent line at $x = 1$.

(d) g has a local maximum at $x = 1$.

(e) g has a local minimum at $x = 1$.

(f) The graph of g has an inflection point at $x = 1$.

(g) The graph of dg/dx crosses the x-axis at $x = 1$.

47. Suppose $F(x)$ is an antiderivative of $f(x) = \sqrt{1 + x^4}$. Express $\int_0^1 \sqrt{1 + x^4}\,dx$ in terms of F.

48. Express the function $y(x)$ with

$$\frac{dy}{dx} = \frac{\sin x}{x} \quad \text{and} \quad y(5) = 3$$

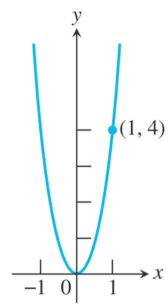
as a definite integral.

49. Show that $y = x^2 + \int_1^x 1/t\,dt + 1$ satisfies both of the following conditions:

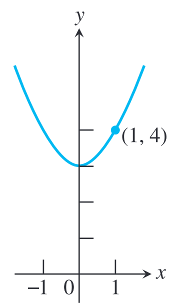
i. $y'' = 2 - \dfrac{1}{x^2}$

ii. $y = 2$ and $y' = 3$ when $x = 1$.

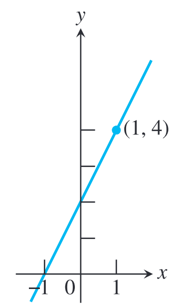
50. Writing to Learn Which of the following is the graph of the function whose derivative is $dy/dx = 2x$ and whose value at $x = 1$ is 4? Explain your answer.



(a)



(b)



(c)

51. *Fuel Efficiency* An automobile computer gives a digital readout of fuel consumption in gallons per hour. During a trip, a passenger recorded the fuel consumption every 5 minutes for a full hour of travel.

time	gal/h	time	gal/h
0	2.5	35	2.5
5	2.4	40	2.4
10	2.3	45	2.3
15	2.4	50	2.4
20	2.4	55	2.4
25	2.5	60	2.3
30	2.6		

(a) Use the Trapezoidal Rule to approximate the total fuel consumption during the hour.

(b) If the automobile covered 60 miles in the hour, what was its fuel efficiency (in miles per gallon) for that portion of the trip?

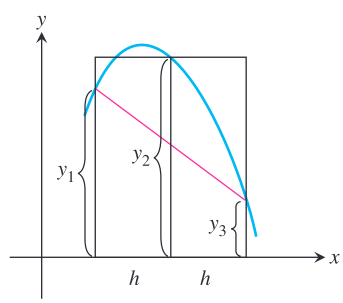
52. ***Skydiving*** Skydivers A and B are in a helicopter hovering at 6400 feet. Skydiver A jumps and descends for 4 sec before opening her parachute. The helicopter then climbs to 7000 feet and hovers there. Forty-five seconds after A leaves the aircraft, B jumps and descends for 13 sec before opening her parachute. Both skydivers descend at 16 ft/sec with parachutes open. Assume that the skydivers fall freely (with acceleration -32 ft/sec^2) before their parachutes open.

(a) At what altitude does A's parachute open?

(b) At what altitude does B's parachute open?

(c) Which skydiver lands first?

53. ***Relating Simpson's Rule, MRAM, and T*** The figure below shows an interval of length $2h$ with a trapezoid, a midpoint rectangle, and a parabolic region on it.



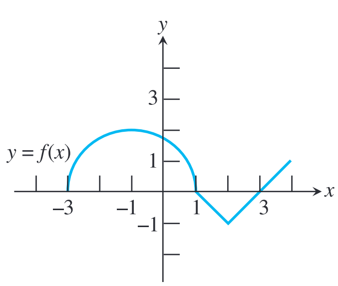
(a) Show that the area of the trapezoid plus twice the area of the rectangle equals

$$h(y_1 + 4y_2 + y_3).$$

(b) Use the result in part (a) to prove that

$$S_{2n} = \frac{2 \cdot \text{MRAM}_n + T_n}{3}.$$

54. The graph of a function f consists of a semicircle and two line segments as shown below.



Let $g(x) = \int_1^x f(t)\,dt$.

(a) Find $g(1)$.

(b) Find $g(3)$.

(c) Find $g(-1)$.

(d) Find all values of x on the open interval $(-3, 4)$ at which g has a relative maximum.

(e) Write an equation for the line tangent to the graph of g at $x = -1$.

(f) Find the x-coordinate of each point of inflection of the graph of g on the open interval $(-3, 4)$.

(g) Find the range of g.

55. What is the total area under the curve $y = e^{-x^2/2}$?

The graph approaches the x-axis as an asymptote both to the left and the right, but quickly enough so that the total area is a finite number. In fact,

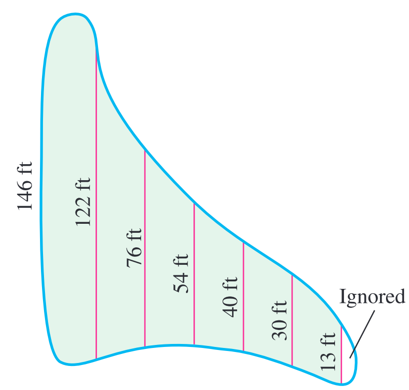
$$\text{NINT}\,(e^{-x^2/2}, x, -10, 10)$$

computes all but a negligible amount of the area.

(a) Find this number on your calculator. Verify that NINT $(e^{-x^2/2}, x, -20, 20)$ does not increase the number enough for the calculator to distinguish the difference.

(b) This area has an interesting relationship to π. Perform various (simple) algebraic operations on the number to discover what it is.

56. ***Filling a Swamp*** A town wants to drain and fill the small polluted swamp shown below. The swamp averages 5 ft deep. About how many cubic yards of dirt will it take to fill the area after the swamp is drained?



Horizontal spacing = 20 ft

57. ***Household Electricity*** We model the voltage V in our homes with the sine function

$$V = V_{\max} \sin(120\,\pi t),$$

which expresses V in volts as a function of time t in seconds. The function runs through 60 cycles each second. The number $V_{\max}$ is the *peak voltage*.

To measure the voltage effectively, we use an instrument that measures the square root of the average value of the square of the voltage over a 1-second interval:

$$V_{\text{rms}} = \sqrt{(V^2)_{\text{av}}}.$$

The subscript "rms" stands for "root mean square." It turns out that

$$V_{\text{rms}} = \frac{V_{\max}}{\sqrt{2}}. \qquad (1)$$

The familiar phrase "115 volts ac" means that the rms voltage is 115. The peak voltage, obtained from Equation 1 as $V_{\max} = 115\sqrt{2}$, is about 163 volts.

(a) Find the average value of V^2 over a 1-sec interval. Then find V_{rms}, and verify Equation 1.

(b) The circuit that runs your electric stove is rated 240 volts rms. What is the peak value of the allowable voltage?

AP *Examination Preparation

You may use a graphing calculator to solve the following problems.

58. The rate at which water flows out of a pipe is given by a differentiable function R of time t. The table below records the rate at 4-hour intervals for a 24-hour period.

t (hours)	$R(t)$ (gallons per hour)
0	9.6
4	10.3
8	10.9
12	11.1
16	10.9
20	10.5
24	9.6

(a) Use the Trapezoid Rule with 6 subdivisions of equal length to approximate $\int_0^{24} R(t)dt$. Explain the meaning of your answer in terms of water flow, using correct units.

(b) Is there some time t between 0 and 24 such that $R'(t) = 0$? Justify your answer.

(c) Suppose the rate of water flow is approximated by $Q(t) = 0.01(950 + 25x - x^2)$. Use $Q(t)$ to approximate the average rate of water flow during the 24-hour period. Indicate units of measure.

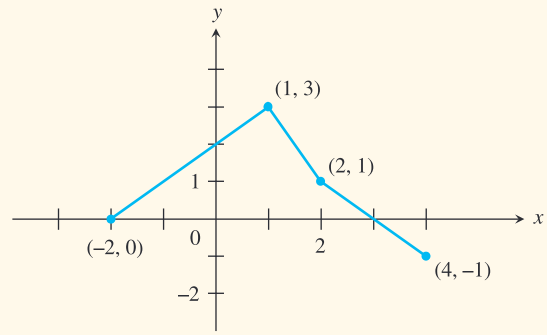
59. Let f be a differentiable function with the following properties.

i. $f'(x) = ax^2 + bx$ **ii.** $f'(1) = -6$ and $f''(x) = 6$

iii. $\int_1^2 f(x)\,dx = 14$

Find $f(x)$. Show your work.

60. The graph of the function f, consisting of three line segments, is shown below.



Let $g(x) = \int_1^x f(t)dt$.

(a) Compute $g(4)$ and $g(-2)$.

(b) Find the instantaneous rate of change of g, with respect to x, at $x = 2$.

(c) Find the absolute minimum value of g on the closed interval $[-2, 4]$. Justify your answer.

(d) The second derivative of g is not defined at $x = 1$ and $x = 2$. Which of these values are x-coordinates of points of inflection of the graph of g? Justify your answer.

Calculus at Work

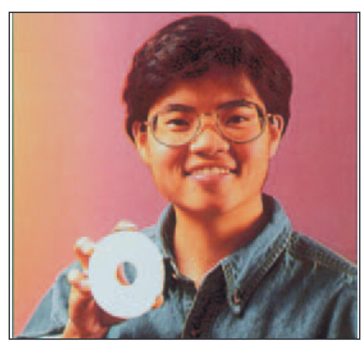
I have a degree in Mechanical Engineering with a minor in Psychology. I am a Research and Development Engineer at Komag, which designs and manufactures hard disks in Santa Clara, California. My job is to test the durability and reliability of the disks, measuring the rest friction between the read/write heads and the disk surface, which is called "Contact-Start-Stop" testing.

I use calculus to evaluate the moment of inertia of different disk stacks, which consist of disks on a spindle, separated by spacer rings. Because the rings vary in size as well as material, the mass of each ring must be determined. For such problems, I refer to my college calculus textbook and its tables of summations and integrals. For instance, I use:

Moment of Inertia =

$$\left[\sum_{i=1}^{n} M_i L_i^2\right] + \frac{1}{3} M_{rod} L_{rod}^2$$

where i = components 1 to n;
M_i = mass of component i such as the disk and/or ring stack;
L_i = distance of component i from a reference point;
M_{rod} = mass of the spindle that rotates;
L_{rod} = length of the spindle.



Andrea Woo

Chapter 6

Differential Equations and Mathematical Modeling

One way to measure how light in the ocean diminishes as water depth increases involves using a Secchi disk. This white disk is 30 centimeters in diameter, and is lowered into the ocean until it disappears from view. The depth of this point (in meters), divided into 1.7, yields the coefficient k used in the equation $I_x = I_0 e^{-kx}$. This equation estimates the intensity I_x of light at depth x using I_0, the intensity of light at the surface.

In an ocean experiment, if the Secchi disk disappears at 55 meters, at what depth will only 1% of surface radiation remain? Section 6.4 will help you answer this question.

Chapter 6 Overview

One of the early accomplishments of calculus was predicting the future position of a planet from its present position and velocity. Today this is just one of a number of occasions on which we deduce everything we need to know about a function from one of its known values and its rate of change. From this kind of information, we can tell how long a sample of radioactive polonium will last; whether, given current trends, a population will grow or become extinct; and how large major league baseball salaries are likely to be in the year 2010. In this chapter, we examine the analytic, graphical, and numerical techniques on which such predictions are based.

6.1 Slope Fields and Euler's Method

What you'll learn about

- Differential Equations
- Slope Fields
- Euler's Method

. . . and why

Differential equations have always been a prime motivation for the study of calculus and remain so to this day.

Differential Equations

We have already seen how the discovery of calculus enabled mathematicians to solve problems that had befuddled them for centuries because the problems involved moving objects. Leibniz and Newton were able to model these problems of motion by using equations involving derivatives—what we call *differential equations* today, after the notation of Leibniz. Much energy and creativity has been spent over the years on techniques for solving such equations, which continue to arise in all areas of applied mathematics.

DEFINITION Differential Equation

An equation involving a derivative is called a **differential equation.** The **order of a differential equation** is the order of the highest derivative involved in the equation.

EXAMPLE 1 Solving a Differential Equation

Find all functions y that satisfy $dy/dx = \sec^2 x + 2x + 5$.

SOLUTION

We first encountered this sort of differential equation (called *exact* because it gives the derivative exactly) in Chapter 4. The solution can be any antiderivative of $\sec^2 x + 2x + 5$, which can be any function of the form $y = \tan x + x^2 + 5x + C$. That family of functions is the *general* solution to the differential equation. ***Now try Exercise 1.***

Notice that we cannot find a unique solution to a differential equation unless we are given further information. If the general solution to a first-order differential equation is continuous, the only additional information needed is the value of the function at a single point, called an *initial condition*. A differential equation with an initial condition is called an *initial value problem*. It has a unique solution, called the *particular solution* to the differential equation.

EXAMPLE 2 Solving an Initial Value Problem

Find the particular solution to the equation $dy/dx = e^x - 6x^2$ whose graph passes through the point $(1, 0)$.

SOLUTION

The general solution is $y = e^x - 2x^3 + C$. Applying the initial condition, we have $0 = e - 2 + C$, from which we conclude that $C = 2 - e$. Therefore, the particular solution is $y = e^x - 2x^3 + 2 - e$. ***Now try Exercise 13.***

An initial condition determines a particular solution by requiring that a solution curve pass through a given point. If the curve is continuous, this pins down the solution on the entire domain. If the curve is discontinuous, the initial condition only pins down the continuous *piece of the curve* that passes through the given point. In this case, the domain of the solution must be specified.

EXAMPLE 3 Handling Discontinuity in an Initial Value Problem

Find the particular solution to the equation $dy/dx = 2x - \sec^2 x$ whose graph passes through the point (0, 3).

SOLUTION

The general solution is $y = x^2 - \tan x + C$. Applying the initial condition, we have $3 = 0 - 0 + C$, from which we conclude that $C = 3$. Therefore, the particular solution is $y = x^2 - \tan x + 3$. Since the point (0, 3) only pins down the continuous piece of the general solution over the interval $(-\pi/2, \pi/2)$, we add the domain stipulation $-\pi/2 < x < \pi/2$.

Now try Exercise 15.

Sometimes we are unable to find an antiderivative to solve an initial value problem, but we can still find a solution using the Fundamental Theorem of Calculus.

EXAMPLE 4 Using the Fundamental Theorem to Solve an Initial Value Problem

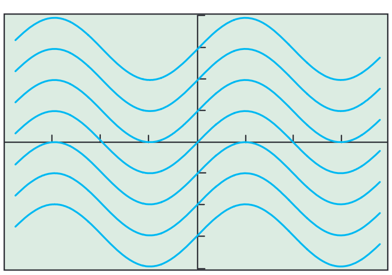
Find the solution to the differential equation $f'(x) = e^{-x^2}$ for which $f(7) = 3$.

SOLUTION

This almost seems too simple, but $f(x) = \int_7^x e^{-t^2}\,dt + 3$ has both of the necessary properties! Clearly, $f(7) = \int_7^7 e^{-t^2}\,dt + 3 = 0 + 3 = 3$, and $f'(x) = e^{x^2}$ by the Fundamental Theorem.

The integral form of the solution in Example 4 might seem less desirable than the explicit form of the solutions in Examples 2 and 3, but (thanks to modern technology) it does enable us to find $f(x)$ for any x. For example, $f(-2) = \int_7^{-2} e^{-t^2}\,dt + 3 =$ fnInt $(e\text{^}(-t^2), t, 7, -2) + 3 \approx 1.2317$.

Now try Exercise 21.



[−2π, 2π] by [−4, 4]

Figure 6.1 A graph of the family of functions $Y_1 = \sin(x) + L_1$, where $L_1 = \{-3, -2, -1, 0, 1, 2, 3\}$. This graph shows some of the functions that satisfy the differential equation $dy/dx = \cos x$. (Example 5)

EXAMPLE 5 Graphing a General Solution

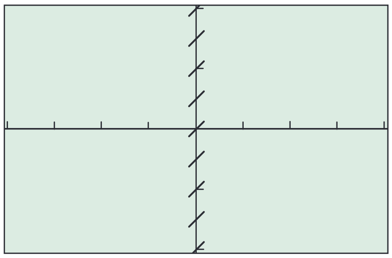
Graph the family of functions that solve the differential equation $dy/dx = \cos x$.

SOLUTION

Any function of the form $y = \sin x + C$ solves the differential equation. We cannot graph them all, but we can graph enough of them to see what a family of solutions would look like. The command $\{-3, -2, -1, 0, 1, 2, 3,\} \to L_1$ stores seven values of C in the list L_1. Figure 6.1 shows the result of graphing the function $Y_1 = \sin(x) + L_1$.

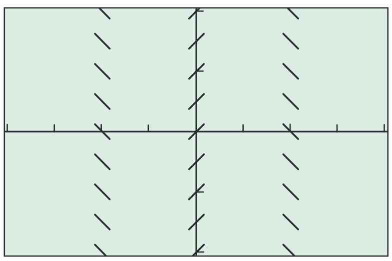
Now try Exercises 25–28.

Notice that the graph in Figure 6.1 consists of a family of parallel curves. This should come as no surprise, since functions of the form $\sin(x) + C$ are all vertical translations of the basic sine curve. It might be less obvious that we could have predicted the appearance of this family of curves from *the differential equation itself*. Exploration 1 gives you a new way to look at the solution graph.



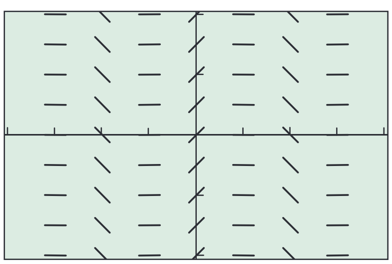
[–2π, 2π] by [–4, 4]

(a)



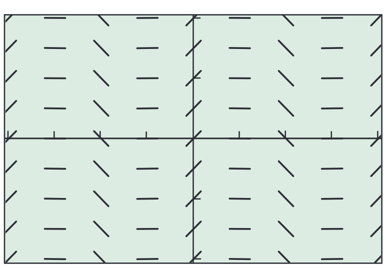
[–2π, 2π] by [–4, 4]

(b)



[–2π, 2π] by [–4, 4]

(c)



[–2π, 2π] by [–4, 4]

(d)

Figure 6.2 The steps in constructing a slope field for the differential equation $dy/dx = \cos x$. (Example 6)

EXPLORATION 1 **Seeing the Slopes**

Figure 6.1 shows the general solution to the exact differential equation $dy/dx = \cos x$.

1. Since $\cos x = 0$ at odd multiples of $\pi/2$, we should "see" that $dy/dx = 0$ at the odd multiples of $\pi/2$ in Figure 6.1. Is that true? How can you tell?
2. Algebraically, the y-coordinate does not affect the value of $dy/dx = \cos x$. Why not?
3. Does the graph show that the y-coordinate does not affect the value of dy/dx? How can you tell?
4. According to the differential equation $dy/dx = \cos x$, what should be the slope of the solution curves when $x = 0$? Can you see this in the graph?
5. According to the differential equation $dy/dx = \cos x$, what should be the slope of the solution curves when $x = \pi$? Can you see this in the graph?
6. Since $\cos x$ is an even function, the slope at any point should be the same as the slope at its reflection across the y-axis. Is this true? How can you tell?

Exploration 1 suggests the interesting possibility that we could have produced the family of curves in Figure 6.1 without even solving the differential equation, simply by looking carefully at slopes. That is exactly the idea behind *slope fields*.

Slope Fields

Suppose we want to produce Figure 6.1 without actually solving the differential equation $dy/dx = \cos x$. Since the differential equation gives the *slope* at any point (x, y), we can use that information to draw a small piece of the linearization at that point, which (thanks to local linearity) approximates the solution curve that passes through that point. Repeating that process at many points yields an approximation of Figure 6.1 called a slope field. Example 6 shows how this is done.

EXAMPLE 6 Constructing a Slope Field

Construct a slope field for the differential equation $dy/dx = \cos x$.

SOLUTION

We know that the slope at any point $(0, y)$ will be $\cos 0 = 1$, so we can start by drawing tiny segments with slope 1 at several points along the y-axis (Figure 6.2a). Then, since the slope at any point (π, y) or $(-\pi, y)$ will be -1, we can draw tiny segments with slope -1 at several points along the vertical lines $x = \pi$ and $x = -\pi$ (Figure 6.2b). The slope at all odd multiples of $\pi/2$ will be zero, so we draw tiny horizontal segments along the lines $x = \pm\pi/2$ and $x = \pm 3\pi/2$ (Figure 6.2c). Finally, we add tiny segments of slope 1 along the lines $x = \pm 2\pi$ (Figure 6.2d).

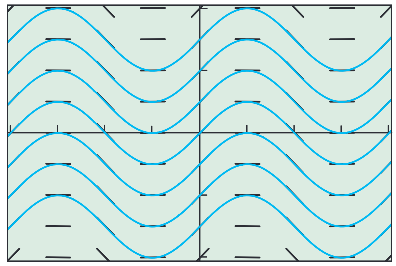
Now try Exercise 29.

To illustrate how a family of solution curves conforms to a slope field, we superimpose the solutions in Figure 6.1 on the slope field in Figure 6.2d. The result is shown in Figure 6.3 on the next page.

We could get a smoother-looking slope field by drawing shorter line segments at more points, but that can get tedious. Happily, the algorithm is simple enough to be programmed into a graphing calculator. One such program, using a lattice of 150 sample points, produced in a matter of seconds the graph in Figure 6.4 on the next page.

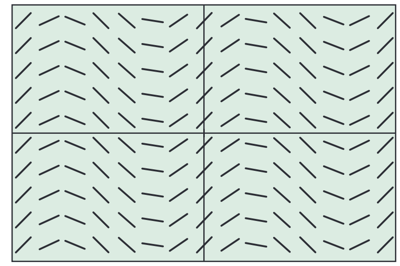
Differential Equation Mode

If your calculator has a *differential equation mode* for graphing, it is intended for graphing slope fields. The usual "Y=" turns into a "$dy/dx =$" screen, and you can enter a function of x and/or y. The grapher draws a slope field for the differential equation when you press the GRAPH button.



[–2π, 2π] by [–4, 4]

Figure 6.3 The graph of the general solution in Figure 6.1 conforms nicely to the slope field of the differential equation. (Example 6)



[–2π, 2π] by [–4, 4]

Figure 6.4 A slope field produced by a graphing calculator program.

It is also possible to produce slope fields for differential equations that are not of the form $dy/dx = f(x)$. We will study analytic techniques for solving certain types of these nonexact differential equations later in this chapter, but you should keep in mind that you can graph the general solution with a slope field even if you cannot find it analytically.

Can We Solve the Differential Equation in Example 7?

Although it looks harmless enough, the differential equation $dy/dx = x + y$ is not easy to solve until you have seen how it is done. It is an example of a *first-order linear differential equation,* and its general solution is

$$y = Ce^x - x - 1$$

(which you can easily check by verifying that $dy/dx = x + y$). We will defer the analytic solution of such equations to a later course.

EXAMPLE 7 Constructing a Slope Field for a Nonexact Differential Equation

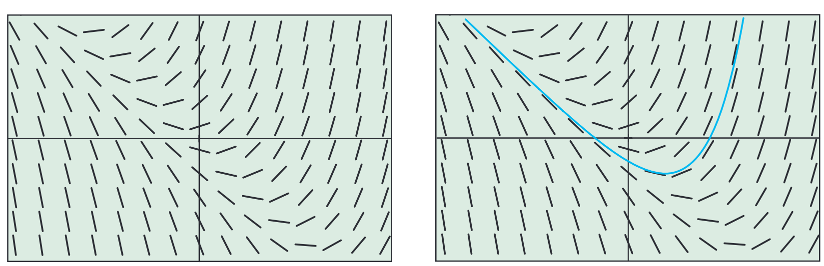
Use a calculator to construct a slope field for the differential equation $dy/dx = x + y$ and sketch a graph of the particular solution that passes through the point (2, 0).

SOLUTION

The calculator produces a graph like the one in Figure 6.5a. Notice the following properties of the graph, all of them easily predictable from the differential equation:

1. The slopes are zero along the line $x + y = 0$.
2. The slopes are -1 along the line $x + y = -1$.
3. The slopes get steeper as x increases.
4. The slopes get steeper as y increases.

The particular solution can be found by drawing a smooth curve through the point (2, 0) that follows the slopes in the slope field, as shown in Figure 6.5b.



[–4.7, 4.7] by [–3.1, 3.1]
(a)

[–4.7, 4.7] by [–3.1, 3.1]
(b)

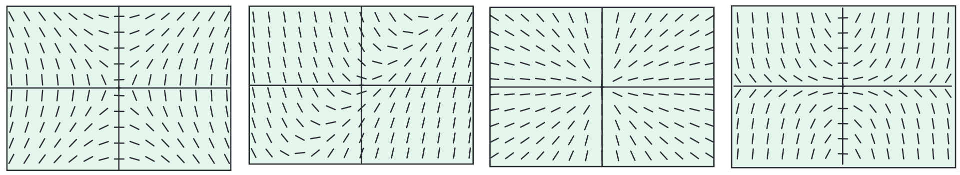
Figure 6.5 (a) A slope field for the differential equation $dy/dx = x + y$, and (b) the same slope field with the graph of the particular solution through (2, 0) superimposed. (Example 7)

Now try Exercise 35.

EXAMPLE 8 Matching Slope Fields with Differential Equations

Use slope analysis to match each of the following differential equations with one of the slope fields (a) through (d). (Do not use your graphing calculator.)

1. $\frac{dy}{dx} = x - y$ **2.** $\frac{dy}{dx} = xy$ **3.** $\frac{dy}{dx} = \frac{x}{y}$ **4.** $\frac{dy}{dx} = \frac{y}{x}$



(a) (b) (c) (d)

SOLUTION

To match Equation 1, we look for a graph that has zero slope along the line $x - y = 0$. That is graph (b).

To match Equation 2, we look for a graph that has zero slope along both axes. That is graph (d).

To match Equation 3, we look for a graph that has horizontal segments when $x = 0$ and vertical segments when $y = 0$. That is graph (a).

To match Equation 4, we look for a graph that has vertical segments when $x = 0$ and horizontal segments when $y = 0$. That is graph (c). *Now try Exercise 39.*

Euler's Method

In Example 7 we graphed the particular solution to an initial value problem by first producing a slope field and then finding a smooth curve through the slope field that passed through the given point. In fact, we could have graphed the particular solution directly, by starting at the given point and piecing together little line segments to build a continuous approximation of the curve. This clever application of local linearity to graph a solution without knowing its equation is called **Euler's Method.**

Euler's Method For Graphing a Solution to an Initial Value Problem

1. Begin at the point (x, y) specified by the initial condition. This point will be on the graph, as required.
2. Use the differential equation to find the slope dy/dx at the point.
3. Increase x by a small amount Δx. Increase y by a small amount Δy, where $\Delta y = (dy/dx)\Delta x$. This defines a new point $(x + \Delta x, y + \Delta y)$ that lies along the linearization (Figure 6.6).
4. Using this new point, return to step 2. Repeating the process constructs the graph to the right of the initial point.
5. To construct the graph moving to the left from the initial point, repeat the process using negative values for Δx.

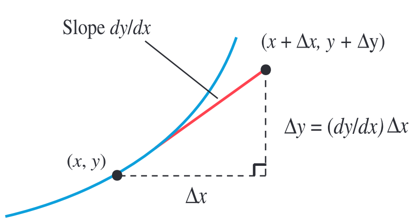


Figure 6.6 How Euler's Method moves along the linearization at the point (x, y) to define a new point $(x + \Delta x, y + \Delta y)$. The process is then repeated, starting with the new point.

We illustrate the method in Example 9.

EXAMPLE 9 Applying Euler's Method

Let f be the function that satisfies the initial value problem in Example 6 (that is, $dy/dx = x + y$ and $f(2) = 0$). Use Euler's method and increments of $\Delta x = 0.2$ to approximate $f(3)$.

SOLUTION

We use Euler's Method to construct an approximation of the curve from $x = 2$ to $x = 3$, pasting together five small linearization segments (Figure 6.7). Each segment will extend from a point (x, y) to a point $(x + \Delta x, y + \Delta y)$, where $\Delta x = 0.2$ and $\Delta y = (dy/dx)\Delta x$. The following table shows how we construct each new point from the previous one.

(x, y)	$dy/dx = x + y$	Δx	$\Delta y = (dy/dx)\Delta x$	$(x + \Delta x, y + \Delta y)$
(2, 0)	2	0.2	0.4	(2.2, 0.4)
(2.2, 0.4)	2.6	0.2	0.52	(2.4, 0.92)
(2.4, 0.92)	3.32	0.2	0.664	(2.6, 1.584)
(2.6, 1.584)	4.184	0.2	0.8368	(2.8, 2.4208)
(2.8, 2.4208)	5.2208	0.2	1.04416	(3, 3.46496)

Euler's Method leads us to an approximation $f(3) \approx 3.46496$, which we would more reasonably report as $f(3) \approx 3.465$. ***Now try Exercise 41.***

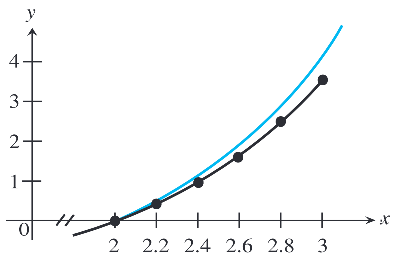


Figure 6.7 Euler's Method is used to construct an approximate solution to an initial value problem between $x = 2$ and $x = 3$. (Example 9)

You can see from Figure 6.7 that Euler's Method leads to an underestimate when the curve is concave up, just as it will lead to an overestimate when the curve is concave down. You can also see that the error increases as the distance from the original point increases. In fact, the true value of $f(3)$ is about 4.155, so the approximation error is about 16.6%. We could increase the accuracy by taking smaller increments; a reasonable option if we have a calculator program to do the work. For example, 100 increments of 0.01 give an estimate of 4.1144, cutting the error to about 1%.

EXAMPLE 10 Moving Backward with Euler's Method

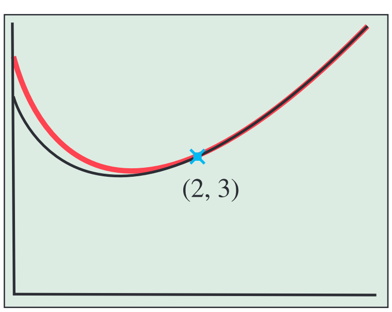
If $dy/dx = 2x - y$ and if $y = 3$ when $x = 2$, use Euler's Method with five equal steps to approximate y when $x = 1.5$.

SOLUTION

Starting at $x = 2$, we need five equal steps of $\Delta x = -0.1$.

(x, y)	$dy/dx = 2x - y$	Δx	$\Delta y = (dy/dx)\Delta x$	$(x + \Delta x, y + \Delta y)$
(2, 3)	1	−0.1	−0.1	(1.9, 2.9)
(1.9, 2.9)	0.9	−0.1	−0.09	(1.8, 2.81)
(1.8, 2.81)	0.79	−0.1	−0.079	(1.7, 2.731)
(1.7, 2.731)	0.669	−0.1	−0.0669	(1.6, 2.6641)
(1.6, 2.6641)	0.5359	−0.1	−0.05359	(1.5, 2.61051)

The value at $x = 1.5$ is approximately 2.61. (The actual value is about 2.649, so the percentage error in this case is about 1.4%.) ***Now try Exercise 45.***



[0, 4] by [0, 6]

Figure 6.8 A grapher program using Euler's Method and increments of 0.1 produced this approximation to the solution curve for the initial value problem in Example 10. The actual solution curve is shown in red.

If we program a grapher to do the work of finding the points, Euler's Method can be used to graph (approximately) the solution to an initial value problem without actually solving it. For example, a graphing calculator program starting with the initial value problem in Example 9 produced the graph in Figure 6.8 using increments of 0.1. The graph of the actual solution is shown in red. Notice that Euler's Method does a better job of approximating the curve when the curve is nearly straight, as should be expected.

Euler's Method is one example of a *numerical method* for solving differential equations. The table of values is the *numerical solution*. The analysis of error in a numerical solution and the investigation of methods to reduce it are important, but appropriate for a more advanced course (which would also describe more accurate numerical methods than the one shown here).

Quick Review 6.1

In Exercises 1–8, determine whether or not the function y satisfies the differential equation.

1. $\dfrac{dy}{dx} = y \qquad y = e^x$

2. $\dfrac{dy}{dx} = 4y \qquad y = e^{4x}$

3. $\dfrac{dy}{dx} = 2xy \qquad y = x^2e^x$

4. $\dfrac{dy}{dx} = 2xy \qquad y = e^{x^2}$

5. $\dfrac{dy}{dx} = 2xy \qquad y = e^{x^2} + 5$

6. $\dfrac{dy}{dx} = \dfrac{1}{y} \qquad y = \sqrt{2x}$

7. $\dfrac{dy}{dx} = y \tan x \qquad y = \sec x$

8. $\dfrac{dy}{dx} = y^2 \qquad y = x^{-1}$

In Exercises 9–12, find the constant C.

9. $y = 3x^2 + 4x + C$ and $y = 2$ when $x = 1$

10. $y = 2\sin x - 3\cos x + C$ and $y = 4$ when $x = 0$

11. $y = e^{2x} + \sec x + C$ and $y = 5$ when $x = 0$

12. $y = \tan^{-1} x + \ln(2x - 1) + C$ and $y = \pi$ when $x = 1$

Section 6.1 Exercises

In Exercises 1–10, find the general solution to the exact differential equation.

1. $\dfrac{dy}{dx} = 5x^4 - \sec^2 x$

2. $\dfrac{dy}{dx} = \sec x \tan x - e^x$

3. $\dfrac{dy}{dx} = \sin x - e^{-x} + 8x^3$

4. $\dfrac{dy}{dx} = \dfrac{1}{x} - \dfrac{1}{x^2} \; (x > 0)$

5. $\dfrac{dy}{dx} = 5^x \ln 5 + \dfrac{1}{x^2 + 1}$

6. $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1 - x^2}} - \dfrac{1}{\sqrt{x}}$

7. $\dfrac{dy}{dt} = 3t^2 \cos(t^3)$

8. $\dfrac{dy}{dt} = (\cos t)\, e^{\sin t}$

9. $\dfrac{du}{dx} = (\sec^2 x^5)(5x^4)$

10. $\dfrac{dy}{du} = 4(\sin u)^3(\cos u)$

In Exercises 11–20, solve the initial value problem explicitly.

11. $\dfrac{dy}{dx} = 3\sin x$ and $y = 2$ when $x = 0$

12. $\dfrac{dy}{dx} = 2e^x - \cos x$ and $y = 3$ when $x = 0$

13. $\dfrac{du}{dx} = 7x^6 - 3x^2 + 5$ and $u = 1$ when $x = 1$

14. $\dfrac{dA}{dx} = 10x^9 + 5x^4 - 2x + 4$ and $A = 6$ when $x = 1$

15. $\dfrac{dy}{dx} = -\dfrac{1}{x^2} - \dfrac{3}{x^4} + 12$ and $y = 3$ when $x = 1$

16. $\dfrac{dy}{dx} = 5\sec^2 x - \dfrac{3}{2}\sqrt{x}$ and $y = 7$ when $x = 0$

17. $\dfrac{dy}{dt} = \dfrac{1}{1 + t^2} + 2^t \ln 2$ and $y = 3$ when $t = 0$

18. $\dfrac{dx}{dt} = \dfrac{1}{t} - \dfrac{1}{t^2} + 6$ and $x = 0$ when $t = 1$

19. $\dfrac{dv}{dt} = 4\sec t \tan t + e^t + 6t$ and $v = 5$ when $t = 0$

20. $\dfrac{ds}{dt} = t(3t - 2)$ and $s = 0$ when $t = 1$

In Exercises 21–24, solve the initial value problem using the Fundamental Theorem. (Your answer will contain a definite integral.)

21. $\dfrac{dy}{dx} = \sin(x^2)$ and $y = 5$ when $x = 1$

22. $\dfrac{du}{dx} = \sqrt{2 + \cos x}$ and $u = -3$ when $x = 0$

23. $F'(x) = e^{\cos x}$ and $F(2) = 9$

24. $G'(s) = \sqrt[3]{\tan s}$ and $G(0) = 4$

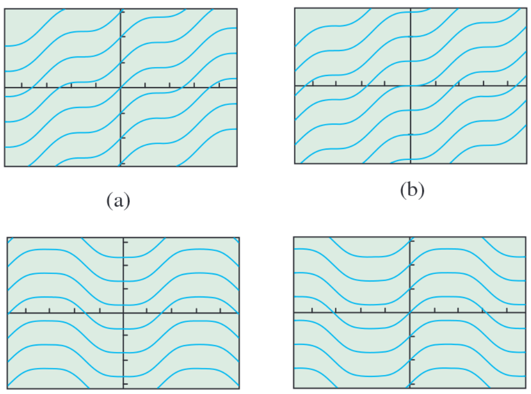
In Exercises 25–28, match the differential equation with the graph of a family of functions (a)–(d) that solve it. Use slope analysis, not your graphing calculator.

25. $\frac{dy}{dx} = (\sin x)^2$ Graph (b)

26. $\frac{dy}{dx} = (\sin x)^3$ Graph (c)

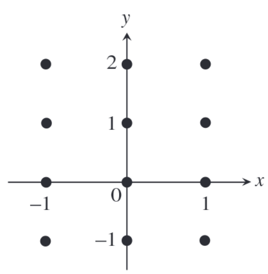
27. $\frac{dy}{dx} = (\cos x)^2$ Graph (a)

28. $\frac{dy}{dx} = (\cos x)^3$ Graph (d)



(a) (b) (c) (d)

In Exercises 29–34, construct a slope field for the differential equation. In each case, copy the graph at the right and draw tiny segments through the twelve lattice points shown in the graph. Use slope analysis, not your graphing calculator.



29. $\frac{dy}{dx} = x$

30. $\frac{dy}{dx} = y$

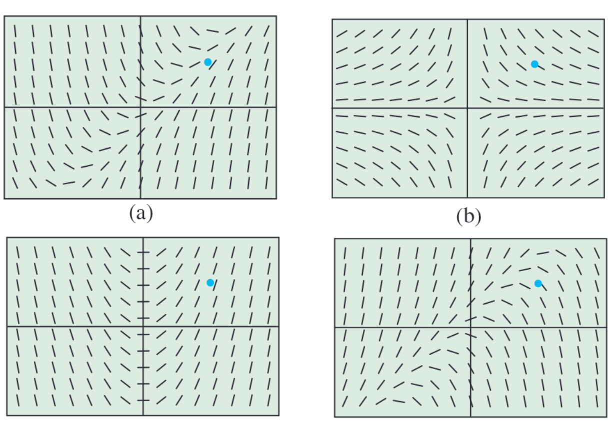
31. $\frac{dy}{dx} = 2x + y$

32. $\frac{dy}{dx} = 2x - y$

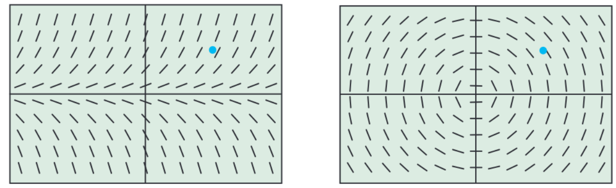
33. $\frac{dy}{dx} = x + 2y$

34. $\frac{dy}{dx} = x - 2y$

In Exercises 35–40, match the differential equation with the appropriate slope field. Then use the slope field to sketch the graph of the particular solution through the highlighted point (3, 2). (All slope fields are shown in the window [−6, 6] by [−4, 4].)



(a) (b) (c) (d)



(e) (f)

35. $\frac{dy}{dx} = x$

36. $\frac{dy}{dx} = y$

37. $\frac{dy}{dx} = x - y$

38. $\frac{dy}{dx} = y - x$

39. $\frac{dy}{dx} = -\frac{y}{x}$

40. $\frac{dy}{dx} = -\frac{x}{y}$

In Exercises 41–44, use Euler's Method with increments of $\Delta x = 0.1$ to approximate the value of y when $x = 1.3$.

41. $\frac{dy}{dx} = x - 1$ and $y = 2$ when $x = 1$ 2.03

42. $\frac{dy}{dx} = y - 1$ and $y = 3$ when $x = 1$ 3.662

43. $\frac{dy}{dx} = y - x$ and $y = 2$ when $x = 1$ 2.3

44. $\frac{dy}{dx} = 2x - y$ and $y = 0$ when $x = 1$ 0.6

In Exercises 45–48, use Euler's Method with increments of $\Delta x = -0.1$ to approximate the value of y when $x = 1.7$.

45. $\frac{dy}{dx} = 2 - x$ and $y = 1$ when $x = 2$ 0.97

46. $\frac{dy}{dx} = 1 + y$ and $y = 0$ when $x = 2$ −0.271

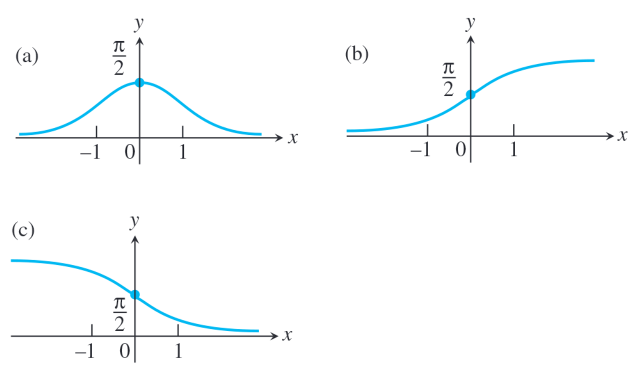
47. $\frac{dy}{dx} = x - y$ and $y = 2$ when $x = 2$ 2.031

48. $\frac{dy}{dx} = x - 2y$ and $y = 1$ when $x = 2$ 1.032

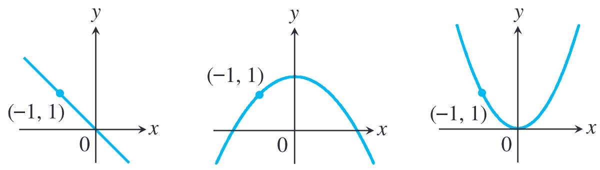
In Exercises 49 and 50, **(a)** determine which graph shows the solution of the initial value problem without actually solving the problem.
(b) Writing to Learn Explain how you eliminated two of the possibilities.

49. $\frac{dy}{dx} = \frac{1}{1 + x^2}, \quad y(0) = \frac{\pi}{2}$

(a) Graph (b)
(b) The slope is always positive, so graphs (a) and (c) can be ruled out.

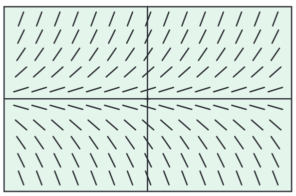


50. $\dfrac{dy}{dx} = -x, \quad y(-1) = 1$



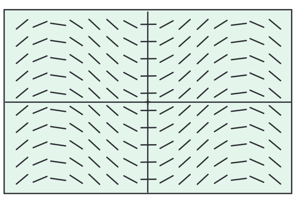
(a) (b) (c)

51. Writing to Learn Explain why $y = x^2$ could not be a solution to the differential equation with slope field shown below.



[–4.7, 4.7] by [–3.1, 3.1]

52. Writing to Learn Explain why $y = \sin x$ could not be a solution to the differential equation with slope field shown below.

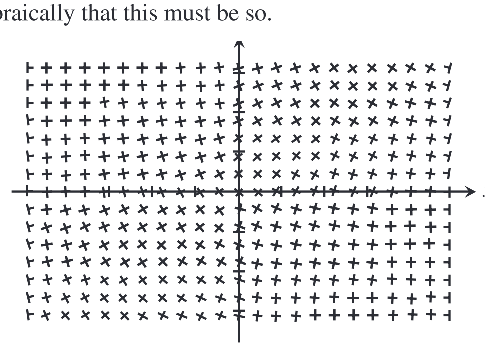


[–4.7, 4.7] by [–3.1, 3.1]

53. *Percentage Error* Let $y = f(x)$ be the solution to the initial value problem $dy/dx = 2x + 1$ such that $f(1) = 3$. Find the percentage error if Euler's Method with $\Delta x = 0.1$ is used to approximate $f(1.4)$.

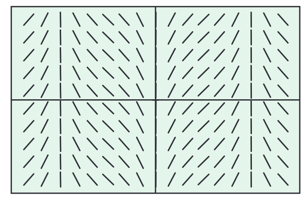
54. *Percentage Error* Let $y = f(x)$ be the solution to the initial value problem $dy/dx = 2x - 1$ such that $f(2) = 3$. Find the percentage error if Euler's Method with $\Delta x = -0.1$ is used to approximate $f(1.6)$.

55. *Perpendicular Slope Fields* The figure below shows the slope fields for the differential equations $dy/dx = e^{(x-y)/2}$ and $dy/dx = -e^{(y-x)/2}$ superimposed on the same grid. It appears that the slope lines are perpendicular wherever they intersect. Prove algebraically that this must be so.



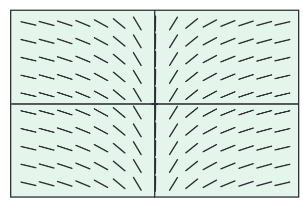
56. *Perpendicular Slope Fields* If the slope fields for the differential equations $dy/dx = \sec x$ and $dy/dx = g(x)$ are perpendicular (as in Exercise 55), find $g(x)$.

57. *Plowing Through a Slope Field* The slope field for the differential equation $dy/dx = \csc x$ is shown below. Find a function that will be *perpendicular* to every line it crosses in the slope field. (*Hint:* First find a differential equation that will produce a perpendicular slope field.)



[–4.7, 4.7] by [–3.1, 3.1]

58. *Plowing Through a Slope Field* The slope field for the differential equation $dy/dx = 1/x$ is shown below. Find a function that will be *perpendicular* to every line it crosses in the slope field. (*Hint:* First find a differential equation that will produce a perpendicular slope field.)



[–4.7, 4.7] by [–3.1, 3.1]

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

59. True or False Any two solutions to the differential equation $dy/dx = 5$ are parallel lines. Justify your answer.

60. True or False If $f(x)$ is a solution to $dy/dx = 2x$, then $f^{-1}(x)$ is a solution to $dy/dx = 2y$. Justify your answer.

61. Multiple Choice A slope field for the differential equation $dy/dx = 42 - y$ will show

(A) a line with slope –1 and y-intercept 42.

(B) a vertical asymptote at $x = 42$.

(C) a horizontal asymptote at $y = 42$.

(D) a family of parabolas opening downward.

(E) a family of parabolas opening to the left.

62. Multiple Choice For which of the following differential equations will a slope field show nothing but negative slopes in the fourth quadrant?

(A) $\dfrac{dy}{dx} = -\dfrac{x}{y}$ **(B)** $\dfrac{dy}{dx} = xy + 5$ **(C)** $\dfrac{dy}{dx} = xy^2 - 2$

(D) $\dfrac{dy}{dx} = \dfrac{x^3}{y^2}$ **(E)** $\dfrac{dy}{dx} = \dfrac{y}{x^2} - 3$

63. Multiple Choice If $dy/dx = 2xy$ and $y = 1$ when $x = 0$, then $y =$

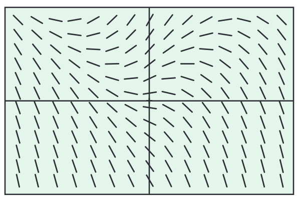
(A) y^{2x} **(B)** e^{x^2} **(C)** x^2y **(D)** $x^2y + 1$ **(E)** $\frac{x^2y^2}{2} + 1$

64. Multiple Choice Which of the following differential equations would produce the slope field shown below?

(A) $\frac{dy}{dx} = y - |x|$ **(B)** $\frac{dy}{dx} = |y| - x$

(C) $\frac{dy}{dx} = |y - x|$ **(D)** $\frac{dy}{dx} = |y + x|$

(E) $\frac{dy}{dx} = |y| - |x|$



[–3, 3] by [–1.98, 1.98]

Explorations

65. *Solving Differential Equations* Let $\frac{dy}{dx} = x - \frac{1}{x^2}$.

(a) Find a solution to the differential equation in the interval (0,) that satisfies $y(1) = 2$.

(b) Find a solution to the differential equation in the interval $(-\infty, 0)$ that satisfies $y(-1) = 1$.

(c) Show that the following piecewise function is a solution to the differential equation for any values of C_1 and C_2.

$$y = \begin{cases} \frac{1}{x} + \frac{x^2}{2} + C_1, & x < 0 \\ \frac{1}{x} + \frac{x^2}{2} + C_2, & x > 0 \end{cases}$$

(d) Choose values for C_1 and C_2 so that the solution in part (c) agrees with the solutions in parts (a) and (b).

(e) Choose values for C_1 and C_2 so that the solution in part (c) satisfies $y(2) = -1$ and $y(-2) = 2$.

66. *Solving Differential Equations* Let $\frac{dy}{dx} = \frac{1}{x}$.

(a) Show that $y = \ln x + C$ is a solution to the differential equation in the interval $(0, \infty)$.

(b) Show that $y = \ln(-x) + C$ is a solution to the differential equation in the interval $(-\infty, 0)$.

(c) Writing to Learn Explain why $y = \ln|x| + C$ is a solution to the differential equation in the domain $(-\infty, 0) \cup (0, \infty)$.

(d) Show that the function

$$y = \begin{cases} \ln(-x) + C_1, & x < 0 \\ \ln x + C_2, & x > 0 \end{cases}$$

is a solution to the differential equation for any values of C_1 and C_2.

Extending the Ideas

67. *Second-Order Differential Equations* Find the general solution to each of the following second-order differential equations by first finding dy/dx and then finding y. The general solution will have two unknown constants.

(a) $\frac{d^2y}{dx^2} = 12x + 4$ **(b)** $\frac{d^2y}{dx^2} = e^x + \sin x$ **(c)** $\frac{d^2y}{dx^2} = x^3 + x^{-3}$

68. *Second-Order Differential Equations* Find the specific solution to each of the following second-order initial value problems by first finding dy/dx and then finding y.

(a) $\frac{d^2y}{dx^2} = 24x^2 - 10$. When $x = 1$, $\frac{dy}{dx} = 3$ and $y = 5$.

(b) $\frac{d^2y}{dx^2} = \cos x - \sin x$. When $x = 0$, $\frac{dy}{dx} = 2$ and $y = 0$

(c) $\frac{d^2y}{dx^2} = e^x - x$. When $x = 0$, $\frac{dy}{dx} = 0$ and $y = 1$.

69. *Differential Equation Potpourri* For each of the following differential equations, find at least one particular solution. You will need to call on past experience with functions you have differentiated. For a greater challenge, find the general solution.

(a) $y' = x$ **(b)** $y' = -x$ **(c)** $y' = y$

(d) $y' = -y$ **(e)** $y' = xy$

70. *Second-Order Potpourri* For each of the following second-order differential equations, find at least one particular solution. You will need to call on past experience with functions you have differentiated. For a significantly greater challenge, find the general solution (which will involve two unknown constants).

(a) $y'' = x$ **(b)** $y'' = -x$ **(c)** $y'' = -\sin x$

(d) $y'' = y$ **(e)** $y'' = -y$

6.2 Antidifferentiation by Substitution

What you'll learn about

- Indefinite Integrals
- Leibniz Notation and Antiderivatives
- Substitution in Indefinite Integrals
- Substitution in Definite Integrals

. . . and why

Antidifferentiation techniques were historically crucial for applying the results of calculus.

Indefinite Integrals

If $y = f(x)$ we can denote the derivative of f by either dy/dx or $f'(x)$. What can we use to denote the *antiderivative* of f? We have seen that the general solution to the differential equation $dy/dx = f(x)$ actually consists of an infinite family of functions of the form $F(x) + C$, where $F'(x) = f(x)$. Both the name for this family of functions and the symbol we use to denote it are closely related to the definite integral because of the Fundamental Theorem of Calculus.

DEFINITION Indefinite Integral

The family of all antiderivatives of a function $f(x)$ is the **indefinite integral of f with respect to x** and is denoted by $\int f(x)dx$.

If F is any function such that $F'(x) = f(x)$, then $\int f(x)dx = F(x) + C$, where C is an arbitrary constant, called the **constant of integration.**

As in Chapter 5, the symbol $\int$ is an **integral sign,** the function f is the **integrand** of the integral, and x is the **variable of integration.**

Notice that an indefinite integral is not at all like a definite integral, despite the similarities in notation and name. A definite integral is a *number*, the limit of a sequence of Riemann sums. An indefinite integral is a *family of functions* having a common derivative. If the Fundamental Theorem of Calculus had not provided such a dramatic link between antiderivatives and integration, we would surely be using a different name and symbol for the general antiderivative today.

EXAMPLE 1 Evaluating an Indefinite Integral

Evaluate $\int (x^2 - \sin x)\, dx$.

SOLUTION

Evaluating this definite integral is just like solving the differential equation $dy/dx = x^2 - \sin x$. Our past experience with derivatives leads us to conclude that

$$\int (x^2 - \sin x)\,dx = \frac{x^3}{3} + \cos x + C$$

(as you can check by differentiating).

Now try Exercise 3.

You have actually been finding antiderivatives since Section 5.3, so Example 1 should hardly have seemed new. Indeed, each derivative formula in Chapter 3 could be turned around to yield a corresponding indefinite integral formula. We list some of the most useful such indefinite integral formulas below. Be sure to familiarize yourself with these before moving on to the next section, in which function composition becomes an issue. (Incidentally, it is in anticipation of the next section that we give some of these formulas in terms of the variable u rather than x.)

Properties of Indefinite Integrals

$$\int k\,f(x)dx = k\int f(x)\,dx \quad \text{for any constant } k$$

$$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Power Formulas

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C \text{ when } n \neq -1$$

$$\int u^{-1}\,du = \int \frac{1}{u}\,du = \ln|u| + C$$
(see Example 2)

Trigonometric Formulas

$$\int \cos u\,du = \sin u + C \qquad \int \sin u\,du = -\cos u + C$$

$$\int \sec^2 u\,du = \tan u + C \qquad \int \csc^2 u\,du = -\cot u + C$$

$$\int \sec u \tan u\,du = \sec u + C \qquad \int \csc u \cot u\,du = -\csc u + C$$

Exponential and Logarithmic Formulas

$$\int e^u\,du = e^u + C \qquad \int a^u\,du = \frac{a^u}{\ln a} + C$$

$$\int \ln u\,du = u\ln u - u + C \quad \text{(See Example 2)}$$

$$\int \log_a u\,du = \int \frac{\ln u}{\ln a}\,du = \frac{u\ln u - u}{\ln a} + C$$

A Note on Absolute Value

Since the indefinite integral does not specify a domain, you should always use the absolute value when finding $\int 1/u\,du$. The function $\ln u + C$ is only defined on positive u-intervals, while the function $\ln |u| + C$ is defined on both the positive *and* negative intervals in the domain of $1/u$ (see Example 2).

EXAMPLE 2 Verifying Antiderivative Formulas

Verify the antiderivative formulas:

(a) $\int u^{-1}\,du = \int \frac{1}{u}\,du = \ln|u| + C$ **(b)** $\int \ln u\,du = u\ln u - u + C$

SOLUTION

We can verify antiderivative formulas by differentiating.

(a) For $u > 0$, we have $\frac{d}{du}(\ln|u| + C) = \frac{d}{du}(\ln u + C) = \frac{1}{u} + 0 = \frac{1}{u}$.

For $u < 0$, we have $\frac{d}{du}(\ln|u| + C) = \frac{d}{du}(\ln(-u) + C) = \frac{1}{-u}(-1) + 0 = \frac{1}{u}$.

Since $\frac{d}{du}(\ln|u| + C) = \frac{1}{u}$ in either case, $\ln|u| + C$ is the general antiderivative of the function $\frac{1}{u}$ on its entire domain.

(b) $\frac{d}{du}(u\ln u - u + C) = 1 \cdot \ln u + u\left(\frac{1}{u}\right) - 1 + 0 = \ln u + 1 - 1 = \ln u.$

Now try Exercise 11.

Leibniz Notation and Antiderivatives

The appearance of the differential "dx" in the definite integral $\int_a^b f(x)dx$ is easily explained by the fact that it is the limit of a Riemann sum of the form $\sum_{k=1}^{n} f(x_k) \cdot \Delta x$ (see Section 5.2). The same "dx" almost seems unnecessary when we use the indefinite integral $\int f(x)dx$ to represent the general antiderivative of f, but in fact it is quite useful for *dealing with the effects of the Chain Rule* when function composition is involved. Exploration 1 will show you why this is an important consideration.

EXPLORATION 1 **Are $\int f(u)\, du$ and $\int f(u)\, dx$ the Same Thing?**

Let $u = x^2$ and let $f(u) = u^3$.

1. Find $\int f(u)\, du$ as a function of u.
2. Use your answer to question 1 to write $\int f(u)\, du$ as a function of x.
3. Show that $f(u) = x^6$ and find $\int f(u)\, dx$ as a function of x.
4. Are the answers to questions 2 and 3 the same?

Exploration 1 shows that the notation $\int f(u)$ is not sufficient to describe an antiderivative when u is a function of another variable. Just as du/du is different from du/dx when differentiating, $\int f(u)\, du$ is different from $\int f(u)\, dx$ when antidifferentiating. We will use this fact to our advantage in the next section, where the importance of "dx" or "du" in the integral expression will become even more apparent.

EXAMPLE 3 **Paying Attention to the Differential**

Let $f(x) = x^3 + 1$ and let $u = x^2$. Find each of the following antiderivatives in terms of x:

(a) $\int f(x)\, dx$ **(b)** $\int f(u)\, du$ **(c)** $\int f(u)\, dx$

SOLUTION

(a) $\int f(x)\, dx = \int (x^3 + 1)\, dx = \frac{x^4}{4} + x + C$

(b) $\int f(u)\, du = \int (u^3 + 1)\, du = \frac{u^4}{4} + u + C = \frac{(x^2)^4}{4} + x^2 + C = \frac{x^8}{4} + x^2 + C$

(c) $\int f(u)\, dx = \int (u^3 + 1)\, dx = \int ((x^2)^3 + 1)\, dx = \int (x^6 + 1)\, dx = \frac{x^7}{7} + x + C$

Now try Exercise 15.

Substitution in Indefinite Integrals

A change of variables can often turn an unfamiliar integral into one that we can evaluate. The important point to remember is that it is *not sufficient* to change an integral of the form $\int f(x)\, dx$ into an integral of the form $\int g(u)\, dx$. The differential matters. A complete substitution changes the integral $\int f(x)\, dx$ into an integral of the form $\int g(u)\, du$.

EXAMPLE 4 **Using Substitution**

Evaluate $\int \sin x\, e^{\cos x}\, dx$.

continued

SOLUTION

Let $u = \cos x$. Then $du/dx = -\sin x$, from which we conclude that $du = -\sin x\, dx$. We rewrite the integral and proceed as follows:

$$\begin{aligned}\int \sin x\, e^{\cos x}\, dx &= -\int (-\sin x)e^{\cos x}\, dx \\ &= -\int e^{\cos x} \cdot (-\sin x)dx \\ &= -\int e^{u}\, du \\ &= -e^{u} + C \\ &= -e^{\cos x} + C\end{aligned}$$

Now try Exercise 19.

If you differentiate $-e^{\cos x} + C$, you will find that a factor of $-\sin x$ appears when you apply the Chain Rule. The technique of *antidifferentiation by substitution* reverses that effect by absorbing the $-\sin x$ into the differential du when you change $\int \sin x\, e^{\cos x}\, dx$ into $-\int e^{u}\, du$. That is why a "u-substitution" always involves a "du-substitution" to convert the integral into a form ready for antidifferentiation.

EXAMPLE 5 Using Substitution

Evaluate $\int x^2\sqrt{5 + 2x^3}\, dx$.

SOLUTION

This integral invites the substitution $u = 5 + 2x^3$, $du = 6x^2\, dx$.

$$\begin{aligned}\int x^2\sqrt{5 + 2x^3}\, dx. &= \int (5 + 2x^3)^{1/2} \cdot x^2\, dx \\ &= \frac{1}{6}\int (5 + 2x^3)^{1/2} \cdot 6x^2\, dx \\ &= \frac{1}{6}\int u^{1/2}\, du \\ &= \frac{1}{6}\left(\frac{2}{3}\right)u^{3/2} + C \\ &= \frac{1}{9}(5 + 2x^3)^{3/2} + C\end{aligned}$$

Now try Exercise 27.

EXAMPLE 6 Using Substitution

Evaluate $\int \cot 7x\, dx$.

SOLUTION

We do not recall a function whose derivative is $\cot 7x$, but a basic trigonometric identity changes the integrand into a form that invites the substitution $u = \sin 7x$, $du = 7\cos 7x\, dx$. We rewrite the integrand as shown on the next page.

$$\int \cot 7x\,dx = \int \frac{\cos 7x}{\sin 7x}\,dx$$

$$= \frac{1}{7}\int \frac{7\cos 7x\,dx}{\sin x}$$

$$= \frac{1}{7}\int \frac{du}{u}$$

$$= \frac{1}{7}\ln|u| + C$$

$$= \frac{1}{7}\ln|\sin 7x| + C$$

Now try Exercise 29.

EXAMPLE 7 Setting Up a Substitution with a Trigonometric Identity

Find the indefinite integrals. In each case you can use a trigonometric identity to set up a substitution.

(a) $\int \frac{dx}{\cos^2 2x}$ **(b)** $\int \cot^2 3x\,dx$ **(c)** $\int \cos^3 x\,dx$

SOLUTION

(a)
$$\int \frac{dx}{\cos^2 2x} = \int \sec^2 2x\,dx$$

$$= \frac{1}{2}\int \sec^2 2x \cdot 2dx$$

$$= \frac{1}{2}\int \sec^2 u\,du$$

$$= \frac{1}{2}\tan u + C$$

$$= \frac{1}{2}\tan 2x + C$$

(b)
$$\int \cot^2 3x\,dx = \int (\csc^2 3x - 1)\,dx$$

$$= \frac{1}{3}\int (\csc^2 3x - 1)\cdot 3\,dx$$

$$= \frac{1}{3}\int (\csc^2 u - 1)\cdot du$$

$$= \frac{1}{3}\int (-\cot u - u) + C$$

$$= \frac{1}{3}(-\cot 3x - 3x) + C$$

$$= -\frac{1}{3}\cot 3x - x + C$$

continued

(c) $\int \cos^3 x\,dx = \int (\cos^2 x)\cos x\,dx$

$$= \int (1 - \sin^2 x)\cos x\,dx$$

$$= \int (1 - u^2)\,du$$

$$= u - \frac{u^3}{3} + C$$

$$= \sin x - \frac{\sin^3 x}{3} + C$$

Now try Exercise 47.

Substitution in Definite Integrals

Antiderivatives play an important role when we evaluate a definite integral by the Fundamental Theorem of Calculus, and so, consequently, does substitution. In fact, if we make full use of our substitution of variables and change the interval of integration to match the u-substitution in the integrand, we can avoid the "resubstitution" step in the previous four examples.

EXAMPLE 8 Evaluating a Definite Integral by Substitution

Evaluate $\int_0^{\pi/3} \tan x \sec^2 x\,dx$.

SOLUTION

Let $u = \tan x$ and $du = \sec^2 x\,dx$.

Note also that $u(0) = \tan 0 = 0$ and $u(\pi/3) = \tan(\pi/3) = \sqrt{3}$.

So

$$\int_0^{\pi/3} \tan x \sec^2 x\,dx = \int_0^{\sqrt{3}} u\,du$$

$$= \frac{u^2}{2}\bigg|_0^{\sqrt{3}}$$

$$= \frac{3}{2} - 0 = \frac{3}{2}$$

Now try Exercise 55.

EXAMPLE 9 That Absolute Value Again

Evaluate $\int_0^1 \frac{x}{x^2 - 4}\,dx$.

SOLUTION

Let $u = x^2 - 4$ and $du = 2x\,dx$. Then $u(0) = 0^2 - 4 = -4$ and $u(1) = 1^2 - 4 = -3$.

continued

So

$$\int_0^1 \frac{x}{x^2-4}\,dx = \frac{1}{2}\int_0^1 \frac{2x\;dx}{x^2-4}$$

$$= \frac{1}{2}\int_{-4}^{-3} \frac{du}{u}$$

$$= \frac{1}{2}\ln|u|\Big|_{-4}^{-3}$$

$$= \frac{1}{2}(\ln 3 - \ln 4) = \frac{1}{2}\ln\left(\frac{3}{4}\right)$$

Notice that $\ln u$ would not have existed over the interval of integration $[-4, -3]$. The absolute value in the antiderivative is important. ***Now try Exercise 63.***

Finally, consider this historical note. The technique of u-substitution derived its importance from the fact that it was a powerful tool for antidifferentiation. Antidifferentiation derived its importance from the Fundamental Theorem, which established it as the way to evaluate definite integrals. Definite integrals derived their importance from real-world applications. While the applications are no less important today, the fact that the definite integrals can be easily evaluated by technology has made the world less reliant on antidifferentiation, and hence less reliant on u-substitution. Consequently, you have seen in this book only a sampling of the substitution tricks calculus students would have routinely studied in the past. You may see more of them in a differential equations course.

Quick Review 6.2 *(For help, go to Sections 3.6 and 3.9.)*

In Exercises 1 and 2, evaluate the definite integral.

1. $\int_0^2 x^4\,dx$

2. $\int_1^5 \sqrt{x-1}\,dx$

In Exercises 3–10, find dy/dx.

3. $y = \int_2^x 3^t\,dt$

4. $y = \int_0^x 3^t\,dt$

5. $y = (x^3 - 2x^2 + 3)^4$

6. $y = \sin^2(4x - 5)$

7. $y = \ln \cos x$

8. $y = \ln \sin x$

9. $y = \ln(\sec x + \tan x)$

10. $y = \ln(\csc x + \cot x)$

Section 6.2 Exercises

In Exercises 1–6, find the indefinite integral.

1. $\int (\cos x - 3x^2)\,dx$

2. $\int x^{-2}\,dx$

3. $\int \left(t^2 - \frac{1}{t^2}\right) dt$

4. $\int \frac{dt}{t^2+1}$

5. $\int (3x^4 - 2x^{-3} + \sec^2 x)\,dx$

6. $\int (2e^x + \sec x \tan x - \sqrt{x})\,dx$

In Exercises 7–12, use differentiation to verify the antiderivative formula.

7. $\int \csc^2 u\,du = -\cot u + C$

8. $\int \csc u \cot u = -\csc u + C$

9. $\int e^{2x}\,dx = \frac{1}{2}e^{2x} + C$

10. $\int 5^x\,dx = \frac{1}{\ln 5}5^x + C$

11. $\int \frac{1}{1+u^2}\,du = \tan^{-1} u + C$

12. $\int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1} u + C$

In Exercises 13–16, verify that $\int f(u)\,du \neq \int f(u)\,dx$

13. $f(u) = \sqrt{u}$ and $u = x^2\ (x > 0)$

14. $f(u) = u^2$ and $u = x^5$

15. $f(u) = e^u$ and $u = 7x$

16. $f(u) = \sin u$ and $u = 4x$

In Exercises 17–24, use the indicated substitution to evaluate the integral. Confirm your answer by differentiation.

17. $\int \sin 3x\,dx, \quad u = 3x$

18. $\int x\cos(2x^2)\,dx, \quad u = 2x^2$

19. $\int \sec 2x\tan 2x\,dx, \quad u = 2x$

20. $\int 28(7x-2)^3\,dx, \quad u = 7x - 2$

21. $\int \frac{dx}{x^2+9}, \quad u = \frac{x}{3}$

22. $\int \frac{9r^2\,dr}{\sqrt{1-r^3}}, \quad u = 1 - r^3$

23. $\int \left(1-\cos\frac{t}{2}\right)^2 \sin\frac{t}{2}\,dt, \quad u = 1 - \cos\frac{t}{2}$

24. $\int 8(y^4+4y^2+1)^2(y^3+2y)\,dy, \quad u = y^4 + 4y^2 + 1$

In Exercises 25–46, use substitution to evaluate the integral.

25. $\int \frac{dx}{(1-x)^2}$

26. $\int \sec^2(x+2)\,dx$

27. $\int \sqrt{\tan x}\,\sec^2 x\,dx$

28. $\int \sec\left(\theta+\frac{\pi}{2}\right)\tan\left(\theta+\frac{\pi}{2}\right)d\theta$

29. $\int \tan(4x+2)\,dx$

30. $\int 3(\sin x)^{-2}\,dx$

31. $\int \cos(3z+4)\,dz$

32. $\int \sqrt{\cot x}\,\csc^2 x\,dx$

33. $\int \frac{\ln^6 x}{x}\,dx$

34. $\int \tan^7\frac{x}{2}\sec^2\frac{x}{2}\,dx$

35. $\int s^{1/3}\cos(s^{4/3}-8)\,ds$

36. $\int \frac{dx}{\sin^2 3x}$

37. $\int \frac{\sin(2t+1)}{\cos^2(2t+1)}\,dt$

38. $\int \frac{6\cos t}{(2+\sin t)^2}\,dt$

39. $\int \frac{dx}{x\ln x}$

40. $\int \tan^2 x\sec^2 x\,dx$

41. $\int \frac{x\,dx}{x^2+1}$

42. $\int \frac{40\,dx}{x^2+25}$

43. $\int \frac{dx}{\cot 3x}$

44. $\int \frac{dx}{\sqrt{5x+8}}$

45. $\int \sec x\,dx$ (*Hint:* Multiply the integrand by
$$\frac{\sec x + \tan x}{\sec x + \tan x}$$
and then use a substitution to integrate the result.)

46. $\int \csc x\,dx$ (*Hint:* Multiply the integrand by
$$\frac{\csc x + \cot x}{\csc x + \cot x}$$
and then use a substitution to integrate the result.)

In Exercises 47–52, use the given trigonometric identity to set up a u-substitution and then evaluate the indefinite integral.

47. $\int \sin^3 2x\,dx, \quad \sin^2 2x = 1 - \cos^2 2x$

48. $\int \sec^4 x\,dx, \quad \sec^2 x = 1 + \tan^2 x$

49. $\int 2\sin^2 x\,dx, \quad \cos 2x = 2\sin^2 x - 1$

50. $\int 4\cos^2 x\,dx, \quad \cos 2x = 1 - 2\cos^2 x$

51. $\int \tan^4 x\,dx, \quad \tan^2 x = \sec^2 x - 1$

52. $\int (\cos^4 x - \sin^4 x)dx, \quad \cos 2x = \cos^2 x - \sin^2 x$

In Exercises 53–66, make a u-substitution and integrate from $u(a)$ to $u(b)$.

53. $\int_0^3 \sqrt{y+1}\,dy$

54. $\int_0^1 r\sqrt{1-r^2}\,dr$

55. $\int_{-\pi/4}^0 \tan x\sec^2 x\,dx$

56. $\int_{-1}^1 \frac{5r}{(4+r^2)^2}\,dr$

57. $\int_0^1 \frac{10\sqrt{\theta}}{(1+\theta^{3/2})^2}\,d\theta$

58. $\int_{-\pi}^{\pi} \frac{\cos x}{\sqrt{4+3\sin x}}\,dx$

59. $\int_0^1 \sqrt{t^5+2t}\,(5t^4+2)\,dt$

60. $\int_0^{\pi/6} \cos^{-3}2\theta\sin 2\theta\,d\theta$

61. $\int_0^7 \frac{dx}{x+2}$

62. $\int_2^5 \frac{dx}{2x-3}$

63. $\int_1^2 \frac{dt}{t-3}$

64. $\int_{\pi/4}^{3\pi/4} \cot x\,dx$

65. $\int_{-1}^3 \frac{x\,dx}{x^2+1}$

66. $\int_0^2 \frac{e^x\,dx}{3+e^x}$

Two Routes to the Integral In Exercises 67 and 68, make a substitution $u = \cdots$ (an expression in x), $du = \cdots$. Then

(a) integrate with respect to u from $u(a)$ to $u(b)$.

(b) find an antiderivative with respect to u, replace u by the expression in x, then evaluate from a to b.

67. $\displaystyle\int_0^1 \frac{x^3}{\sqrt{x^4+9}}\,dx$

68. $\displaystyle\int_{\pi/6}^{\pi/3} (1 - \cos 3x)\sin 3x\,dx$

69. Show that

$$y = \ln\left|\frac{\cos 3}{\cos x}\right| + 5$$

is the solution to the initial value problem

$$\frac{dy}{dx} = \tan x, \quad f(3) = 5.$$

(See the discussion following Example 4, Section 5.4.)

70. Show that

$$y = \ln\left|\frac{\sin x}{\sin 2}\right| + 6$$

is the solution to the initial value problem

$$\frac{dy}{dx} = \cot x, \quad f(2) = 6.$$

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

71. True or False By u-substitution, $\int_0^{\pi/4} \tan^3 x \sec^2 x\,dx = \int_0^{\pi/4} u^3\,du$. Justify your answer.

72. True or False If f is positive and differentiable on $[a, b]$, then $\displaystyle\int_a^b \frac{f'(x)dx}{f(x)} = \ln\left(\frac{f(b)}{f(a)}\right)$. Justify your answer.

73. Multiple Choice $\int \tan x\,dx =$

(A) $\dfrac{\tan^2 x}{2} + C$

(B) $\ln|\cot x| + C$

(C) $\ln|\cos x| + C$

(D) $-\ln|\cos x| + C$

(E) $-\ln|\cot x| + C$

74. Multiple Choice $\int_0^2 e^{2x}\,dx =$

(A) $\dfrac{e^4}{2}$ **(B)** $e^4 - 1$ **(C)** $e^4 - 2$ **(D)** $2e^4 - 2$ **(E)** $\dfrac{e^4 - 1}{2}$

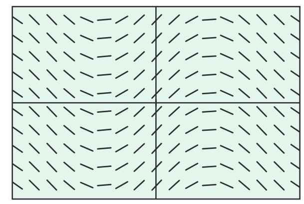
75. Multiple Choice If $\int_3^5 f(x - a)\,dx = 7$ where a is a constant, then $\int_{3-a}^{5-a} f(x)\,dx =$

(A) $7 + a$ **(B)** 7 **(C)** $7 - a$ **(D)** $a - 7$ **(E)** -7

76. Multiple Choice If the differential equation $dy/dx = f(x)$ leads to the slope field shown below, which of the following could be $\int f(x)\,dx$?

(A) $\sin x + C$ **(B)** $\cos x + C$ **(C)** $-\sin x + C$

(D) $-\cos x + C$ **(E)** $\dfrac{\sin^2 x}{2} + C$



Explorations

77. *Constant of Integration* Consider the integral

$$\int \sqrt{x+1}\,dx.$$

(a) Show that $\displaystyle\int \sqrt{x+1}\,dx = \frac{2}{3}(x+1)^{3/2} + C.$

(b) Writing to Learn Explain why

$$y_1 = \int_0^x \sqrt{t+1}\,dt \quad \text{and} \quad y_2 = \int_3^x \sqrt{t+1}\,dt$$

are antiderivatives of $\sqrt{x+1}$.

(c) Use a table of values for $y_1 - y_2$ to find the value of C for which $y_1 = y_2 + C$.

(d) Writing to Learn Give a convincing argument that

$$C = \int_0^3 \sqrt{x+1}\,dx.$$

78. Group Activity *Making Connections* Suppose that

$$\int f(x)\,dx = F(x) + C.$$

(a) Explain how you can use the derivative of $F(x) + C$ to confirm the integration is correct.

(b) Explain how you can use a slope field of f and the graph of $y = F(x)$ to support your evaluation of the integral.

(c) Explain how you can use the graphs of $y_1 = F(x)$ and $y_2 = \int_0^x f(t)\,dt$ to support your evaluation of the integral.

(d) Explain how you can use a table of values for $y_1 - y_2$, y_1 and y_2 defined as in part (c), to support your evaluation of the integral.

(e) Explain how you can use graphs of f and NDER of $F(x)$ to support your evaluation of the integral.

(f) Illustrate parts (a)–(e) for $f(x) = \dfrac{x}{\sqrt{x^2+1}}$.

79. ***Different Solutions?*** Consider the integral $\int 2 \sin x \cos x \, dx$.

(a) Evaluate the integral using the substitution $u = \sin x$.

(b) Evaluate the integral using the substitution $u = \cos x$.

(c) **Writing to Learn** Explain why the different-looking answers in parts (a) and (b) are actually equivalent.

80. ***Different Solutions?*** Consider the integral $\int 2 \sec^2 x \tan x \, dx$.

(a) Evaluate the integral using the substitution $u = \tan x$.

(b) Evaluate the integral using the substitution $u = \sec x$.

(c) **Writing to Learn** Explain why the different-looking answers in parts (a) and (b) are actually equivalent.

Extending the Ideas

81. ***Trigonometric Substitution*** Suppose $u = \sin^{-1} x$. Then $\cos u > 0$.

(a) Use the substitution $x = \sin u$, $dx = \cos u \, du$ to show that

$$\int \frac{dx}{\sqrt{1 - x^2}} = \int 1 \, du.$$

(b) Evaluate $\int 1 \, du$ to show that $\int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x + C.$

82. ***Trigonometric Substitution*** Suppose $u = \tan^{-1} x$.

(a) Use the substitution $x = \tan u$, $dx = \sec^2 u \, du$ to show that

$$\int \frac{dx}{1 + x^2} = \int 1 \, du.$$

(b) Evaluate $\int 1 \, du$ to show that $\int \frac{dx}{1 + x^2} = \tan^{-1} x + C.$

83. ***Trigonometric Substitution*** Suppose $\sqrt{x} = \sin y$.

(a) Use the substitution $x = \sin^2 y$, $dx = 2 \sin y \cos y \, dy$ to show that

$$\int_0^{1/2} \frac{\sqrt{x}\, dx}{\sqrt{1 - x}} = \int_0^{\pi/4} 2 \sin^2 y \, dy.$$

(b) Use the identity given in Exercise 49 to evaluate the definite integral without a calculator.

84. ***Trigonometric Substitution*** Suppose $u = \tan^{-1} x$.

(a) Use the substitution $x = \tan u$, $dx = \sec^2 u \, du$ to show that

$$\int_0^{\sqrt{3}} \frac{dx}{\sqrt{1 + x^2}} = \int_0^{\pi/3} \sec u \, du.$$

(b) Use the hint in Exercise 45 to evaluate the definite integral without a calculator.

6.3 Antidifferentiation by Parts

What you'll learn about

- Product Rule in Integral Form
- Solving for the Unknown Integral
- Tabular Integration
- Inverse Trigonometric and Logarithmic Functions

. . . and why

The Product Rule relates to derivatives as the technique of parts relates to antiderivatives.

Product Rule in Integral Form

When u and v are differentiable functions of x, the Product Rule for differentiation tells us that

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Integrating both sides with respect to x and rearranging leads to the integral equation

$$\int\left(u\frac{dv}{dx}\right)dx = \int\left(\frac{d}{dx}(uv)\right)dx - \int\left(v\frac{du}{dx}\right)dx$$

$$= uv - \int\left(v\frac{du}{dx}\right)dx.$$

When this equation is written in the simpler differential notation we obtain the following formula.

Integration by Parts Formula

$$\int u\,dv = uv - \int v\,du$$

This formula expresses one integral, $\int u\,dv$, in terms of a second integral, $\int v\,du$. With a proper choice of u and v, the second integral may be easier to evaluate than the first. This is the reason for the importance of the formula. When faced with an integral that we cannot handle analytically, we can replace it by one with which we might have more success.

L I P E T

If you are wondering what to choose for u, here is what we usually do. Our first choice is a natural logarithm (L), if there is one. If there isn't, we look for an inverse trigonometric function (I). If there isn't one of these either, look for a polynomial (P). Barring that, look for an exponential (E) or a trigonometric function (T). That's the preference order: **L I P E T.**

In general, we want u to be something that simplifies when differentiated, and dv to be something that remains manageable when integrated.

EXAMPLE 1 Using Integration by Parts

Evaluate $\int x\cos x\,dx$.

SOLUTION

We use the formula $\int u\,dv = uv - \int v\,du$ with

$$u = x, \qquad dv = \cos x\,dx.$$

To complete the formula, we take the differential of u and find the simplest antiderivative of $\cos x$.

$$du = dx \qquad v = \sin x$$

Then,

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C.$$

Now try Exercise 1.

Let's examine the choices available for u and v in Example 1.

EXPLORATION 1 **Choosing the Right u and dv**

Not every choice of u and dv leads to success in antidifferentiation by parts. There is always a trade-off when we replace $\int u\,dv$ with $\int v\,du$, and we gain nothing if $\int v\,du$ is no easier to find than the integral we started with. Let us look at the other choices we might have made in Example 1 to find $\int x \cos x\,dx$.

1. Apply the parts formula to $\int x \cos x\,dx$, letting $u = 1$ and $dv = x \cos x\,dx$. Analyze the result to explain why the choice of $u = 1$ is never a good one.
2. Apply the parts formula to $\int x \cos x\,dx$, letting $u = x \cos x$ and $dv = dx$. Analyze the result to explain why this is not a good choice for this integral.
3. Apply the parts formula to $\int x \cos x\,dx$, letting $u = \cos x$ and $dv = x\,dx$. Analyze the result to explain why this is not a good choice for this integral.
4. What makes x a good choice for u and $\cos x\,dx$ a good choice for dv?

The goal of integration by parts is to go from an integral $\int u\,dv$ that we don't see how to evaluate to an integral $\int v\,du$ that we can evaluate. Keep in mind that integration by parts does not always work.

Sometimes we have to use integration by parts more than once to evaluate an integral.

EXAMPLE 2 Repeated Use of Integration by Parts

Evaluate $\int x^2 e^x\,dx$.

SOLUTION

With $u = x^2$, $dv = e^x\,dx$, $du = 2x\,dx$, and $v = e^x$, we have

$$\int x^2 e^x\,dx = x^2 e^x - 2\int x e^x\,dx.$$

The new integral is less complicated than the original because the exponent on x is reduced by one. To evaluate the integral on the right, we integrate by parts again with $u = x$, $dv = e^x\,dx$. Then $du = dx$, $v = e^x$, and

$$\int x e^x\,dx = x e^x - \int e^x\,dx = x e^x - e^x + C.$$

Hence,

$$\begin{aligned}\int x^2 e^x\,dx &= x^2 e^x - 2\int x e^x\,dx \\ &= x^2 e^x - 2x e^x + 2e^x + C.\end{aligned}$$

The technique of Example 2 works for any integral $\int x^n e^x\,dx$ in which n is a positive integer, because differentiating x^n will eventually lead to zero and integrating e^x is easy. We will say more on this later in this section when we discuss *tabular integration.*

Now try Exercise 5.

EXAMPLE 3 Solving an Initial Value Problem

Solve the differential equation $dy/dx = x \ln(x)$ subject to the initial condition $y = -1$ when $x = 1$. Confirm the solution graphically by showing that it conforms to the slope field.

continued

SOLUTION

We find the antiderivative of $x \ln(x)$ by using parts. It is usually a better idea to differentiate $\ln(x)$ than to antidifferentiate it (do you see why?), so we let $u = \ln(x)$ and $dv = x\,dx$.

$$y = \int x \ln(x)\,dx$$
$$= \left(\frac{x^2}{2}\right)\ln(x) - \int \left(\frac{x^2}{2}\right)\left(\frac{1}{x}\right)dx$$
$$= \left(\frac{x^2}{2}\right)\ln(x) - \int \left(\frac{x}{2}\right)dx$$
$$= \left(\frac{x^2}{2}\right)\ln(x) - \frac{x^2}{4} + C$$

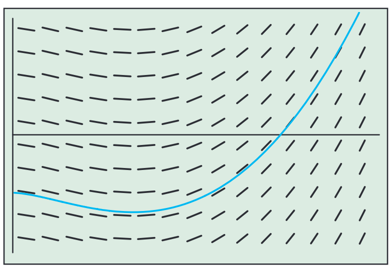
Using the initial condition,

$$-1 = \left(\frac{1}{2}\right)\ln(1) - \frac{1}{4} + C$$
$$-\frac{3}{4} = 0 + C$$
$$C = -\frac{3}{4}.$$

Thus

$$y = \left(\frac{x^2}{2}\right)\ln(x) - \frac{x^2}{4} - \frac{3}{4}.$$

Figure 6.9 shows a graph of this function superimposed on a slope field for $dy/dx = x\ln(x)$, to which it conforms nicely. ***Now try Exercise 11.***



[0, 3] by [–1.5, 1.5]

Figure 6.9 The solution to the initial value problem in Example 3 conforms nicely to a slope field of the differential equation. (Example 3)

Solving for the Unknown Integral

Integrals like the one in the next example occur in electrical engineering. Their evaluation requires two integrations by parts, followed by solving for the unknown integral.

EXAMPLE 4 Solving for the Unknown Integral

Evaluate $\int e^x \cos x\,dx$.

SOLUTION

Let $u = e^x$, $dv = \cos x\,dx$. Then $du = e^x\,dx$, $v = \sin x$, and

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx.$$

The second integral is like the first, except it has $\sin x$ in place of $\cos x$. To evaluate it, we use integration by parts with

$$u = e^x, \quad dv = \sin x\,dx, \quad v = -\cos x, \quad du = e^x\,dx.$$

Then

$$\int e^x \cos x\,dx = e^x \sin x - \left(-e^x \cos x - \int (-\cos x)(e^x\,dx)\right)$$
$$= e^x \sin x + e^x \cos x - \int e^x \cos x\,dx.$$

continued

The unknown integral now appears on both sides of the equation. Adding the integral to both sides gives

$$2\int e^x \cos x\, dx = e^x \sin x + e^x \cos x + C.$$

Dividing by 2 and renaming the constant of integration gives

$$\int e^x \cos x\, dx = \frac{e^x \sin x + e^x \cos x}{2} + C.$$

Now try Exercise 17.

When making repeated use of integration by parts in circumstances like Example 4, once a choice for u and dv is made, it is usually not a good idea to switch choices in the second stage of the problem. Doing so will result in undoing the work. For example, if we had switched to the substitution $u = \sin x$, $dv = e^x dx$ in the second integration, we would have obtained

$$\int e^x \cos x\, dx = e^x \sin x - \left(e^x \sin x - \int e^x \cos x\, dx\right)$$
$$= \int e^x \cos x\, dx,$$

undoing the first integration by parts.

Tabular Integration

We have seen that integrals of the form $\int f(x)g(x)dx$, in which f can be differentiated repeatedly to become zero and g can be integrated repeatedly without difficulty, are natural candidates for integration by parts. However, if many repetitions are required, the calculations can be cumbersome. In situations like this, there is a way to organize the calculations that saves a great deal of work. It is **tabular integration,** as shown in Examples 5 and 6.

EXAMPLE 5 Using Tabular Integration

Evaluate $\int x^2 e^x\, dx$.

SOLUTION

With $f(x) = x^2$ and $g(x) = e^x$, we list:

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^2		e^x
$2x$	$(-)$	e^x
2		e^x
0		e^x

We combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^2 e^x\, dx = x^2 e^x - 2xe^x + 2e^x + C.$$

Compare this with the result in Example 2. ***Now try Exercise 21.***

EXAMPLE 6 Using Tabular Integration

Evaluate $\int x^3 \sin x\, dx$.

SOLUTION

With $f(x) = x^3$ and $g(x) = \sin x$, we list:

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^3		$\sin x$
$3x^2$	$(-)$	$-\cos x$
$6x$		$-\sin x$
6	$(-)$	$\cos x$
0		$\sin x$

Again we combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^3 \sin x\, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C.$$

Now try Exercise 23.

Inverse Trigonometric and Logarithmic Functions

The method of parts is only useful when the integrand can be written as a product of two functions (u and dv). In fact, *any* integrand $f(x)\, dx$ satisfies that requirement, since we can let $u = f(x)$ and $dv = dx$. There are not many antiderivatives of the form $\int f(x)\, dx$ that you would want to find by parts, but there are some, most notably the antiderivatives of logarithmic and inverse trigonometric functions.

EXAMPLE 7 Antidifferentiating ln *x*

Find $\int \ln x\, dx$.

SOLUTION

If we want to use parts, we have little choice but to let $u = \ln x$ and $dv = dx$.

$$\begin{aligned}\int \ln x\, dx &= (\ln x)(x) - \int (x)\left(\frac{1}{x}\right) dx \\ &= x \ln x - \int 1\, dx \\ &= x \ln x - x + C\end{aligned}$$

EXAMPLE 8 Antidifferentiating $\sin^{-1} x$

Find the solution to the differential equation $dy/dx = \sin^{-1} x$ if the graph of the solution passes through the point $(0, 0)$.

SOLUTION

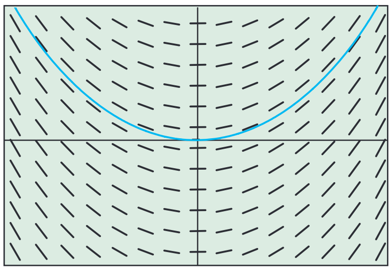
We find $\int \sin^{-1} x\, dx$, letting $u = \sin^{-1} x$, $dv = dx$.

continued

$$\int \sin^{-1} x\,dx = (\sin^{-1} x)(x) - \int (x)\left(\frac{1}{\sqrt{1-x^2}}\right) dx$$
$$= x\sin^{-1} x - \int \frac{x\,dx}{\sqrt{1-x^2}}$$
$$= x\sin^{-1} x + \frac{1}{2}\int \frac{-2x\,dx}{\sqrt{1-x^2}}$$
$$= x\sin^{-1} x + \frac{1}{2}\int u^{-1/2}du$$
$$= x\sin^{-1} x + u^{1/2} + C$$
$$= x\sin^{-1} x + \sqrt{1-x^2} + C$$

Applying the initial condition $y = 0$ when $x = 0$, we conclude that the particular solution is $y = x\sin^{-1} x + \sqrt{1-x^2} - 1$.

A graph of $y = x\sin^{-1} x + \sqrt{1-x^2} - 1$ conforms nicely to the slope field for $dy/dx = \sin^{-1} x$, as shown in Figure 6.10.



[–1, 1] by [–0.5, 0.5]

Figure 6.10 The solution to the initial value problem in Example 8 conforms nicely to the slope field of the differential equation. (Example 8)

Quick Review 6.3 *(For help, go to Sections 3.8 and 3.9.)*

In Exercises 1–4, find dy/dx.

1. $y = x^3 \sin 2x$

2. $y = e^{2x} \ln (3x + 1)$

3. $y = \tan^{-1} 2x$

4. $y = \sin^{-1} (x + 3)$

In Exercises 5 and 6, solve for x in terms of y.

5. $y = \tan^{-1} 3x$

6. $y = \cos^{-1} (x + 1)$

7. Find the area under the arch of the curve $y = \sin \pi x$ from $x = 0$ to $x = 1$.

8. Solve the differential equation $dy/dx = e^{2x}$.

9. Solve the initial value problem $dy/dx = x + \sin x$, $y(0) = 2$.

10. Use differentiation to confirm the integration formula

$$\int e^x \sin x\,dx = \frac{1}{2}e^x (\sin x - \cos x).$$

Section 6.3 Exercises

In Exercises 1–10, find the indefinite integral.

1. $\int x \sin x\,dx$

2. $\int x\,e^x\,dx$

3. $\int 3t\,e^{2t}\,dt$

4. $\int 2t \cos(3t)\,dt$

5. $\int x^2 \cos x\,dx$

6. $\int x^2 e^{-x}\,dx$

7. $\int 3x^2\,e^{2x}\,dx$

8. $\int x^2 \cos\left(\frac{x}{2}\right) dx$

9. $\int y \ln y\,dy$

10. $\int t^2 \ln t\,dt$

In Exercises 11–16, solve the initial value problem. Confirm your answer by checking that it conforms to the slope field of the differential equation.

11. $\frac{dy}{dx} = (x + 2) \sin x$ and $y = 2$ when $x = 0$

12. $\frac{dy}{dx} = 2xe^{-x}$ and $y = 3$ when $x = 0$

13. $\frac{du}{dx} = x \sec^2 x$ and $u = 1$ when $x = 0$

14. $\frac{dz}{dx} = x^3 \ln x$ and $z = 5$ when $x = 1$

15. $\frac{dy}{dx} = x\sqrt{x - 1}$ and $y = 2$ when $x = 1$

16. $\frac{dy}{dx} = 2x\sqrt{x + 2}$ and $y = 0$ when $x = -1$

In Exercises 17–20, use parts and solve for the unknown integral.

17. $\int e^x \sin x\, dx$ **18.** $\int e^{-x} \cos x\, dx$

19. $\int e^x \cos 2x\, dx$ **20.** $\int e^{-x} \sin 2x\, dx$

In Exercises 21–24, use tabular integration to find the antiderivative.

21. $\int x^4 e^{-x}\, dx$ **22.** $\int (x^2 - 5x)e^x\, dx$

23. $\int x^3 e^{-2x}\, dx$ **24.** $\int x^3 \cos 2x\, dx$

In Exercises 25–28, evaluate the integral analytically. Support your answer using NINT.

25. $\int_0^{\pi/2} x^2 \sin 2x\, dx$ **26.** $\int_0^{\pi/2} x^3 \cos 2x\, dx$

27. $\int_{-2}^{3} e^{2x} \cos 3x\, dx$ **28.** $\int_{-3}^{2} e^{-2x} \sin 2x\, dx$

In Exercises 29–32, solve the differential equation.

29. $\frac{dy}{dx} = x^2 e^{4x}$ **30.** $\frac{dy}{dx} = x^2 \ln x$

31. $\frac{dy}{d\theta} = \theta \sec^{-1} \theta, \quad \theta > 1$ **32.** $\frac{dy}{d\theta} = \theta \sec \theta \tan \theta$

33. ***Finding Area*** Find the area of the region enclosed by the x-axis and the curve $y = x \sin x$ for

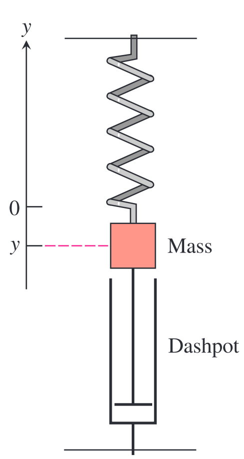
(a) $0 \le x \le \pi$, **(b)** $\pi \le x \le 2\pi$, **(c)** $0 \le x \le 2\pi$.

34. ***Finding Area*** Find the area of the region enclosed by the y-axis and the curves $y = x^2$ and $y = (x^2 + x + 1)e^{-x}$.

35. ***Average Value*** A retarding force, symbolized by the dashpot in the figure, slows the motion of the weighted spring so that the mass's position at time t is

$$y = 2e^{-t} \cos t, \quad t \ge 0.$$

Find the average value of y over the interval $0 \le t \le 2\pi$.



Standardized Test Questions

You should solve the following problems without using a graphing calculator.

36. **True or False** If $f'(x) = g(x)$, then $\int x\, g(x)\, dx = x\, f(x) - \int f(x)\, dx$. Justify your answer.

37. **True or False** If $f'(x) = g(x)$, then $\int x^2\, g(x)\, dx = x^2 f(x) - 2\int x\, f(x)\, dx$. Justify your answer.

38. **Multiple Choice** If $\int x^2 \cos x\, dx = h(x) - \int 2x \sin x\, dx$, then $h(x) =$

(A) $2 \sin x + 2x \cos x + C$

(B) $x^2 \sin x + C$

(C) $2x \cos x - x^2 \sin x + C$

(D) $4 \cos x - 2x \sin x + C$

(E) $(2 - x^2) \cos x - 4 \sin x + C$

39. **Multiple Choice** $\int x \sin(5x)\, dx =$

(A) $-x \cos (5x) + \sin (5x) + C$

(B) $-\frac{x}{5}\cos (5x) + \frac{1}{25}\sin (5x) + C$

(C) $-\frac{x}{5}\cos (5x) + \frac{1}{5}\sin (5x) + C$

(D) $\frac{x}{5}\cos (5x) + \frac{1}{25}\sin (5x) + C$

(E) $5x \cos (5x) - \sin (5x) + C$

40. **Multiple Choice** $\int x \csc^2 x\, dx =$

(A) $\frac{x^2 \csc^3 x}{6} + C$

(B) $x \cot x - \ln |\sin x| + C$

(C) $-x \cot x + \ln |\sin x| + C$

(D) $-x \cot x - \ln |\sin x| + C$

(E) $-x \sec^2 x - \tan x + C$

41. **Multiple Choice** The graph of $y = f(x)$ conforms to the slope field for the differential equation $dy/dx = 4x \ln x$, as shown in the graph below. Which of the following could be $f(x)$?

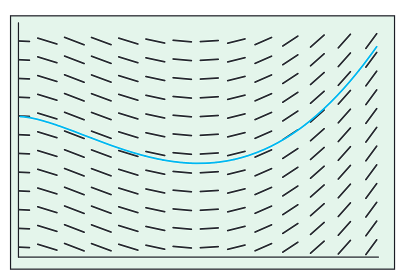
(A) $2x^2 (\ln x)^2 + 3$

(B) $x^3 \ln x + 3$

(C) $2x^2 \ln x - x^2 + 3$

(D) $(2x^2 + 3) \ln x - 1$

(E) $2x (\ln x)^2 - \frac{4}{3}(\ln x)^3 + 3$



[0, 2] by [0, 5]

Explorations

42. Consider the integral $\int x^n e^x \, dx$. Use integration by parts to evaluate the integral if

(a) $n = 1$.

(b) $n = 2$.

(c) $n = 3$.

(d) Conjecture the value of the integral for any positive integer n.

(e) **Writing to Learn** Give a convincing argument that your conjecture in part (d) is true.

In Exercises 43–46, evaluate the integral by using a substitution prior to integration by parts.

43. $\int \sin \sqrt{x} \, dx$

44. $\int e^{\sqrt{3x+9}} \, dx$

45. $\int x^7 e^{x^2} \, dx$

46. $\int \sin (\ln r) \, dr$

In Exercises 47–50, use integration by parts to establish the *reduction formula.*

47. $\int x^n \cos x \, dx = x^n \sin x - n \int x^{n-1} \sin x \, dx$

48. $\int x^n \sin x \, dx = -x^n \cos x + n \int x^{n-1} \cos x \, dx$

49. $\int x^n e^{ax} \, dx = \dfrac{x^n e^{ax}}{a} - \dfrac{n}{a} \int x^{n-1} e^{ax} \, dx, \quad a \neq 0$

50. $\int (\ln x)^n \, dx = x(\ln x)^n - n \int (\ln x)^{n-1} \, dx$

Extending the Ideas

51. ***Integrating Inverse Functions*** Assume that the function f has an inverse.

(a) Show that $\int f^{-1}(x) \, dx = \int y f'(y) \, dy$. (*Hint:* Use the substitution $y = f^{-1}(x)$.)

(b) Use integration by parts on the second integral in part (a) to show that

$$\int f^{-1}(x) \, dx = \int y f'(y) \, dy = x f^{-1}(x) - \int f(y) \, dy.$$

52. ***Integrating Inverse Functions*** Assume that the function f has an inverse. Use integration by parts directly to show that

$$\int f^{-1}(x) \, dx = x f^{-1}(x) - \int x \left(\frac{d}{dx} f^{-1}(x) \right) dx.$$

In Exercises 53–56, evaluate the integral using

(a) the technique of Exercise 51.

(b) the technique of Exercise 52.

(c) Show that the expressions (with $C = 0$) obtained in parts (a) and (b) are the same.

53. $\int \sin^{-1} x \, dx$

54. $\int \tan^{-1} x \, dx$

55. $\int \cos^{-1} x \, dx$

56. $\int \log_2 x \, dx$

Quick Quiz for AP* Preparation: Sections 6.1–6.3

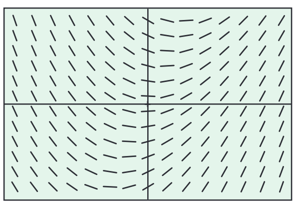
You should solve the following problems without using a graphing calculator.

1. Multiple Choice Which of the following differential equations would produce the slope field shown below?

(A) $\dfrac{dy}{dx} = y - 3x$ **(B)** $\dfrac{dy}{dx} = y - \dfrac{x}{3}$

(C) $\dfrac{dy}{dx} = y + \dfrac{x}{3}$ **(D)** $\dfrac{dy}{dx} = y + \dfrac{x}{3}$

(E) $\dfrac{dy}{dx} = x - \dfrac{y}{3}$



2. Multiple Choice If the substitution $\sqrt{x} = \sin y$ is made in the integrand of $\displaystyle\int_0^{1/2} \frac{\sqrt{x}}{\sqrt{1-x}}\,dx$, the resulting integral is

(A) $\displaystyle\int_0^{1/2} \sin^2 y\, dy$ **(B)** $\displaystyle 2\int_0^{1/2} \frac{\sin^2 y}{\cos y}\, dy$

(C) $\displaystyle 2\int_0^{\pi/4} \sin^2 y\, dy$ **(D)** $\displaystyle\int_0^{\pi/4} \sin^2 y\, dy$

(E) $\displaystyle 2\int_0^{\pi/6} \sin^2 y\, dy$

3. Multiple Choice $\int x\, e^{2x}\, dx =$

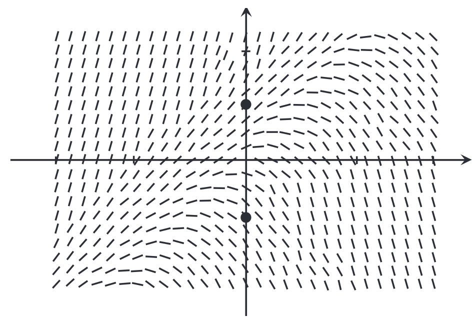
(A) $\dfrac{x\,e^{2x}}{2} - \dfrac{e^{2x}}{4} + C$ **(B)** $\dfrac{x\,e^{2x}}{2} - \dfrac{e^{2x}}{2} + C$

(C) $\dfrac{x\,e^{2x}}{2} + \dfrac{e^{2x}}{4} + C$ **(D)** $\dfrac{x\,e^{2x}}{2} + \dfrac{e^{2x}}{2} + C$

(E) $\dfrac{x^2\,e^{2x}}{4} + C$

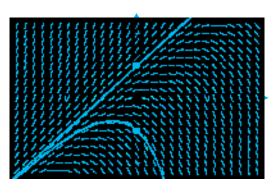
4. Free Response Consider the differential equation $dy/dx = 2y - 4x$.

(a) The slope field for the differential equation is shown below. Sketch the solution curve that passes through the point (0, 1) and sketch the solution curve that goes through the point (0, −1).



(b) There is a value of b for which $y = 2x + b$ is a solution to the differential equation. Find this value of b. Justify your answer.

(c) Let g be the function that satisfies the given differential equation with the initial condition $g(0) = 0$. It appears from the slope field that g has a local maximum at the point (0, 0). Using the differential equation, prove analytically that this is so.



6.4 Exponential Growth and Decay

What you'll learn about

- Separable Differential Equations
- Law of Exponential Change
- Continuously Compounded Interest
- Radioactivity
- Modeling Growth with Other Bases
- Newton's Law of Cooling

. . . and why

Understanding the differential equation $dy/dx = ky$ gives us new insight into exponential growth and decay.

Separable Differential Equations

Before we revisit the topic of exponential growth (last seen as a precalculus topic in Chapter P), we need to introduce the concept of separable differential equations.

DEFINITION Separable Differential Equation

A differential equation of the form $dy/dx = f(y)g(x)$ is called **separable.** We **separate the variables** by writing it in the form

$$\frac{1}{f(y)}\,dy = g(x)\,dx.$$

The solution is found by antidifferentiating each side with respect to its thusly isolated variable.

EXAMPLE 1 Solving by Separation of Variables

Solve for y if $dy/dx = (xy)^2$ and $y = 1$ when $x = 1$.

SOLUTION

The equation is separable because it can be written in the form $dy/dx = y^2x^2$, where $f(y) = y^2$ and $g(x) = x^2$. We separate the variables and antidifferentiate as follows.

$$y^{-2}\,dy = x^2\,dx$$

$$\int y^{-2}\,dy = \int x^2\,dx$$

$$-y^{-1} = \frac{x^3}{3} + C$$

We then apply the initial condition to find C.

$$-1 = \frac{1}{3} + C \Rightarrow C = -\frac{4}{3}$$

$$-y^{-1} = \frac{x^3}{3} - \frac{4}{3}$$

$$y^{-1} = \frac{4 - x^3}{3}$$

$$y = \frac{3}{4 - x^3}$$

This solution is valid for the continuous section of the function that goes through the point (1, 1), that is, on the domain $(-\infty, \sqrt[3]{4})$.

It is apparent that $y = 1$ when $x = 1$, but it is worth checking that $dy/dx = (xy)^2$.

$$y = \frac{3}{4 - x^3}$$

$$\frac{dy}{dx} = -3(4 - x^3)^{-2}(-3x^2)$$

$$\frac{dy}{dx} = \frac{9x^2}{(4 - x^3)^2} = x^2\left(\frac{3}{4 - x^3}\right)^2 = x^2y^2 = (xy)^2$$

Now try Exercise 1.

Law of Exponential Change

You have probably solved enough exponential growth problems by now to recognize that they involve growth in which the rate of change is proportional to the amount present. The more bacteria in the dish, the faster they multiply. The more radioactive material present, the faster it decays. The greater your bank account (assuming it earns compounded interest), the faster it grows.

The differential equation that describes this growth is $dy/dt = ky$, where k is the *growth constant* (if positive) or the *decay constant* (if negative). We can solve this equation by separating the variables.

$$\frac{dy}{dt} = ky$$

$$\frac{1}{y}\,dy = k\,dt$$

$$\ln|y| = kt + C$$

$$|y| = e^{kt+C}$$

$$|y| = e^C e^{kt}$$

$$y = \pm e^C e^{kt}$$

$$y = Ae^{kt}$$

What if $A = 0$?

If $A = 0$, then the solution to $dy/dt = ky$ is the constant function $y = 0$. This function is technically of the form $y = Ae^{kt}$, but we do not consider it to be an exponential function. The initial condition in this case leads to a "trivial" solution.

This solution shows that the *only* growth function that results in a growth rate proportional to the amount present is, in fact, exponential. Note that the constant A is the amount present when $t = 0$, so it is usually denoted y_0.

The Law of Exponential Change

If y changes at a rate proportional to the amount present (that is, if $dy/dt = ky$), and if $y = y_0$ when $t = 0$, then

$$y = y_0 e^{kt}.$$

The constant k is the **growth constant** if $k > 0$ or the **decay constant** if $k < 0$.

Now try Exercise 11.

Continuously Compounded Interest

Suppose that A_0 dollars are invested at a fixed annual interest rate r (expressed as a decimal). If interest is added to the account k times a year, the amount of money present after t years is

$$A(t) = A_0\left(1 + \frac{r}{k}\right)^{kt}.$$

The interest might be added ("compounded," bankers say) monthly ($k = 12$), weekly ($k = 52$), daily ($k = 365$), or even more frequently, by the hour or by the minute.

If, instead of being added at discrete intervals, the interest is added continuously at a rate proportional to the amount in the account, we can model the growth of the account with the initial value problem.

Differential equation: $\dfrac{dA}{dt} = rA$

Initial condition: $A(0) = A_0$

It can be shown that

$$\lim_{k\to\infty} A_0\left(1+\frac{r}{k}\right)^{kt} = A_0e^{rt}.$$

We will see how this limit is evaluated in Section 8.2, Exercise 57.

The amount of money in the account after t years is then

$$A(t) = A_0e^{rt}.$$

Interest paid according to this formula is said to be **compounded continuously.** The number r is the **continuous interest rate.**

EXAMPLE 2 Compounding Interest Continuously

Suppose you deposit \$800 in an account that pays 6.3% annual interest. How much will you have 8 years later if the interest is **(a)** compounded continuously? **(b)** compounded quarterly?

SOLUTION

Here $A_0 = 800$ and $r = 0.063$. The amount in the account to the nearest cent after 8 years is

(a) $A(8) = 800e^{(0.063)(8)} = 1324.26.$

(b) $A(8) = 800\left(1+\dfrac{0.063}{4}\right)^{(4)(8)} = 1319.07.$

You might have expected to generate more than an additional \$5.19 with interest compounded continuously. ***Now try Exercise 19.***

Radioactivity

For radium-226, which used to be painted on watch dials to make them glow at night (a dangerous practice for the painters, who licked their brush-tips), t is measured in years and $k = 4.3 \times 10^{-4}$. For radon-222 gas, t is measured in days and $k = 0.18$. The decay of radium in the earth's crust is the source of the radon we sometimes find in our basements.

When an atom emits some of its mass as radiation, the remainder of the atom reforms to make an atom of some new element. This process of radiation and change is **radioactive decay,** and an element whose atoms go spontaneously through this process is **radioactive.** Radioactive carbon-14 decays into nitrogen. Radium, through a number of intervening radioactive steps, decays into lead.

Experiments have shown that at any given time the rate at which a radioactive element decays (as measured by the number of nuclei that change per unit of time) is approximately proportional to the number of radioactive nuclei present. Thus, the decay of a radioactive element is described by the equation $dy/dt = -ky$, $k > 0$. If y_0 is the number of radioactive nuclei present at time zero, the number still present at any later time t will be

$$y = y_0e^{-kt}, \qquad k > 0.$$

Convention

It is conventional to use $-k$ $(k > 0)$ here instead of k $(k < 0)$ to emphasize that y is decreasing.

The **half-life** of a radioactive element is the time required for half of the radioactive nuclei present in a sample to decay. Example 3 shows the surprising fact that the half-life is a constant that depends only on the radioactive substance and not on the number of radioactive nuclei present in the sample.

EXAMPLE 3 Finding Half-Life

Find the half-life of a radioactive substance with decay equation $y = y_0e^{-kt}$ and show that the half-life depends only on k.

SOLUTION

Model The half-life is the solution to the equation

$$y_0e^{-kt} = \frac{1}{2}y_0.$$

continued

Solve Algebraically

$$e^{-kt} = \frac{1}{2}$$

$$-kt = \ln \frac{1}{2}$$

$$t = -\frac{1}{k}\ln\frac{1}{2} = \frac{\ln 2}{k}$$

Interpret This value of t is the half-life of the element. It depends only on the value of k. Note that the number y_0 does not appear. *Now try Exercise 21.*

DEFINITION Half-life

The **half-life** of a radioactive substance with rate constant k $(k > 0)$ is

$$\text{half-life} = \frac{\ln 2}{k}.$$

Modeling Growth with Other Bases

As we have seen, the differential equation $dy/dt = ky$ leads to the exponential solution

$$y = y_0\, e^{kt},$$

where y_0 is the value of y at $t = 0$. We can also write this solution in the form

$$y = y_0\, b^{ht},$$

where b is any positive number not equal to 1, and h is another rate constant, related to k by the equation $k = h \ln b$. This means that exponential growth can be modeled in *any* positive base not equal to 1, enabling us to choose a convenient base to fit a given growth pattern, as the following exploration shows.

EXPLORATION 1 Choosing a Convenient Base

A certain population y is growing at a continuous rate so that the population doubles every 5 years.

1. Let $y = y_0\, 2^{ht}$. Since $y = 2\, y_0$ when $t = 5$, what is h? What is the relationship of h to the doubling period?
2. How long does it take for the population to triple?

A certain population y is growing at a continuous rate so that the population triples every 10 years.

3. Let $y = y_0\, 3^{ht}$. Since $y = 3\, y_0$ when $t = 10$, what is h? What is the relationship of h to the tripling period?
4. How long does it take for the population to double?

A certain isotope of sodium (Na-24) has a half-life of 15 hours. That is, half the atoms of Na-24 disintegrate into another nuclear form in fifteen hours.

5. Let $A = A_0(1/2)^{ht}$. Since $y = (1/2)\, y_0$ when $t = 15$, what is h? What is the relationship of h to the half-life?
6. How long does it take for the amount of radioactive material to decay to 10% of the original amount?

It is important to note that while the exponential growth model $y = y_0\, b^{ht}$ satisfies the differential equation $dy/dt = ky$ for any positive base b, it is only when $b = e$ that the growth constant k appears in the exponent as the coefficient of t. In general, the coefficient of t is the reciprocal of the time period required for the population to grow (or decay) by a factor of b.

Hornets Aplenty

The hornet population in Example 4 can grow exponentially for a while, but the ecosystem cannot sustain such growth all summer. The model will eventually become logistic, as we will see in the next section. Nevertheless, given the right conditions, a bald-faced hornet nest can grow to be bigger than a basketball and house more than 600 workers.

EXAMPLE 4 Choosing a Base

At the beginning of the summer, the population of a hive of bald-faced hornets (which are actually wasps) is growing at a rate proportional to the population. From a population of 10 on May 1, the number of hornets grows to 50 in thirty days. If the growth continues to follow the same model, how many days after May 1 will the population reach 100?

SOLUTION

Since $dy/dt = ky$, the growth is exponential. Noting that the population grows by a factor of 5 in 30 days, we model the growth in base 5: $y = 10 \times 5^{(1/30)t}$. Now we need only solve the equation $100 = 10 \times 5^{(1/30)t}$ for t:

$$100 = 10 \times 5^{(1/30)t}$$
$$10 = 5^{(1/30)t}$$
$$\ln 10 = (1/30)t \ln 5$$
$$t = 30\left(\frac{\ln 10}{\ln 5}\right) = 42.920$$

Approximately 43 days will pass after May 1 before the population reaches 100.

Now try Exercise 23.

Carbon-14 Dating

The decay of radioactive elements can sometimes be used to date events from earth's past. The ages of rocks more than 2 billion years old have been measured by the extent of the radioactive decay of uranium (half-life 4.5 billion years!). In a living organism, the ratio of radioactive carbon, carbon-14, to ordinary carbon stays fairly constant during the lifetime of the organism, being approximately equal to the ratio in the organism's surroundings at the time. After the organism's death, however, no new carbon is ingested, and the proportion of carbon-14 decreases as the carbon-14 decays. It is possible to estimate the ages of fairly old organic remains by comparing the proportion of carbon-14 they contain with the proportion assumed to have been in the organism's environment at the time it lived. Archeologists have dated shells (which contain $CaCO_3$), seeds, and wooden artifacts this way. The estimate of 15,500 years for the age of the cave paintings at Lascaux, France, is based on carbon-14 dating. After generations of controversy, the Shroud of Turin, long believed by many to be the burial cloth of Christ, was shown by carbon-14 dating in 1988 to have been made after A.D. 1200.

EXAMPLE 5 Using Carbon-14 Dating

Scientists who use carbon-14 dating use 5700 years for its half-life. Find the age of a sample in which 10% of the radioactive nuclei originally present have decayed.

SOLUTION

We model the exponential decay in base 1/2: $A = A_0(1/2)^{t/5700}$. We seek the value of t for which $0.9A_0 = A_0(1/2)^{t/5700}$, or $(1/2)^{t/5700} = 0.9$.

Solving algebraically with logarithms,

$$(1/2)^{t/5700} = 0.9$$
$$(t/5700)\ln(1/2) = \ln(0.9)$$
$$t = 5700\left(\frac{\ln(0.9)}{\ln(0.5)}\right)$$
$$t \approx 866.$$

Interpreting the answer, we conclude that the sample is about 866 years old.

Now try Exercise 25.

Newton's Law of Cooling

Soup left in a cup cools to the temperature of the surrounding air. A hot silver ingot immersed in water cools to the temperature of the surrounding water. In situations like these, the rate at which an object's temperature is changing at any given time is roughly proportional to the difference between its temperature and the temperature of the surrounding medium.

J. Ernest Wilkins, Jr.
(1923–)

By the age of nineteen, J. Ernest Wilkins had earned a Ph.D. degree in Mathematics from the University of Chicago. He then taught, served on the Manhattan project (the goal of which was to build the first atomic bomb), and worked as a mathematician and physicist for several corporations. In 1970, Dr. Wilkins joined the faculty at Howard University and served as head of the electrical engineering, physics, chemistry, and mathematics departments before retiring. He is currently working as Distinguished Professor of Applied Mathematics and Mathematical Physics at Clark Atlanta University.

This observation is *Newton's Law of Cooling,* although it applies to warming as well, and there is an equation for it.

If T is the temperature of the object at time t, and T_s is the surrounding temperature, then

$$\frac{dT}{dt} = -k(T - T_s). \tag{1}$$

Since $dT = d(T - T_s)$, Equation 1 can be written as

$$\frac{d}{dt}(T - T_s) = -k(T - T_s).$$

Its solution, by the law of exponential change, is

$$T - T_s = (T_0 - T_s)e^{-kt},$$

where T_0 is the temperature at time $t = 0$. This equation also bears the name **Newton's Law of Cooling.**

EXAMPLE 6 Using Newton's Law of Cooling

A hard-boiled egg at 98°C is put in a pan under running 18°C water to cool. After 5 minutes, the egg's temperature is found to be 38°C. How much longer will it take the egg to reach 20°C?

SOLUTION

Model Using Newton's Law of Cooling with $T_s = 18$ and $T_0 = 98$, we have

$$T - 18 = (98 - 18)e^{-kt} \quad \text{or} \quad T = 18 + 80e^{-kt}.$$

To find k we use the information that $T = 38$ when $t = 5$.

$$\begin{aligned} 38 &= 18 + 80e^{-5k} \\ e^{-5k} &= \frac{1}{4} \\ -5k &= \ln\frac{1}{4} = -\ln 4 \\ k &= \frac{1}{5}\ln 4 \end{aligned}$$

The egg's temperature at time t is $T = 18 + 80e^{-(0.2 \ln 4)t}$.

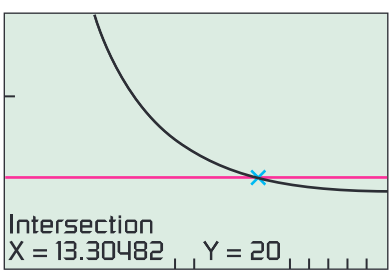
Solve Graphically We can now use a grapher to find the time when the egg's temperature is 20°C. Figure 6.11 shows that the graphs of

$$y = 20 \quad \text{and} \quad y = T = 18 + 80e^{-(0.2 \ln 4)t}$$

intersect at about $t = 13.3$.

Interpret The egg's temperature will reach 20°C in about 13.3 min after it is put in the pan under running water to cool. Because it took 5 min to reach 38°C, it will take slightly more than 8 additional minutes to reach 20°C.

Now try Exercise 31.



[0, 20] by [10, 40]

Figure 6.11 The egg will reach 20°C about 13.3 min after being placed in the pan to cool. (Example 6)

The next example shows how to use exponential regression to fit a function to real data. A CBL™ temperature probe was used to collect the data.

Table 6.1 Experimental Data

Time (sec)	T (°C)	$T - T_s$ (°C)
2	64.8	60.3
5	49.0	44.5
10	31.4	26.9
15	22.0	17.5
20	16.5	12.0
25	14.2	9.7
30	12.0	7.5

EXAMPLE 7 Using Newton's Law of Cooling

A temperature probe (thermometer) is removed from a cup of coffee and placed in water that has a temperature of $T_s = 4.5$°C. Temperature readings T, as recorded in Table 6.1, are taken after 2 sec, 5 sec, and every 5 sec thereafter. Estimate

(a) the coffee's temperature at the time the temperature probe was removed.

(b) the time when the temperature probe reading will be 8°C.

SOLUTION

Model According to Newton's Law of Cooling, $T - T_s = (T_0 - T_s)e^{-kt}$, where $T_s = 4.5$ and T_0 is the temperature of the coffee (probe reading) at $t = 0$.

We use exponential regression to find that

$$T - 4.5 = 61.66(0.9277^t)$$

is a model for the $(t, T - T_s) = (t, T - 4.5)$ data.

Thus,

$$T = 4.5 + 61.66(0.9277^t)$$

is a model for the (t, T) data.

Figure 6.12a shows the graph of the model superimposed on a scatter plot of the (t, T) data.

(a) At time $t = 0$, when the probe was removed, the temperature was

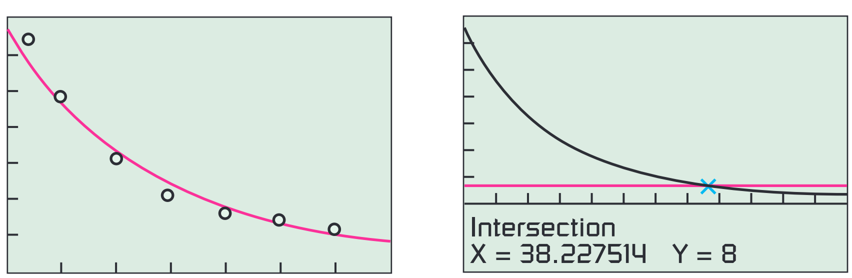
$$T = 4.5 + 61.66(0.9277^0) \approx 66.16\text{°C}.$$

(b) Solve Graphically Figure 6.12b shows that the graphs of

$$y = 8 \quad \text{and} \quad y = T = 4.5 + 61.66(0.9277^t)$$

intersect at about $t = 38$.

Interpret The temperature of the coffee was about 66.2°C when the temperature probe was removed. The temperature probe will reach 8°C about 38 sec after it is removed from the coffee and placed in the water.



[0, 35] by [0, 70]

(a)

[0, 60] by [–20, 70]

(b)

Figure 6.12 (Example 7)

Now try Exercise 33.

Quick Review 6.4

You should solve the following problems without using a graphing calculator.

In Exercises 1 and 2, rewrite the equation in exponential form or logarithmic form.

1. $\ln a = b$ **2.** $e^c = d$

In Exercises 3–8, solve the equation.

3. $\ln(x + 3) = 2$ **4.** $100e^{2x} = 600$

5. $0.85^x = 2.5$ **6.** $2^{k+1} = 3^k$

7. $1.1^t = 10$ **8.** $e^{-2t} = \frac{1}{4}$

In Exercises 9 and 10, solve for y.

9. $\ln(y + 1) = 2x - 3$

10. $\ln|y + 2| = 3t - 1$

Section 6.4 Exercises

In Exercises 1–10, use separation of variables to solve the initial value problem. Indicate the domain over which the solution is valid.

1. $\frac{dy}{dx} = \frac{x}{y}$ and $y = 2$ when $x = 1$

2. $\frac{dy}{dx} = -\frac{x}{y}$ and $y = 3$ when $x = 4$

3. $\frac{dy}{dx} = \frac{y}{x}$ and $y = 2$ when $x = 2$

4. $\frac{dy}{dx} = 2xy$ and $y = 3$ when $x = 0$

5. $\frac{dy}{dx} = (y + 5)(x + 2)$ and $y = 1$ when $x = 0$

6. $\frac{dy}{dx} = \cos^2 y$ and $y = 0$ when $x = 0$

7. $\frac{dy}{dx} = (\cos x)e^{y+\sin x}$ and $y = 0$ when $x = 0$

8. $\frac{dy}{dx} = e^{x-y}$ and $y = 2$ when $x = 0$

9. $\frac{dy}{dx} = -2xy^2$ and $y = 0.25$ when $x = 1$

10. $\frac{dy}{dx} = \frac{4\sqrt{y}\ln x}{x}$ and $y = 1$ when $x = e$

In Exercises 11–14, find the solution of the differential equation $dy/dt = ky$, k a constant, that satisfies the given conditions.

11. $k = 1.5$, $y(0) = 100$ **12.** $k = -0.5$, $y(0) = 200$

13. $y(0) = 50$, $y(5) = 100$ **14.** $y(0) = 60$, $y(10) = 30$

In Exercises 15–18, complete the table for an investment if interest is compounded continuously.

	Initial Deposit ($)	Annual Rate (%)	Doubling Time (yr)	Amount in 30 yr ($)
15.	1000	8.6		
16.	2000		15	
17.		5.25		2898.44
18.	1200			10,405.37

In Exercises 19 and 20, find the amount of time required for a $2000 investment to double if the annual interest rate r is compounded **(a)** annually, **(b)** monthly, **(c)** quarterly, and **(d)** continuously.

19. $r = 4.75\%$ **20.** $r = 8.25\%$

21. ***Half-Life*** The radioactive decay of Sm-151 (an isotope of samarium) can be modeled by the differential equation $dy/dt = -0.0077y$, where t is measured in years. Find the half-life of Sm-151.

22. ***Half-Life*** An isotope of neptunium (Np-240) has a half-life of 65 minutes. If the decay of Np-240 is modeled by the differential equation $dy/dt = -ky$, where t is measured in minutes, what is the decay constant k?

23. ***Growth of Cholera Bacteria*** Suppose that the cholera bacteria in a colony grows unchecked according to the Law of Exponential Change. The colony starts with 1 bacterium and doubles in number every half hour.

(a) How many bacteria will the colony contain at the end of 24 h?

(b) **Writing to Learn** Use part (a) to explain why a person who feels well in the morning may be dangerously ill by evening even though, in an infected person, many bacteria are destroyed.

24. ***Bacteria Growth*** A colony of bacteria is grown under ideal conditions in a laboratory so that the population increases exponentially with time. At the end of 3 h there are 10,000 bacteria. At the end of 5 h there are 40,000 bacteria. How many bacteria were present initially?

25. ***Radon-222*** The decay equation for radon-222 gas is known to be $y = y_0e^{-0.18t}$, with t in days. About how long will it take the amount of radon in a sealed sample of air to decay to 90% of its original value?

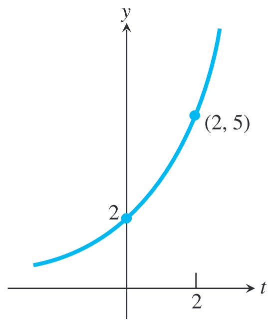
26. ***Polonium-210*** The number of radioactive atoms remaining after t days in a sample of polonium-210 that starts with y_0 radioactive atoms is $y = y_0e^{-0.005t}$.

(a) Find the element's half-life.

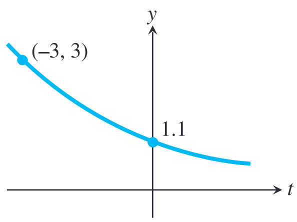
(b) Your sample will not be useful to you after 95% of the radioactive nuclei present on the day the sample arrives have disintegrated. For about how many days after the sample arrives will you be able to use the polonium?

In Exercises 27 and 28, find the exponential function $y = y_0 e^{kt}$ whose graph passes through the two points.

27.



28.



29. *Mean Life of Radioactive Nuclei* Physicists using the radioactive decay equation $y = y_0 e^{-kt}$ call the number $1/k$ the *mean life* of a radioactive nucleus. The mean life of a radon-222 nucleus is about $1/0.18 \approx 5.6$ days. The mean life of a carbon-14 nucleus is more than 8000 years. Show that 95% of the radioactive nuclei originally present in any sample will disintegrate within three mean lifetimes, that is, by time $t = 3/k$. Thus, the mean life of a nucleus gives a quick way to estimate how long the radioactivity of a sample will last.

30. *Finding the Original Temperature of a Beam* An aluminum beam was brought from the outside cold into a machine shop where the temperature was held at 65°F. After 10 min, the beam warmed to 35°F and after another 10 min its temperature was 50°F. Use Newton's Law of Cooling to estimate the beam's initial temperature.

31. *Cooling Soup* Suppose that a cup of soup cooled from 90°C to 60°C in 10 min in a room whose temperature was 20°C. Use Newton's Law of Cooling to answer the following questions.

(a) How much longer would it take the soup to cool to 35°C?

(b) Instead of being left to stand in the room, the cup of 90°C soup is put into a freezer whose temperature is −15°C. How long will it take the soup to cool from 90°C to 35°C?

32. *Cooling Silver* The temperature of an ingot of silver is 60°C above room temperature right now. Twenty minutes ago, it was 70°C above room temperature. How far above room temperature will the silver be

(a) 15 minutes from now?

(b) 2 hours from now?

(c) When will the silver be 10°C above room temperature?

33. *Temperature Experiment* A temperature probe is removed from a cup of coffee and placed in water whose temperature (T_s) is 10°C. The data in Table 6.2 were collected over the next 30 sec with a CBL™ temperature probe.

Table 6.2 Experimental Data

Time (sec)	T (°C)	$T - T_s$ (°C)
2	80.47	70.47
5	69.39	59.39
10	49.66	39.66
15	35.26	25.26
20	28.15	18.15
25	23.56	13.56
30	20.62	10.62

(a) Find an exponential regression equation for the $(t, T - T_s)$ data.

(b) Use the regression equation in part (a) to find a model for the (t, T) data. Superimpose the graph of the model on a scatter plot of the (t, T) data.

(c) Estimate when the temperature probe will read 12°C.

(d) Estimate the coffee's temperature when the temperature probe was removed.

34. *A Very Cool Experiment* A temperature probe is removed from a cup of hot chocolate and placed in ice water (temperature $T_s = 0°C$). The data in Table 6.3 were collected over the next 30 seconds.

Table 6.3 Experimental Data

Time (sec)	Temperature (°C)
2	74.68
5	61.99
10	34.89
15	21.95
20	15.36
25	11.89
30	10.02

(a) Writing to Learn Explain why temperature in this experiment can be modeled as an exponential function of time.

(b) Use exponential regression to find the best exponential model. Superimpose a graph of the model on a scatter plot of the (*time, temperature*) data.

(c) Estimate when the probe will reach 5°C.

(d) Estimate the temperature of the hot chocolate when the probe was removed.

35. *Dating Crater Lake* The charcoal from a tree killed in the volcanic eruption that formed Crater Lake in Oregon contained 44.5% of the carbon-14 found in living matter. About how old is Crater Lake?

36. ***Carbon-14 Dating Measurement Sensitivity*** To see the effect of a relatively small error in the estimate of the amount of carbon-14 in a sample being dated, answer the following questions about this hypothetical situation.

(a) A fossilized bone found in central Illinois in the year A.D. 2000 contains 17% of its original carbon-14 content. Estimate the year the animal died.

(b) Repeat part (a) assuming 18% instead of 17%.

(c) Repeat part (a) assuming 16% instead of 17%.

37. What is the half-life of a substance that decays to 1/3 of its original radioactive amount in 5 years?

38. A savings account earning compound interest triples in value in 10 years. How long will it take for the original investment to quadruple?

39. ***The Inversion of Sugar*** The processing of raw sugar has an "inversion" step that changes the sugar's molecular structure. Once the process has begun, the rate of change of the amount of raw sugar is proportional to the amount of raw sugar remaining. If 1000 kg of raw sugar reduces to 800 kg of raw sugar during the first 10 h, how much raw sugar will remain after another 14 h?

40. ***Oil Depletion*** Suppose the amount of oil pumped from one of the canyon wells in Whittier, California, decreases at the continuous rate of 10% per year. When will the well's output fall to one-fifth of its present level?

41. ***Atmospheric Pressure*** Earth's atmospheric pressure p is often modeled by assuming that the rate dp/dh at which p changes with the altitude h above sea level is proportional to p. Suppose that the pressure at sea level is 1013 millibars (about 14.7 lb/in^2) and that the pressure at an altitude of 20 km is 90 millibars.

(a) Solve the initial value problem

Differential equation: $\dfrac{dp}{dh} = kp,$

Initial condition: $p = p_0$ when $h = 0,$

to express p in terms of h. Determine the values of p_0 and k from the given altitude-pressure data.

(b) What is the atmospheric pressure at $h = 50$ km?

(c) At what altitude does the pressure equal 900 millibars?

42. ***First Order Chemical Reactions*** In some chemical reactions the rate at which the amount of a substance changes with time is proportional to the amount present. For the change of δ-glucono lactone into gluconic acid, for example,

$$\frac{dy}{dt} = -0.6y$$

when y is measured in grams and t is measured in hours. If there are 100 grams of a δ-glucono lactone present when $t = 0$, how many grams will be left after the first hour?

43. ***Discharging Capacitor Voltage*** Suppose that electricity is draining from a capacitor at a rate proportional to the voltage V across its terminals and that, if t is measured in seconds,

$$\frac{dV}{dt} = -\frac{1}{40}V.$$

(a) Solve this differential equation for V, using V_0 to denote the value of V when $t = 0$.

(b) How long will it take the voltage to drop to 10% of its original value?

44. ***John Napier's Answer*** John Napier (1550–1617), the Scottish laird who invented logarithms, was the first person to answer the question, "What happens if you invest an amount of money at 100% yearly interest, compounded continuously?"

(a) **Writing to Learn** What does happen? Explain.

(b) How long does it take to triple your money?

(c) **Writing to Learn** How much can you earn in a year?

45. ***Benjamin Franklin's Will*** The Franklin Technical Institute of Boston owes its existence to a provision in a codicil to Benjamin Franklin's will. In part the codicil reads:

> I wish to be useful even after my Death, if possible, in forming and advancing other young men that may be serviceable to their Country in both Boston and Philadelphia. To this end I devote Two thousand Pounds Sterling, which I give, one thousand thereof to the Inhabitants of the Town of Boston in Massachusetts, and the other thousand to the inhabitants of the City of Philadelphia, in Trust and for the Uses, Interests and Purposes hereinafter mentioned and declared.

Franklin's plan was to lend money to young apprentices at 5% interest with the provision that each borrower should pay each year along

> … with the yearly Interest, one tenth part of the Principal, which sums of Principal and Interest shall be again let to fresh Borrowers. … If this plan is executed and succeeds as projected without interruption for one hundred Years, the Sum will then be one hundred and thirty-one thousand Pounds of which I would have the Managers of the Donation to the Inhabitants of the Town of Boston, then lay out at their discretion one hundred thousand Pounds in Public Works. … The remaining thirty-one thousand Pounds, I would have continued to be let out on Interest in the manner above directed for another hundred Years. … At the end of this second term if no unfortunate accident has prevented the operation the sum will be Four Millions and Sixty-one Thousand Pounds.

It was not always possible to find as many borrowers as Franklin had planned, but the managers of the trust did the best they could. At the end of 100 years from the receipt of the Franklin gift, in January 1894, the fund had grown from 1000 pounds to almost 90,000 pounds. In 100 years the original capital had multiplied about 90 times instead of the 131 times Franklin had imagined.

(a) What annual rate of interest, compounded continuously for 100 years, would have multiplied Benjamin Franklin's original capital by 90?

(b) In Benjamin Franklin's estimate that the original 1000 pounds would grow to 131,000 in 100 years, he was using an annual rate of 5% and compounding once each year. What rate of interest per year when compounded continuously for 100 years would multiply the original amount by 131?

46. ***Rules of 70 and 72*** The rules state that it takes about $70/i$ or $72/i$ years for money to double at i percent, compounded continuously, using whichever of 70 or 72 is easier to divide by i.

(a) Show that it takes $t = (\ln 2)/r$ years for money to double if it is invested at annual interest rate r (in decimal form) compounded continuously.

(b) Graph the functions

$$y_1 = \frac{\ln 2}{r}, \quad y_2 = \frac{70}{100r}, \quad \text{and} \quad y_3 = \frac{72}{100r}$$

in the $[0, 0.1]$ by $[0, 100]$ viewing window.

(c) Writing to Learn Explain why these two rules of thumb for mental computation are reasonable.

(d) Use the rules to estimate how long it takes to double money at 5% compounded continuously.

(e) Invent a rule for estimating the number of years needed to triple your money.

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

47. True or False If $dy/dx = ky$, then $y = e^{kx} + C$. Justify your answer.

48. True or False The general solution to $dy/dt = 2y$ can be written in the form $y = C(3^{kt})$ for some constants C and k. Justify your answer.

49. Multiple Choice A bank account earning continuously compounded interest doubles in value in 7.0 years. At the same interest rate, how long would it take the value of the account to triple?

(A) 4.4 years **(B)** 9.8 years **(C)** 10.5 years
(D) 11.1 years **(E)** 21.0 years

50. Multiple Choice A sample of Ce-143 (an isotope of cerium) loses 99% of its radioactive matter in 199 hours. What is the half-life of Ce-143?

(A) 4 hours **(B)** 6 hours **(C)** 30 hours
(D) 100.5 hours **(E)** 143 hours

51. Multiple Choice In which of the following models is dy/dt directly proportional to y?

I. $y = e^{kt} + C$
II. $y = Ce^{kt}$
III. $y = 28^{kt}$

(A) I only **(B)** II only **(C)** I and II only
(D) II and III only **(E)** I, II, and III

52. Multiple Choice An apple pie comes out of the oven at 425°F and is placed on a counter in a 68°F room to cool. In 30 minutes it has cooled to 195°F. According to Newton's Law of Cooling, how many additional minutes must pass before it cools to 100°F?

(A) 12.4 **(B)** 15.4 **(C)** 25.0 **(D)** 35.0 **(E)** 40.0

Explorations

53. ***Resistance Proportional to Velocity*** It is reasonable to assume that the air resistance encountered by a moving object, such as a car coasting to a stop, is proportional to the object's velocity. The resisting force on an object of mass m moving with velocity v is thus $-kv$ for some positive constant k.

(a) Use the law *Force* = *Mass* × *Acceleration* to show that the velocity of an object slowed by air resistance (and no other forces) satisfies the differential equation

$$m\frac{dy}{dt} = -kv.$$

(b) Solve the differential equation to show that $v = v_0 e^{-(k/m)t}$, where v_0 is the velocity of the object at time $t = 0$.

(c) If k is the same for two objects of different masses, which one will slow to half its starting velocity in the shortest time? Justify your answer.

54. ***Coasting to a Stop*** Assume that the resistance encountered by a moving object is proportional to the object's velocity so that its velocity is $v = v_0 e^{-(k/m)t}$.

(a) Integrate the velocity function with respect to t to obtain the distance function s. Assume that $s(0) = 0$ and show that

$$s(t) = \frac{v_0 m}{k}\left(1 - e^{-(k/m)t}\right).$$

(b) Show that the total coasting distance traveled by the object as it coasts to a complete stop is $v_0 m/k$.

55. ***Coasting to a Stop*** Table 6.4 shows the distance s (meters) coasted on in-line skates in terms of time t (seconds) by Kelly Schmitzer. Find a model for her position in the form given in Exercise 54(a) and superimpose its graph on a scatter plot of the data. Her initial velocity was $v_0 = 0.80$ m/sec, her mass $m = 49.90$ kg (110 lb), and her total coasting distance was 1.32 m.

Table 6.4 Kelly Schmitzer Skating Data

t (sec)	s (m)	t (sec)	s (m)	t (sec)	s (m)
0	0	1.5	0.89	3.1	1.30
0.1	0.07	1.7	0.97	3.3	1.31
0.3	0.22	1.9	1.05	3.5	1.32
0.5	0.36	2.1	1.11	3.7	1.32
0.7	0.49	2.3	1.17	3.9	1.32
0.9	0.60	2.5	1.22	4.1	1.32
1.1	0.71	2.7	1.25	4.3	1.32
1.3	0.81	2.9	1.28	4.5	1.32

Source: Valerie Sharrits, St. Francis de Sales H.S., Columbus, OH.

56. ***Coasting to a Stop*** Table 6.5 shows the distance s (meters) coasted on in-line skates in t seconds by Johnathon Krueger. Find a model for his position in the form given in Exercise 54(a) and superimpose its graph on a scatter plot of the data. His initial velocity was $v_0 = 0.86$ m/sec, his mass $m = 30.84$ kg (he weighed 68 lb), and his total coasting distance 0.97 m.

Table 6.5 Johnathon Krueger Skating Data

t (sec)	s (m)	t (sec)	s (m)	t (sec)	s (m)
0	0	0.93	0.61	1.86	0.93
0.13	0.08	1.06	0.68	2.00	0.94
0.27	0.19	1.20	0.74	2.13	0.95
0.40	0.28	1.33	0.79	2.26	0.96
0.53	0.36	1.46	0.83	2.39	0.96
0.67	0.45	1.60	0.87	2.53	0.97
0.80	0.53	1.73	0.90	2.66	0.97

Source: Valerie Sharrits, St. Francis de Sales H.S., Columbus, OH.

Extending the Ideas

57. ***Continuously Compounded Interest***

(a) Use tables to give a numerical argument that

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e.$$

Support your argument graphically.

(b) For several different values of r, give numerical and graphical evidence that

$$\lim_{x\to\infty}\left(1+\frac{r}{x}\right)^x = e^r.$$

(c) **Writing to Learn** Explain why compounding interest over smaller and smaller periods of time leads to the concept of interest compounded continuously.

58. ***Skydiving*** If a body of mass m falling from rest under the action of gravity encounters an air resistance proportional to the square of the velocity, then the body's velocity $v(t)$ is modeled by the initial value problem

Differential equation: $m\dfrac{dv}{dt} = mg - kv^2,$

Initial condition: $v(0) = 0,$

where t represents time in seconds, g is the acceleration due to gravity, and k is a constant that depends on the body's aerodynamic properties and the density of the air. (We assume that the fall is short enough so that variation in the air's density will not affect the outcome.)

(a) Show that the function

$$v(t) = \sqrt{\frac{mg}{k}}\,\frac{e^{at}-e^{-at}}{e^{at}+e^{-at}},$$

where $a = \sqrt{gk/m}$, is a solution of the initial value problem.

(b) Find the body's limiting velocity, $\lim_{t\to\infty} v(t)$.

(c) For a 160-lb skydiver ($mg = 160$), and with time in seconds and distance in feet, a typical value for k is 0.005. What is the diver's limiting velocity in feet per second? in miles per hour?

Skydivers can vary their limiting velocities by changing the amount of body area opposing the fall. Their velocities can vary from 94 to 321 miles per hour.

6.5 Logistic Growth

What you'll learn about

- How Populations Grow
- Partial Fractions
- The Logistic Differential Equation
- Logistic Growth Models

. . . and why

Populations in the real world tend to grow logistically over extended periods of time.

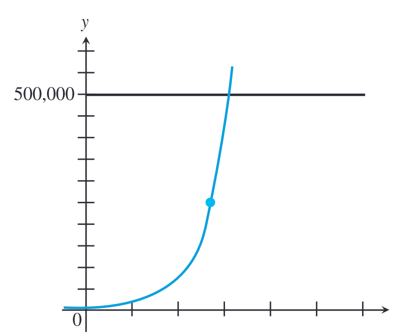
How Populations Grow

In Section 6.4 we showed that when the rate of change of a population is directly proportional to the size of the population, the population grows exponentially. This seems like a reasonable model for population growth in the short term, but populations in nature cannot sustain exponential growth for very long. Available food, habitat, and living space are just a few of the constraints that will eventually impose limits on the growth of any real-world population.

EXPLORATION 1 Exponential Growth Revisited

Almost any algebra book will include a problem like this: A culture of bacteria in a Petri dish is doubling every hour. If there are 100 bacteria at time $t = 0$, how many bacteria will there be in 12 hours?

1. Answer the algebra question.
2. Suppose a textbook editor, seeking to add a little unit conversion to the problem to satisfy a reviewer, changes "12 hours" to "12 days" in the second edition of the textbook. What is the answer to the revised question?
3. Is the new answer reasonable? (*Hint:* It has been estimated that there are about 10^{79} atoms in the entire universe.)
4. Suppose the maximal sustainable population of bacteria in this Petri dish is 500,000 bacteria. How many hours will it take the bacteria to reach this population if the exponential model continues to hold?
5. The graph below shows what the graph of the population would look like if it were to remain exponential until hitting 500,000. Draw a more reasonable graph that shows how the population might approach 500,000 after growing exponentially up to the marked point.



You might recall that we introduced *logistic* curves in Section 4.3 to illustrate points of inflection. Logistic growth, which starts off exponentially and then changes concavity to approach a maximal sustainable population, is a better model for real-world populations, for all the reasons mentioned above.

Partial Fractions

Before we introduce the differential equation that describes logistic growth, we need to review a bit of algebra that is needed to solve it.

Partial Fraction Decomposition with Distinct Linear Denominators

If $f(x) = \frac{P(x)}{Q(x)}$, where P and Q are polynomials with the degree of P less than the degree of Q, and if $Q(x)$ can be written as a product of distinct linear factors, then $f(x)$ can be written as a sum of rational functions with distinct linear denominators.

We will illustrate this principle with examples.

A Question of Degree

Note that the technique of partial fractions only applies to rational functions of the form

$$\frac{P(x)}{Q(x)}$$

where P and Q are polynomials with the degree of P less than the degree of Q. Such a fraction is called *proper*. Example 2 will show you how to handle an *improper* fraction.

A Little on the Heaviside

The substitution technique used to find A and B in Example 1 (and in subsequent examples) is often called the Heaviside Method after English engineer Oliver Heaviside (1850–1925).

EXAMPLE 1 Finding a Partial Fraction Decomposition

Write the function $f(x) = \frac{x - 13}{2x^2 - 7x + 3}$ as a sum of rational functions with linear denominators.

SOLUTION

Since $f(x) = \frac{x - 13}{(2x - 1)(x - 3)}$, we will find numbers A and B so that

$f(x) = \frac{A}{2x - 1} + \frac{B}{x - 3}$.

Note that $\frac{A}{2x - 1} + \frac{B}{x - 3} = \frac{A(x - 3) + B(2x - 1)}{(2x - 1)(x - 3)}$, so it follows that

$$A(x - 3) + B(2x - 1) = x - 13. \qquad (1)$$

Setting $x = 3$ in equation (1), we get

$$A(0) + B(5) = -10, \text{ so } B = -2.$$

Setting $x = \frac{1}{2}$ in equation (1), we get

$$A\left(-\frac{5}{2}\right) + B(0) = -\frac{25}{2}, \text{ so } A = 5.$$

Therefore $f(x) = \frac{x - 13}{(2x - 1)(x - 3)} = \frac{5}{2x - 1} - \frac{2}{x - 3}$.

Now try Exercise 3.

You might already have guessed that partial fraction decomposition can be of great value when antidifferentiating rational functions.

EXAMPLE 2 Antidifferentiating with Partial Fractions

Find $\int \frac{3x^4 + 1}{x^2 - 1}\, dx$.

SOLUTION

First we note that the degree of the denominator is not less than the degree of the numerator. We use the division algorithm to find the quotient and remainder:

$$\begin{array}{r|l} & 3x^2 + 3 \\ x^2 - 1 & 3x^4 \qquad + 1 \\ & \underline{3x^4 - 3x^2} \\ & 3x^2 + 1 \\ & \underline{3x^2 - 3} \\ & 4 \end{array}$$

continued

Thus

$$\int \frac{3x^4+1}{x^2-1}\,dx = \int \left(3x^2+3+\frac{4}{x^2-1}\right) dx$$
$$= x^3+3x+\int \frac{4}{(x-1)(x+1)}\,dx$$
$$= x^3+3x+\int \left(\frac{A}{x-1}+\frac{B}{x+1}\right) dx.$$

We know that $A(x+1)+B(x-1)=4$.

Setting $x=1$,

$$A(2)+B(0)=4, \text{ so } A=2.$$

Setting $x=-1$,

$$A(0)+B(-2)=4, \text{ so } B=-2.$$

Thus

$$\int \frac{3x^4+1}{x^2-1}\,dx = x^3+3x+\int \left(\frac{2}{x-1}+\frac{-2}{x+1}\right) dx$$
$$= x^3+3x+2\ln|x-1|-2\ln|x+1|+C$$
$$= x^3+3x+2\ln\left|\frac{x-1}{x+1}\right|+C.$$

Now try Exercise 7.

EXAMPLE 3 Finding Three Partial Fractions

This example will be our most laborious problem.

Find the general solution to $\dfrac{dy}{dx}=\dfrac{6x^2-8x-4}{(x^2-4)(x-1)}$.

SOLUTION

$$y=\int \frac{6x^2-8x-4}{(x-2)(x+2)(x-1)}\,dx = \int \left(\frac{A}{x-2}+\frac{B}{x+2}+\frac{C}{x-1}\right) dx.$$

We know that $A(x+2)(x-1)+B(x-2)(x-1)+C(x-2)(x+2)=6x^2-8x-4$.

Setting $x=2$:

$$A(4)(1)+B(0)+C(0)=4, \text{ so } A=1.$$

Setting $x=-2$:

$$A(0)+B(-4)(-3)+C(0)=36, \text{ so } B=3.$$

Setting $x=1$,

$$A(0)+B(0)+C(-1)(3)=-6, \text{ so } C=2.$$

Thus

$$\int \frac{6x^2-8x-4}{(x-2)(x+2)(x-1)}\,dx = \int \left(\frac{1}{x-2}+\frac{3}{x+2}+\frac{2}{x-1}\right) dx$$
$$= \ln|x-2|+3\ln|x+2|+2\ln|x-1|+C$$
$$= \ln\left(|x-2||x+2|^3\,|x-1|^2\right)+C.$$

Now try Exercise 17.

The technique of partial fractions can actually be extended to apply to all rational functions, but the method has to be adapted slightly if there are repeated linear factors or irreducible quadratic factors in the denominator. Both of these cases lead to partial fractions with quadratic denominators, and we will not deal with them in this book.

The Logistic Differential Equation

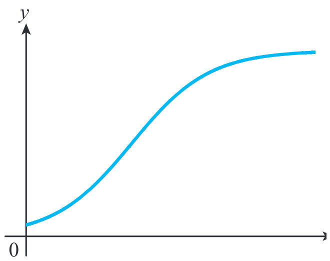


Figure 6.13 A logistic curve.

Now consider the case of a population P with a growth curve as a function of time that begins increasing and concave up (as in exponential growth), then turns increasing and concave down as it approaches the carrying capacity of its habitat. A **logistic curve,** like the one shown in Figure 6.13, has the shape to model this growth.

We have seen that the exponential growth at the beginning can be modeled by the differential equation

$$\frac{dP}{dt} = kP \text{ for some } k > 0.$$

If we want the growth rate to approach zero as P approaches a maximal **carrying capacity** M, we can introduce a limiting factor of $M - P$:

$$\frac{dP}{dt} = kP(M - P)$$

This is the **logistic differential equation.** Before we find its general solution, let us see how much we can learn about logistic growth just by studying the differential equation itself.

EXPLORATION 2 **Learning from the Differential Equation**

Consider a (positive) population P that satisfies $dP/dt = kP(M - P)$, where k and M are positive constants.

1. For what values of P will the growth rate dP/dt be close to zero?
2. As a function of P, $y = kP(M - P)$ has a graph that is an upside-down parabola. What is the value of P at the vertex of that parabola?
3. Use the answer to part (2) to explain why the growth rate is maximized when the population reaches half the carrying capacity.
4. If the initial population is less than M, is the initial growth rate positive or negative?
5. If the initial population is greater than M, is the initial growth rate positive or negative?
6. If the initial population equals M, what is the initial growth rate?
7. What is $\lim_{t\to\infty} P(t)$? Does it depend on the initial population?

You can use the results of Exploration 2 in the following example.

EXAMPLE 4

The growth rate of a population P of bears in a newly established wildlife preserve is modeled by the differential equation $dP/dt = 0.008P(100 - P)$, where t is measured in years.

(a) What is the carrying capacity for bears in this wildlife preserve?

(b) What is the bear population when the population is growing the fastest?

(c) What is the rate of change of the population when it is growing the fastest?

continued

SOLUTION

(a) The carrying capacity is 100 bears.

(b) The bear population is growing the fastest when it is half the carrying capacity, 50 bears.

(c) When $P = 50$, $dP/dt = 0.008(50)(100 - 50) = 20$ bears per year. Although the derivative represents the instantaneous growth rate, it is reasonable to say that the population grows by about 20 bears that year.

Now try Exercise 25.

In this next example we will find the solution to a logistic differential equation with an initial condition.

EXAMPLE 5 Tracking a Moose Population

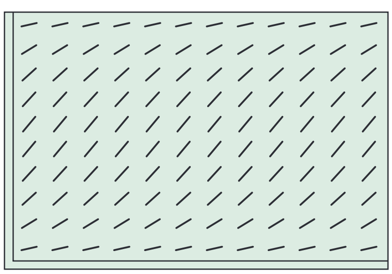
In 1985 and 1987, the Michigan Department of Natural Resources airlifted 61 moose from Algonquin Park, Ontario to Marquette County in the Upper Peninsula. It was originally hoped that the population P would reach carrying capacity in about 25 years with a growth rate of

$$\frac{dP}{dt} = 0.0003P(1000 - P).$$

(a) According to the model, what is the carrying capacity?

(b) With a calculator, generate a slope field for the differential equation.

(c) Solve the differential equation with the initial condition $P(0) = 61$ and show that it conforms to the slope field.



[0, 25] by [0, 1000]

Figure 6.14 The slope field for the moose differential equation in Example 5.

SOLUTION

(a) The carrying capacity is 1000 moose.

(b) The slope field is shown in Figure 6.14. Since the population approaches a horizontal asymptote at 1000 in about 25 years, we use the window [0, 25] by [0, 1000].

(c) After separating the variables, we encounter an antiderivative to be found using partial fractions.

$$\frac{dP}{P(1000 - P)} = 0.0003\,dt$$

$$\int \frac{1}{P(1000 - P)}\,dP = \int 0.0003\,dt$$

$$\int \left(\frac{A}{P} + \frac{B}{1000 - P}\right) dP = \int 0.0003\,dt$$

We know that $A(1000 - P) + B(P) = 1$.

Setting $P = 0$: $A(1000) + B(0) = 1$, so $A = 0.001$.

Setting $P = 1000$: $A(0) + B(1000) = 1$, so $B = 0.001$.

continued

$$\int\left(\frac{0.001}{P} + \frac{0.001}{1000 - P}\right) dP = \int 0.0003\, dt$$

$$\int\left(\frac{1}{P} + \frac{1}{1000 - P}\right) dP = \int 0.3\, dt$$

$$\ln P - \ln(1000 - P) = 0.3t + C$$

$$\ln(1000 - P) - \ln P = -0.3t - C$$

$$\ln\left(\frac{1000 - P}{P}\right) = -0.3t - C$$

$$\frac{1000}{P} - 1 = e^{-0.3t - C}$$

$$\frac{1000}{P} = 1 + e^{-0.3t}\, e^{-C}$$

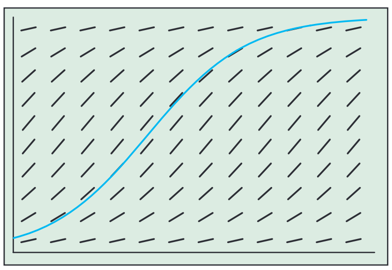
Setting $P = 61$ and $t = 0$, we find that $e^{-C} \approx 15.393$. Thus

$$\frac{1000}{P} = 1 + 15.393e^{-0.3t}$$

$$P = \frac{1000}{1 + 15.393e^{-0.3t}}.$$

The graph conforms nicely to the slope field, as shown in Figure 6.15.

Now try Exercise 29.



[0, 25] by [0, 1000]

Figure 6.15 The particular solution

$$P = \frac{1000}{1 + 15.393e^{-0.3t}}$$

conforms nicely to the slope field for $dP/dt = 0.0003P\,(1000 - P)$. (Example 5)

Logistic Growth Models

We could solve many more logistic differential equations and the algebra would look the same every time. In fact, it is almost as simple to solve the equation using letters for all the constants, thereby arriving at a general formula. In Exercise 35 we will ask you to verify the result in the box below.

The General Logistic Formula

The solution of the general logistic differential equation

$$\frac{dP}{dt} = kP(M - P)$$

is

$$P = \frac{M}{1 + Ae^{-(Mk)t}}$$

where A is a constant determined by an appropriate initial condition. The **carrying capacity** M and the **growth constant** k are positive constants.

We introduced logistic regression in Example 6 of Section 4.3. We close this section with another logistic regression example that makes use of our additional understanding.

EXAMPLE 6 Using Logistic Regression

Table 6.6 shows the population of Aurora, CO for selected years between 1950 and 2003.

Table 6.6 Population of Aurora, CO

Years after 1950	Population
0	11,421
20	74,974
30	158,588
40	222,103
50	275,923
53	290,418

Source: Bureau of the Census, U.S. Department of Commerce, as reported in The World Almanac and Book of Facts, 2005.

(a) Use logistic regression to find a logistic curve to model the data and superimpose it on a scatter plot of population against years after 1950.

(b) Based on the regression equation, what will the Aurora population approach in the long run?

(c) Based on the regression equation, when will the population of Aurora first exceed 300,000 people?

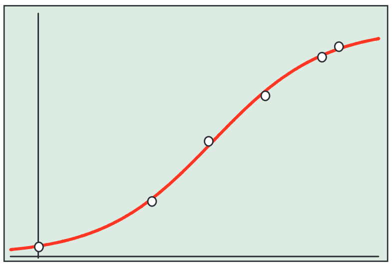
(d) Write a logistic differential equation in the form $dP/dt = kP(M - P)$ that models the growth of the Aurora data in Table 6.6.

SOLUTION

(a) The regression equation is

$$P = \frac{316440.7}{1 + 23.577e^{-0.1026t}}.$$

The graph is shown superimposed on the scatter plot in Figure 6.16. The fit is almost perfect.



[–5, 60] by [–3600, 337000]

Figure 6.16 The logistic regression curve fitted to the data for population growth in Aurora, CO from 1950 to 2003. (Example 6)

(b) Approximately 316,441 people. (The carrying capacity is the numerator of the regression equation.)

(c) Set

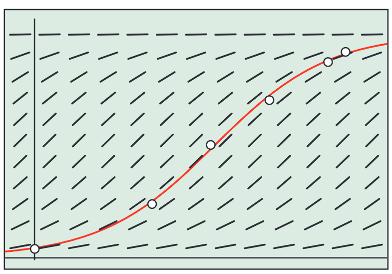
$$\frac{316440.7}{1 + 23.577e^{-0.1026t}} = 300{,}000.$$

The regression line crosses the 300,000 mark sometime in the 59$^{\text{th}}$ year, that is, in 2009.

(d) We see from the regression equation that $M = 316440.7$ and $Mk = 0.1026$. Therefore $k \approx 3.24 \times 10^{-7}$. The logistic growth model is

$$\frac{dP}{dt} = (3.24 \times 10^{-7})\,P(316440.7 - P).$$

Figure 6.17 shows the slope field for this differential equation superimposed on the scatter plot and the regression equation. ***Now try Exercise 37.***



[–5, 60] by [–3600, 337000]

Figure 6.17 The slope field for the differential equation derived from the regression curve fits the data and the regression curve nicely. (Example 6)

We caution readers once again not to assume that logistic models work perfectly in all real-world, population-growth problems; there are too many unpredictable variables that can and will change the growth conditions over time.

Graphing Calculator Logistics

Unfortunately, some graphing calculators allow for a *vertical shift* when fitting a logistic curve to a set of data points. The regression equation for such a curve would have the form

$$y = \frac{c}{1 + ae^{-bx}} + d.$$

While the curve might fit the data better, this function cannot be a solution to the logistic differential equation if d is not zero. Since our definition of a logistic function begins with the differential equation, we will consistently use only logistic regression equations of the form

$$y = \frac{c}{1 + ae^{-bx}}.$$

Quick Review 6.5 *(For help, go to Sections 2.2 and 2.3.)*

In Exercises 1–4, use the polynomial division algorithm (as in Example 2 of this section) to write the rational function in the form $Q(x) + \frac{R(x)}{D(x)}$, where the degree of R is less than the degree of D.

1. $\frac{x^2}{x - 1}$

2. $\frac{x^2}{x^2 - 4}$

3. $\frac{x^2 + x + 1}{x^2 + x - 2}$

4. $\frac{x^3 - 5}{x^2 - 1}$

In Exercises 5–10, let $f(x) = \frac{60}{1 + 5e^{-0.1x}}$.

5. Find where f is continuous.

6. Find $\lim_{x \to \infty} f(x)$.

7. Find $\lim_{x \to -\infty} f(x)$.

8. Find the y-intercept of the graph of f.

9. Find all horizontal asymptotes of the graph of f.

10. Draw the graph of $y = f(x)$.

Section 6.5 Exercises

In Exercises 1–4, find the values of A and B that complete the partial fraction decomposition.

1. $\frac{x - 12}{x^2 - 4x} = \frac{A}{x} + \frac{B}{x - 4}$

2. $\frac{2x + 16}{x^2 + x - 6} = \frac{A}{x + 3} + \frac{B}{x - 2}$

3. $\frac{16 - x}{x^2 + 3x - 10} = \frac{A}{x - 2} + \frac{B}{x + 5}$

4. $\frac{3}{x^2 - 9} = \frac{A}{x - 3} + \frac{B}{x + 3}$

In Exercises 5–14, evaluate the integral.

5. $\int \frac{x - 12}{x^2 - 4x}\,dx$

6. $\int \frac{2x + 16}{x^2 + x - 6}\,dx$

7. $\int \frac{2x^3}{x^2 - 4}\,dx$

8. $\int \frac{x^2 - 6}{x^2 - 9}\,dx$

9. $\int \frac{2\,dx}{x^2 + 1}$

10. $\int \frac{3\,dx}{x^2 + 9}$

11. $\int \frac{7\,dx}{2x^2 - 5x - 3}$

12. $\int \frac{1 - 3x}{3x^2 - 5x + 2}\,dx$

13. $\int \frac{8x - 7}{2x^2 - x - 3}\,dx$

14. $\int \frac{5x + 14}{x^2 + 7x}\,dx$

In Exercises 15–18, solve the differential equation.

15. $\frac{dy}{dx} = \frac{2x - 6}{x^2 - 2x}$

16. $\frac{du}{dx} = \frac{2}{x^2 - 1}$

17. $F'(x) = \frac{2}{x^3 - x}$

18. $G'(t) = \frac{2t^3}{t^3 - t}$

In Exercises 19–22, find the integral *without* using the technique of partial fractions.

19. $\int \frac{2x}{x^2 - 4}\,dx$

20. $\int \frac{4x - 3}{2x^2 - 3x + 1}\,dx$

21. $\int \frac{x^2 + x - 1}{x^2 - x}\,dx$

22. $\int \frac{2x^3}{x^2 - 1}\,dx$

In Exercises 23–26, the logistic equation describes the growth of a population P, where t is measured in years. In each case, find **(a)** the carrying capacity of the population, **(b)** the size of the population when it is growing the fastest, and **(c)** the rate at which the population is growing when it is growing the fastest.

23. $\frac{dP}{dt} = 0.006P(200 - P)$

24. $\frac{dP}{dt} = 0.0008P(700 - P)$

25. $\frac{dP}{dt} = 0.0002P(1200 - P)$

26. $\frac{dP}{dt} = 10^{-5}P(5000 - P)$

In Exercises 27–30, solve the initial value problem using partial fractions. Use a graphing utility to generate a slope field for the differential equation and verify that the solution conforms to the slope field.

27. $\frac{dP}{dt} = 0.006P(200 - P)$ and $P = 8$ when $t = 0$.

28. $\frac{dP}{dt} = 0.0008P(700 - P)$ and $P = 10$ when $t = 0$.

29. $\frac{dP}{dt} = 0.0002P(1200 - P)$ and $P = 20$ when $t = 0$.

30. $\frac{dP}{dt} = 10^{-5}P(5000 - P)$ and $P = 50$ when $t = 0$.

In Exercises 31 and 32, a population function is given.

(a) Show that the function is a solution of a logistic differential equation. Identify k and the carrying capacity.

(b) **Writing to Learn** Estimate $P(0)$. Explain its meaning in the context of the problem.

31. ***Rabbit Population*** A population of rabbits is given by the formula

$$P(t) = \frac{1000}{1 + e^{4.8 - 0.7t}},$$

where t is the number of months after a few rabbits are released.

32. ***Spread of Measles*** The number of students infected by measles in a certain school is given by the formula

$$P(t) = \frac{200}{1 + e^{5.3 - t}},$$

where t is the number of days after students are first exposed to an infected student.

33. ***Guppy Population*** A 2000-gallon tank can support no more than 150 guppies. Six guppies are introduced into the tank. Assume that the rate of growth of the population is

$$\frac{dP}{dt} = 0.0015P(150 - P),$$

where time t is in weeks.

(a) Find a formula for the guppy population in terms of t.

(b) How long will it take for the guppy population to be 100? 125?

34. ***Gorilla Population*** A certain wild animal preserve can support no more than 250 lowland gorillas. Twenty-eight gorillas were known to be in the preserve in 1970. Assume that the rate of growth of the population is

$$\frac{dP}{dt} = 0.0004P(250 - P),$$

where time t is in years.

(a) Find a formula for the gorilla population in terms of t.

(b) How long will it take for the gorilla population to reach the carrying capacity of the preserve?

35. ***Logistic Differential Equation*** Show that the solution of the differential equation

$$\frac{dP}{dt} = kP(M - P) \quad \text{is} \quad P = \frac{M}{1 + Ae^{-Mkt}},$$

where A is a constant determined by an appropriate initial condition.

36. ***Limited Growth Equation*** Another differential equation that models limited growth of a population P in an environment with carrying capacity M is $dP/dt = k(M - P)$ (where $k > 0$ and $M > 0$).

(a) Show that $P = M - Ae^{-kt}$, where A is a constant determined by an appropriate initial condition.

(b) What is $\lim_{t\to\infty} P(t)$?

(c) For what time $t \geq 0$ is the population growing the fastest?

(d) **Writing to Learn** How does the growth curve in this model differ from the growth curve in the logistic model?

37. ***Population Growth*** Table 6.7 shows the population of Laredo, Texas for selected years between 1950 and 2003.

Table 6.7 Population of Laredo, TX

Years after 1950	Population
0	10,571
20	81,437
30	138,857
40	180,650
50	215,794
53	218,027

Source: Bureau of the Census, U.S. Department of Commerce, as reported in The World Almanac and Book of Facts, 2005.

(a) Use logistic regression to find a curve to model the data and superimpose it on a scatter plot of population against years after 1950.

(b) Based on the regression equation, what number will the Laredo population approach in the long run?

(c) Based on the regression equation, when will the Laredo population first exceed 225,000 people?

(d) Write a logistic differential equation in the form $dP/dt = kP(M - P)$ that models the growth of the Laredo data in Table 6.7.

38. ***Population Growth*** Table 6.8 shows the population of Virginia Beach, VA for selected years between 1950 and 2003.

Table 6.8 Population of Virginia Beach

Years after 1950	Population
0	5,390
20	172,106
30	262,199
40	393,069
50	425,257
53	439,467

Source: Bureau of the Census, U.S. Department of Commerce, as reported in The World Almanac and Book of Facts, 2005.

(a) Use logistic regression to find a curve to model the data and superimpose it on a scatter plot of population against years after 1950.

(b) Based on the regression equation, what number will the Virginia Beach population approach in the long run?

(c) Based on the regression equation, when will the Virginia Beach population first exceed 450,000 people?

(d) Write a logistic differential equation in the form $dP/dt = kP(M - P)$ that models the growth of the Virginia Beach data in Table 6.8.

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

39. **True or False** For small values of t, the solution to logistic differential equation $dP/dt = kP(100 - P)$ that passes through the point (0, 10) resembles the solution to the differential equation $dP/dt = kP$ that passes through the point (0, 10). Justify your answer.

40. **True or False** The graph of any solution to the differential equation $dP/dt = kP(100 - P)$ has asymptotes $y = 0$ and $y = 100$. Justify your answer.

41. **Multiple Choice** The spread of a disease through a community can be modeled with the logistic equation

$$\frac{dy}{dt} = \frac{600}{1 + 59e^{-0.1t}},$$

where y is the number of people infected after t days. How many people are infected when the disease is spreading the fastest?

(A) 10 **(B)** 59 **(C)** 60 **(D)** 300 **(E)** 600

42. Multiple Choice The spread of a disease through a community can be modeled with the logistic equation

$$\frac{dy}{dt} = \frac{0.9}{1 + 45e^{-0.15t}},$$

where y is the proportion of people infected after t days. According to the model, what percentage of the people in the community will not become infected?

(A) 2% **(B)** 10% **(C)** 15% **(D)** 45% **(E)** 90%

43. Multiple Choice $\int_2^3 \frac{3}{(x-1)(x+2)}\,dx =$

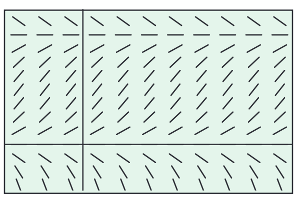
(A) $-\frac{33}{20}$ **(B)** $-\frac{9}{20}$ **(C)** $\ln\left(\frac{5}{2}\right)$ **(D)** $\ln\left(\frac{8}{5}\right)$ **(E)** $\ln\left(\frac{2}{5}\right)$

44. Multiple Choice Which of the following differential equations would produce the slope field shown below?

(A) $\frac{dy}{dx} = 0.01x(120 - x)$ **(B)** $\frac{dy}{dx} = 0.01y(120 - y)$

(C) $\frac{dy}{dx} = 0.01y(100 - x)$ **(D)** $\frac{dy}{dx} = \frac{120}{1 + 60e^{-1.2x}}$

(E) $\frac{dy}{dx} = \frac{120}{1 + 60e^{-1.2y}}$



[–3, 8] by [–50, 150]

Explorations

45. *Extinct Populations* One theory states that if the size of a population falls below a minimum m, the population will become extinct. This condition leads to the *extended* logistic differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)\left(1 - \frac{m}{P}\right)$$

$$= \frac{k}{M}(M - P)(P - m),$$

with $k > 0$ the proportionality constant and M the population maximum.

(a) Show that dP/dt is positive for $m < P < M$ and negative if $P < m$ or $P > M$.

(b) Let $m = 100$, $M = 1200$, and assume that $m < P < M$. Show that the differential equation can be rewritten in the form

$$\left[\frac{1}{1200 - P} + \frac{1}{P - 100}\right]\frac{dP}{dt} = \frac{11}{12}k.$$

Use a procedure similar to that used in Example 5 in Section 6.5 to solve this differential equation.

(c) Find the solution to part (b) that satisfies $P(0) = 300$.

(d) Superimpose the graph of the solution in part (c) with $k = 0.1$ on a slope field of the differential equation.

(e) Solve the general extended differential equation with the restriction $m < P < M$.

46. *Integral Tables* Antiderivatives of various generic functions can be found as formulas in *integral tables*. See if you can derive the formulas that would appear in an integral table for the following functions. (Here, a is an arbitrary constant.)

(a) $\int \frac{dx}{a^2 + x^2}$ **(b)** $\int \frac{dx}{a^2 - x^2}$ **(c)** $\int \frac{dx}{(a + x)^2}$

Extending the Ideas

47. *Partial Fractions with Repeated Linear Factors*

If

$$f(x) = \frac{P(x)}{(x - r)^m}$$

is a rational function with the degree of P less than m, then the partial fraction decomposition of f is

$$f(x) = \frac{A_1}{x - r} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_m}{(x - r)^m}.$$

For example,

$$\frac{4x}{(x - 2)^2} = \frac{4}{x - 2} + \frac{8}{(x - 2)^2}.$$

Use partial fractions to find the following integrals:

(a) $\int \frac{5x}{(x + 3)^2}\,dx$

(b) $\int \frac{5x}{(x + 3)^3}\,dx$ (*Hint:* Use part (a).)

48. *More on Repeated Linear Factors* The Heaviside Method is not very effective at finding the unknown numerators for partial fraction decompositions with repeated linear factors, but here is another way to find them.

(a) If $\frac{x^2 + 3x + 5}{(x - 1)^3} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{C}{(x - 1)^3}$, show that $A(x - 1)^2 + B(x - 1) + C = x^2 + 3x + 5$.

(b) Expand and equate coefficients of like terms to show that $A = 1$, $-2A + B = 3$, and $A - B + C = 5$. Then find A, B, and C.

(c) Use partial fractions to evaluate $\int \frac{x^2 + 3x + 5}{(x - 1)^3}\,dx$.

Quick Quiz: Sections 6.4 and 6.5

You may use a graphing calculator to solve the following problems.

1. **Multiple Choice** The rate at which acreage is being consumed by a plot of kudzu is proportional to the number of acres already consumed at time t. If there are 2 acres consumed when $t = 1$ and 3 acres consumed when $t = 5$, how many acres will be consumed when $t = 8$?

   (A) 3.750 **(B)** 4.000 **(C)** 4.066 **(D)** 4.132 **(E)** 4.600

2. **Multiple Choice** Let $F(x)$ be an antiderivative of $\cos(x^2)$. If $F(1) = 0$, then $F(5) =$

   (A) -0.099 **(B)** -0.153 **(C)** -0.293 **(D)** -0.992 **(E)** -1.833

3. **Multiple Choice** $\int \frac{dx}{(x-1)(x+3)} =$

   (A) $\frac{1}{4}\ln\left|\frac{x-1}{x+3}\right| + C$ **(B)** $\frac{1}{4}\ln\left|\frac{x+3}{x-1}\right| + C$

   (C) $\frac{1}{2}\ln|(x-1)(x+3)| + C$ **(D)** $\frac{1}{2}\ln\left|\frac{2x+2}{(x-1)(x+3)}\right| + C$

   (E) $\ln|(x-1)(x+3)| + C$

4. **Free Response** A population is modeled by a function P that satisfies the logistic differential equation

$$\frac{dP}{dt} = \frac{P}{5}\left(1 - \frac{P}{10}\right).$$

   (a) If $P(0) = 3$, what is $\lim_{t\to\infty} P(t)$?

   (b) If $P(0) = 20$, what is $\lim_{t\to\infty} P(t)$?

   (c) A different population is modeled by a function Y that satisfies the separable differential equation

$$\frac{dY}{dt} = \frac{Y}{5}\left(1 - \frac{t}{10}\right).$$

   Find $Y(t)$ if $Y(0) = 3$.

   (d) For the function Y found in part (c), what is $\lim_{t\to\infty} Y(t)$?

Chapter 6 Key Terms

antidifferentiation by parts (p. 341)
antidifferentiation by substitution (p. 334)
arbitrary constant of integration (p. 331)
carbon-14 dating (p. 354)
carrying capacity (p. 367)
compounded continuously (p. 352)
constant of integration (p. 331)
continuous interest rate (p. 352)
decay constant (p. 351)
differential equation (p.321)
direction field (p. 323)
Euler's Method (p. 325)
evaluate an integral (p. 331)
exact differential equation (p. 321)
general solution to a differential equation (p. 321)
growth constant (p. 351)
first-order differential equation (p. 321)
first-order linear differential equation (p. 324)
graphical solution of a differential equation (p. 322)
half-life (p. 352)
Heaviside method (p. 363)
indefinite integral (p. 331)
initial condition (p. 321)
initial value problem (p. 321)
integral sign (p. 331)
integrand (p. 331)
integration by parts (p. 341)
Law of Exponential Change (p. 351)
Leibniz notation for integrals (p. 333)
logistic differential equation (p. 365)
logistic growth model (p. 367)
logistic regression (p. 365)
Newton's Law of Cooling (p. 354)
numerical method (p. 327)
numerical solution of a differential equation (p. 327)
order of a differential equation (p. 321)
partial fraction decomposition (p. 363)
particular solution (p. 321)
proper rational function (p. 363)
properties of indefinite integrals (p. 332)
radioactive (p. 352)
radioactive decay (p. 352)
resistance proportional to velocity (p. 360)
second-order differential equation (p. 330)
separable differential equations (p. 350)
separation of variables (p. 350)
slope field (p. 323)
solution of a differential equation (p. 321)
substitution in definite integrals (p. 336)
tabular integration (p. 344)
variable of integration (p. 331)

Chapter 6 Review Exercises

The collection of exercises marked in red could be used as a chapter test.

In Exercises 1–10, evaluate the integral analytically. Then use NINT to support your result.

1. $\displaystyle\int_0^{\pi/3} \sec^2\theta\, d\theta$

2. $\displaystyle\int_1^2 \left(x + \frac{1}{x^2}\right) dx$

3. $\displaystyle\int_0^1 \frac{36\, dx}{(2x+1)^3}$

4. $\displaystyle\int_{-1}^1 2x \sin(1 - x^2)\, dx$

5. $\displaystyle\int_0^{\pi/2} 5 \sin^{3/2} x \cos x\, dx$

6. $\displaystyle\int_{1/2}^4 \frac{x^2 + 3x}{x}\, dx$

7. $\displaystyle\int_0^{\pi/4} e^{\tan x} \sec^2 x\, dx$

8. $\displaystyle\int_1^e \frac{\sqrt{\ln r}}{r}\, dr$

9. $\displaystyle\int_0^1 \frac{x}{x^2 + 5x + 6}\, dx$

10. $\displaystyle\int_1^2 \frac{2x + 6}{x^2 - 3x}\, dx$

In Exercises 11–24, evaluate the integral.

11. $\displaystyle\int \frac{\cos x}{2 - \sin x}\, dx$

12. $\displaystyle\int \frac{dx}{\sqrt[3]{3x + 4}}$

13. $\displaystyle\int \frac{t\, dt}{t^2 + 5}$

14. $\displaystyle\int \frac{1}{\theta^2} \sec\frac{1}{\theta} \tan\frac{1}{\theta}\, d\theta$

15. $\displaystyle\int \frac{\tan(\ln y)}{y}\, dy$

16. $\displaystyle\int e^x \sec(e^x)\, dx$

17. $\displaystyle\int \frac{dx}{x \ln x}$

18. $\displaystyle\int \frac{dt}{t\sqrt{t}}$

19. $\displaystyle\int x^3 \cos x\, dx$

20. $\displaystyle\int x^4 \ln x\, dx$

21. $\displaystyle\int e^{3x} \sin x\, dx$

22. $\displaystyle\int x^2 e^{-3x}\, dx$

23. $\displaystyle\int \frac{25}{x^2 - 25}\, dx$

24. $\displaystyle\int \frac{5x + 2}{2x^2 + x - 1}\, dx$

In Exercises 25–34, solve the initial value problem analytically. Support your solution by overlaying its graph on a slope field of the differential equation.

25. $\dfrac{dy}{dx} = 1 + x + \dfrac{x^2}{2}, \quad y(0) = 1$

26. $\dfrac{dy}{dx} = \left(x + \dfrac{1}{x}\right)^2, \quad y(1) = 1$

27. $\dfrac{dy}{dt} = \dfrac{1}{t + 4}, \quad y(-3) = 2$

28. $\dfrac{dy}{d\theta} = \csc 2\theta \cot 2\theta, \quad y(\pi/4) = 1$

29. $\dfrac{d^2y}{dx^2} = 2x - \dfrac{1}{x^2}, \quad x > 0, \quad y'(1) = 1, \quad y(1) = 0$

30. $\dfrac{d^3r}{dt^3} = -\cos t, \quad r''(0) = r'(0) = r(0) = -1$

31. $\dfrac{dy}{dx} = y + 2, \quad y(0) = 2$

32. $\dfrac{dy}{dx} = (2x + 1)(y + 1), \quad y(-1) = 1$

33. $\dfrac{dy}{dt} = y(1 - y), \quad y(0) = 0.1$

34. $\dfrac{dy}{dx} = 0.001y(100 - y), \quad y(0) = 5$

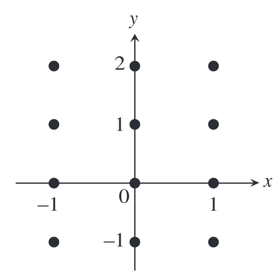
35. Find an integral equation $y = \int_a^x f(t)dt$ such that $dy/dx = \sin^3 x$ and $y = 5$ when $x = 4$.

36. Find an integral equation $y = \int_a^x f(t)dt$ such that $dy/dx = \sqrt{1 + x^4}$ and $y = 2$ when $x = 1$.

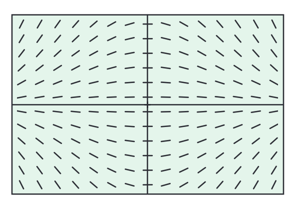
In Exercises 37 and 38, construct a slope field for the differential equation. In each case, copy the graph shown and draw tiny segments through the twelve lattice points shown in the graph. Use slope analysis, not your graphing calculator.

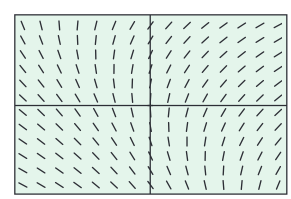
37. $\dfrac{dy}{dx} = -x$

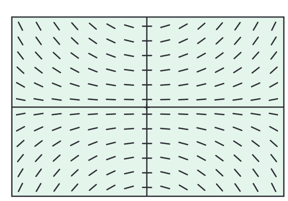
38. $\dfrac{dy}{dx} = 1 - y$

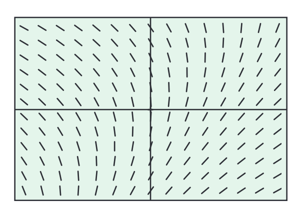


In Exercises 39–42, match the differential equation with the appropriate slope field. (All slope fields are shown in the window $[-6, 6]$ by $[-4, 4]$.)


(a)


(b)


(c)


(d)

39. $\dfrac{dy}{dx} = \dfrac{5}{x + y}$

40. $\dfrac{dy}{dx} = \dfrac{5}{x - y}$

41. $\dfrac{dy}{dx} = \dfrac{xy}{10}$

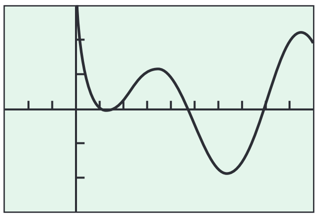
42. $\dfrac{dy}{dx} = -\dfrac{xy}{10}$

43. Suppose $dy/dx = x + y - 1$ and $y = 1$ when $x = 1$. Use Euler's Method with increments of $\Delta x = 0.1$ to approximate the value of y when $x = 1.3$.

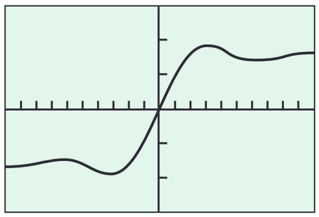
44. Suppose $dy/dx = x - y$ and $y = 2$ when $x = 1$. Use Euler's Method with increments of $\Delta x = -0.1$ to approximate the value of y when $x = 0.7$.

In Exercises 45 and 46, match the indefinite integral with the graph of one of the antiderivatives of the integrand.

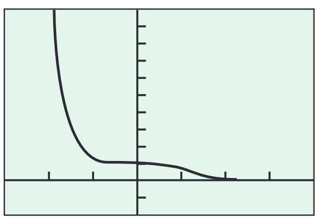
45. $\displaystyle\int \frac{\sin x}{x}\,dx$ **46.** $\displaystyle\int e^{-x^2}\,dx$


[–3, 10] by [–3, 3]

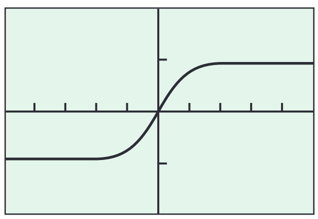
(a)


[–10, 10] by [–3, 3]

(b)


[–3, 4] by [–2, 10]

(c)


[–5, 5] by [–2, 2]

(d)

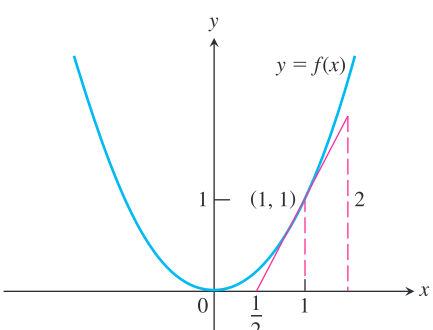
47. **Writing to Learn** The figure shows the graph of the function $y = f(x)$ that is the solution of one of the following initial value problems. Which one? How do you know?

i. $dy/dx = 2x, \quad y(1) = 0$

ii. $dy/dx = x^2, \quad y(1) = 1$

iii. $dy/dx = 2x + 2, \quad y(1) = 1$

iv. $dy/dx = 2x, \quad y(1) = 1$



48. **Writing to Learn** Does the following initial value problem have a solution? Explain.

$$\frac{d^2y}{dx^2} = 0, \quad y'(0) = 1, \quad y(0) = 0$$

49. ***Moving Particle*** The acceleration of a particle moving along a coordinate line is

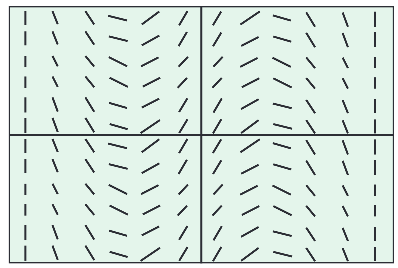
$$\frac{d^2s}{dt^2} = 2 + 6t \text{ m/sec}^2.$$

At $t = 0$ the velocity is 4 m/sec.

(a) Find the velocity as a function of time t.

(b) How far does the particle move during the first second of its trip, from $t = 0$ to $t = 1$?

50. ***Sketching Solutions*** Draw a possible graph for the function $y = f(x)$ with slope field given in the figure that satisfies the initial condition $y(0) = 0$.


[–10, 10] by [–10, 10]

51. ***Californium-252*** What costs \$27 million per gram and can be used to treat brain cancer, analyze coal for its sulfur content, and detect explosives in luggage? The answer is californium-252, a radioactive isotope so rare that only about 8 g of it have been made in the western world since its discovery by Glenn Seaborg in 1950. The half-life of the isotope is 2.645 years—long enough for a useful service life and short enough to have a high radioactivity per unit mass. One microgram of the isotope releases 170 million neutrons per second.

(a) What is the value of k in the decay equation for this isotope?

(b) What is the isotope's mean life? (See Exercise 19, Section 6.4.)

52. ***Cooling a Pie*** A deep-dish apple pie, whose internal temperature was 220°F when removed from the oven, was set out on a 40°F breezy porch to cool. Fifteen minutes later, the pie's internal temperature was 180°F. How long did it take the pie to cool from there to 70°F?

53. ***Finding Temperature*** A pan of warm water (46°C) was put into a refrigerator. Ten minutes later, the water's temperature was 39°C; 10 minutes after that, it was 33°C. Use Newton's Law of Cooling to estimate how cold the refrigerator was.

54. ***Art Forgery*** A painting attributed to Vermeer (1632–1675), which should contain no more than 96.2% of its original carbon-14, contains 99.5% instead. About how old is the forgery?

55. ***Carbon-14*** What is the age of a sample of charcoal in which 90% of the carbon-14 that was originally present has decayed?

56. ***Appreciation*** A violin made in 1785 by John Betts, one of England's finest violin makers, cost \$250 in 1924 and sold for \$7500 in 1988. Assuming a constant relative rate of appreciation, what was that rate?

57. *Working Underwater* The intensity $L(x)$ of light x feet beneath the surface of the ocean satisfies the differential equation

$$\frac{dL}{dx} = -kL,$$

where k is a constant. As a diver you know from experience that diving to 18 ft in the Caribbean Sea cuts the intensity in half. You cannot work without artificial light when the intensity falls below a tenth of the surface value. About how deep can you expect to work without artificial light?

58. *Transport through a Cell Membrane* Under certain conditions, the result of the movement of a dissolved substance across a cell's membrane is described by the equation

$$\frac{dy}{dt} = k\frac{A}{V}(c - y),$$

where y is the concentration of the substance inside the cell, and dy/dt is the rate with which y changes over time. The letters k, A, V, and c stand for constants, k being the *permeability coefficient* (a property of the membrane), A the surface area of the membrane, V the cell's volume, and c the concentration of the substance outside the cell. The equation says that the rate at which the concentration changes within the cell is proportional to the difference between it and the outside concentration.

(a) Solve the equation for $y(t)$, using $y_0 = y(0)$.

(b) Find the steady-state concentration, $\lim_{t\to\infty} y(t)$.

59. *Logistic Equation* The spread of flu in a certain school is given by the formula

$$P(t) = \frac{150}{1 + e^{4.3-t}},$$

where t is the number of days after students are first exposed to infected students.

(a) Show that the function is a solution of a logistic differential equation. Identify k and the carrying capacity.

(b) Writing to Learn Estimate $P(0)$. Explain its meaning in the context of the problem.

(c) Estimate the number of days it will take for a total of 125 students to become infected.

60. *Confirming a Solution* Show that

$$y = \int_0^x \sin(t^2)\,dt + x^3 + x + 2$$

is the solution of the initial value problem.

Differential equation: $y'' = 2x\cos(x^2) + 6x$

Initial conditions: $y'(0) = 1, \quad y(0) = 2$

61. *Finding an Exact Solution* Use analytic methods to find the exact solution to

$$\frac{dP}{dt} = 0.002P\left(1 - \frac{P}{800}\right), \quad P(0) = 50.$$

62. *Supporting a Solution* Give two ways to provide graphical support for the integral formula

$$\int x^2 \ln x\,dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C.$$

63. *Doubling Time* Find the amount of time required for \$10,000 to double if the 6.3% annual interest is compounded **(a)** annually, **(b)** continuously.

64. *Constant of Integration* Let

$$f(x) = \int_0^x u(t)\,dt \quad \text{and} \quad g(x) = \int_3^x u(t)\,dt.$$

(a) Show that f and g are antiderivatives of $u(x)$.

(b) Find a constant C so that $f(x) = g(x) + C$.

65. *Population Growth* Table 6.9 shows the population of Anchorage, AK for selected years between 1950 and 2003.

Table 6.9 Population of Anchorage, AK

Years after 1950	Population
0	11,254
20	48,081
30	174,431
53	270,951

Source: Bureau of the Census, U.S. Department of Commerce, as reported in The World Almanac and Book of Facts, 2005.

(a) Use logistic regression to find a curve to model the data and superimpose it on a scatter plot of population against years after 1950.

(b) Based on the regression equation, what number will the Anchorage population approach in the long run?

(c) Write a logistic differential equation in the form $dp/dt = kP(M - P)$ that models the growth of the Anchorage data in Table 6.9.

(d) Writing to Learn The population of Anchorage in 2000 was 260,283. If this point is included in the data, how does it affect carrying capacity predicted by the regression equation? Is there reason to be concerned about our model?

66. *Temperature Experiment* A temperature probe is removed from a cup of hot chocolate and placed in water whose temperature (T_s) is 0°C. The data in Table 6.10 were collected over the next 30 sec with a CBL™ temperature probe.

Table 6.10 Experimental Data

Time t (sec)	T (°C)
2	74.68
5	61.99
10	34.89
15	21.95
20	15.36
25	11.89
30	10.02

(a) Find an exponential regression equation for the (t, T) data. Superimpose its graph on a scatter plot of the data.

(b) Estimate when the temperature probe will read 40°C.

(c) Estimate the hot chocolate's temperature when the temperature probe was removed.

AP* Examination Preparation

You may use a graphing calculator to solve the following problems.

67. The spread of a rumor through a small town is modeled by $dy/dt = 1.2y(1 - y)$, where y is the proportion of the townspeople who have heard the rumor at time t in days. At time $t = 0$, ten percent of the townspeople have heard the rumor.

(a) What proportion of the townspeople have heard the rumor when it is spreading the fastest?

(b) Find y explicitly as a function of t.

(c) At what time t is the rumor spreading the fastest?

68. A population P of wolves at time t years ($t \geq 0$) is increasing at a rate directly proportional to $600 - P$, where the constant of proportionality is k.

(a) If $P(0) = 200$, find $P(t)$ in terms of t and k.

(b) If $P(2) = 500$, find k.

(c) Find $\lim_{t\to\infty} P(t)$.

69. Let $v(t)$ be the velocity, in feet per second, of a skydiver at time t seconds, $t \geq 0$. After her parachute opens, her velocity satisfies the differential equation $dv/dt = -2(v + 17)$, with initial condition $v(0) = -47$.

(a) Use separation of variables to find an expression for v in terms of t, where t is measured in seconds.

(b) Terminal velocity is defined as $\lim_{t\to\infty} v(t)$. Find the terminal velocity of the skydiver to the nearest foot per second.

(c) It is safe to land when her speed is 20 feet per second. At what time t does she reach this speed?

Calculus at Work

I have a Bachelor's and Master's degree in Aerospace Engineering from the University of California at Davis. I started my professional career as a Facility Engineer managing productivity and maintenance projects in the Unitary Project Wind Tunnel facility at NASA Ames Research Center. I used calculus and differential equations in fluid mechanic analyses of the tunnels. I then moved to the position of Test Manager, still using some fluid mechanics and other mechanical engineering analysis tools to solve problems. For example, the lift and drag forces acting on an airplane wing can be determined by integrating the known pressure distribution on the wing.

I am currently a NASA On-Site Systems Engineer for the Lunar Prospector spacecraft project, at Lockheed Martin Missiles and Space in Sunnyvale, California. Differential equations and integration are used to design some of the flight hardware for the spacecraft. I work on ensuring that the different systems of the spacecraft are adequately integrated together to meet the specified design requirements. This often means doing some analysis to determine if the systems will function properly and within the constraints of the space environment. Some of these analyses require use of differential equations and integration to determine the most exact results, within some margin of error.

Ross Shaw

Chapter 7 Applications of Definite Integrals

The art of pottery developed independently in many ancient civilizations and still exists in modern times. The desired shape of the side of a pottery vase can be described by:

$$y = 5.0 + 2 \sin (x/4) \ (0 \le x \le 8\pi)$$

where x is the height and y is the radius at height x (in inches).

A base for the vase is preformed and placed on a potter's wheel. How much clay should be added to the base to form this vase if the inside radius is always 1 inch less than the outside radius? Section 7.3 contains the needed mathematics.

Chapter 7 Overview

By this point it should be apparent that finding the limits of Riemann sums is not just an intellectual exercise; it is a natural way to calculate mathematical or physical quantities that appear to be irregular when viewed as a whole, but which can be fragmented into regular pieces. We calculate values for the regular pieces using known formulas, then sum them to find a value for the irregular whole. This approach to problem solving was around for thousands of years before calculus came along, but it was tedious work and the more accurate you wanted to be the more tedious it became.

With calculus it became possible to get *exact* answers for these problems with almost no effort, because in the limit these sums became definite integrals and definite integrals could be evaluated with antiderivatives. With calculus, the challenge became one of fitting an integrable function to the situation at hand (the "modeling" step) and then finding an antiderivative for it.

Today we can finesse the antidifferentiation step (occasionally an insurmountable hurdle for our predecessors) with programs like NINT, but the modeling step is no less crucial. Ironically, it is the modeling step that is thousands of years old. Before either calculus or technology can be of assistance, we must still break down the irregular whole into regular parts and set up a function to be integrated. We have already seen how the process works with area, volume, and average value, for example. Now we will focus more closely on the underlying modeling step: how to set up the function to be integrated.

7.1 Integral As Net Change

What you'll learn about

- Linear Motion Revisited
- General Strategy
- Consumption Over Time
- Net Change from Data
- Work

. . . and why

The integral is a tool that can be used to calculate net change and total accumulation.

Linear Motion Revisited

In many applications, the integral is viewed as net change over time. The classic example of this kind is distance traveled, a problem we discussed in Chapter 5.

EXAMPLE 1 Interpreting a Velocity Function

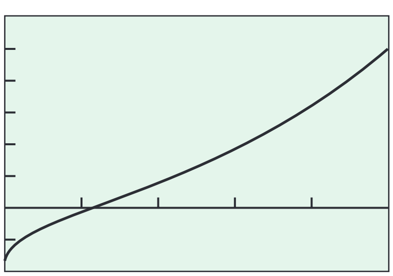
Figure 7.1 shows the velocity

$$\frac{ds}{dt} = v(t) = t^2 - \frac{8}{(t+1)^2}$$

of a particle moving along a horizontal s-axis for $0 \le t \le 5$. Describe the motion.

SOLUTION

Solve Graphically The graph of v (Figure 7.1) starts with $v(0) = -8$, which we interpret as saying that the particle has an initial velocity of 8 cm/sec to the left. It slows to a halt at about $t = 1.25$ sec, after which it moves to the right ($v > 0$) with increasing speed, reaching a velocity of $v(5) \approx 24.8$ cm/sec at the end. *Now try Exercise 1(a).*



[0, 5] by [–10, 30]

Figure 7.1 The velocity function in Example 1.

EXAMPLE 2 Finding Position from Displacement

Suppose the initial position of the particle in Example 1 is $s(0) = 9$. What is the particle's position at **(a)** $t = 1$ sec? **(b)** $t = 5$ sec?

SOLUTION

Solve Analytically

(a) The position at $t = 1$ is the initial position $s(0)$ plus the displacement (the amount, Δs, that the position changed from $t = 0$ to $t = 1$). When velocity is

continued

Reminder from Section 3.4

A change in position is a **displacement.** If $s(t)$ is a body's position at time t, the displacement over the time interval from t to $t + \Delta t$ is $s(t + \Delta t) - s(t)$. The displacement may be positive, negative, or zero, depending on the motion.

constant during a motion, we can find the displacement (change in position) with the formula

$$\text{Displacement} = \text{rate of change} \times \text{time}.$$

But in our case the velocity varies, so we resort instead to partitioning the time interval $[0, 1]$ into subintervals of length Δt so short that the velocity is effectively constant on each subinterval. If t_k is any time in the kth subinterval, the particle's velocity throughout that interval will be close to $v(t_k)$. The change in the particle's position during the brief time this constant velocity applies is

$$v(t_k)\,\Delta t.$$

If $v(t_k)$ is negative, the displacement is negative and the particle will move left. If $v(t_k)$ is positive, the particle will move right. The sum

$$\sum v(t_k)\,\Delta t$$

of all these small position changes approximates the displacement for the time interval $[0, 1]$.

The sum $\sum v(t_k)\,\Delta t$ is a Riemann sum for the continuous function $v(t)$ over $[0, 1]$. As the norms of the partitions go to zero, the approximations improve and the sums converge to the integral of v over $[0, 1]$, giving

$$\begin{aligned}\text{Displacement} &= \int_0^1 v(t)\,dt \\ &= \int_0^1 \left(t^2 - \frac{8}{(t+1)^2}\right) dt \\ &= \left[\frac{t^3}{3} + \frac{8}{t+1}\right]_0^1 \\ &= \frac{1}{3} + \frac{8}{2} - 8 = -\frac{11}{3}.\end{aligned}$$

During the first second of motion, the particle moves $11/3$ cm to the left. It starts at $s(0) = 9$, so its position at $t = 1$ is

$$\text{New position} = \text{initial position} + \text{displacement} = 9 - \frac{11}{3} = \frac{16}{3}.$$

(b) If we model the displacement from $t = 0$ to $t = 5$ in the same way, we arrive at

$$\text{Displacement} = \int_0^5 v(t)\,dt = \left[\frac{t^3}{3} + \frac{8}{t+1}\right]_0^5 = 35.$$

The motion has the net effect of displacing the particle 35 cm to the right of its starting point. The particle's final position is

$$\begin{aligned}\text{Final position} &= \text{initial position} + \text{displacement} \\ &= s(0) + 35 = 9 + 35 = 44.\end{aligned}$$

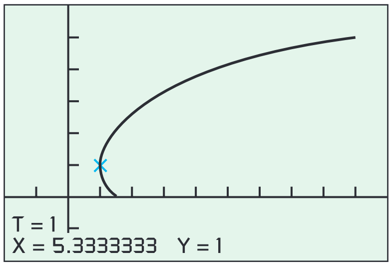
Support Graphically The position of the particle at any time t is given by

$$s(t) = \int_0^t \left[u^2 - \frac{8}{(u+1)^2}\right] du + 9,$$

because $s'(t) = v(t)$ and $s(0) = 9$. Figure 7.2 shows the graph of $s(t)$ given by the parametrization

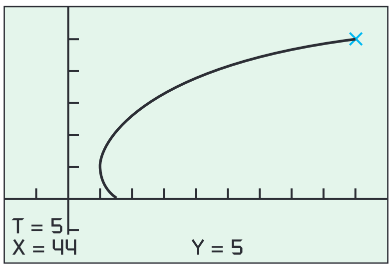
$$x(t) = \text{NINT}\,(v(u), u, 0, t) + 9, \quad y(t) = t, \quad 0 \le t \le 5.$$

continued



[–10, 50] by [–2, 6]

(a)



[–10, 50] by [–2, 6]

(b)

Figure 7.2 Using TRACE and the parametrization in Example 2 you can "see" the left and right motion of the particle.

(a) Figure 7.2a supports that the position of the particle at $t = 1$ is 16/3.

(b) Figure 7.2b shows the position of the particle is 44 at $t = 5$. Therefore, the displacement is $44 - 9 = 35$. ***Now try Exercise 1(b).***

The reason for our method in Example 2 was to illustrate the *modeling step* that will be used throughout this chapter. We can also solve Example 2 using the techniques of Chapter 6 as shown in Exploration 1.

EXPLORATION 1 Revisiting Example 2

The velocity of a particle moving along a horizontal s-axis for $0 \le t \le 5$ is

$$\frac{ds}{dt} = t^2 - \frac{8}{(t+1)^2}.$$

1. Use the indefinite integral of ds/dt to find the solution of the initial value problem

$$\frac{ds}{dt} = t^2 - \frac{8}{(t+1)^2}, \qquad s(0) = 9.$$

2. Determine the position of the particle at $t = 1$. Compare your answer with the answer to Example 2a.
3. Determine the position of the particle at $t = 5$. Compare your answer with the answer to Example 2b.

We know now that the particle in Example 1 was at $s(0) = 9$ at the beginning of the motion and at $s(5) = 44$ at the end. But it did not travel from 9 to 44 directly—it began its trip by moving to the left (Figure 7.2). How much distance did the particle actually travel? We find out in Example 3.

EXAMPLE 3 Calculating Total Distance Traveled

Find the *total distance traveled* by the particle in Example 1.

SOLUTION

Solve Analytically We partition the time interval as in Example 2 but record every position shift as *positive* by taking absolute values. The Riemann sum approximating total distance traveled is

$$\sum |v(t_k)|\,\Delta t,$$

and we are led to the integral

$$\text{Total distance traveled} = \int_0^5 |v(t)|\,dt = \int_0^5 \left| t^2 - \frac{8}{(t+1)^2} \right| dt.$$

Evaluate Numerically We have

$$\text{NINT}\left(\left| t^2 - \frac{8}{(t+1)^2} \right|, t, 0, 5 \right) \approx 42.59.$$

Now try Exercise 1(c).

What we learn from Examples 2 and 3 is this: Integrating velocity gives displacement (net area between the velocity curve and the time axis). Integrating the *absolute value* of velocity gives total distance traveled (total area between the velocity curve and the time axis).

General Strategy

The idea of fragmenting net effects into finite sums of easily estimated small changes is not new. We used it in Section 5.1 to estimate cardiac output, volume, and air pollution. What *is* new is that we can now identify many of these sums as Riemann sums and express their limits as integrals. The advantages of doing so are twofold. First, we can evaluate one of these integrals to get an accurate result in less time than it takes to crank out even the crudest estimate from a finite sum. Second, the integral itself becomes a formula that enables us to solve similar problems without having to repeat the modeling step.

The strategy that we began in Section 5.1 and have continued here is the following:

Strategy for Modeling with Integrals

1. *Approximate what you want to find as a Riemann sum* of values of a continuous function multiplied by interval lengths. If $f(x)$ is the function and $[a, b]$ the interval, and you partition the interval into subintervals of length Δx, the approximating sums will have the form $\sum f(c_k)\,\Delta x$ with c_k a point in the kth subinterval.
2. *Write a definite integral,* here $\int_a^b f(x)\,dx$, to express the limit of these sums as the norms of the partitions go to zero.
3. *Evaluate the integral* numerically or with an antiderivative.

EXAMPLE 4 Modeling the Effects of Acceleration

A car moving with initial velocity of 5 mph accelerates at the rate of $a(t) = 2.4t$ mph per second for 8 seconds.

(a) How fast is the car going when the 8 seconds are up?

(b) How far did the car travel during those 8 seconds?

SOLUTION

(a) We first model the effect of the acceleration on the car's velocity.

Step 1: *Approximate the net change in velocity as a Riemann sum.* When acceleration is constant,

$$\text{velocity change} = \text{acceleration} \times \text{time applied}.$$

To apply this formula, we partition $[0, 8]$ into short subintervals of length Δt. On each subinterval the acceleration is nearly constant, so if t_k is any point in the kth subinterval, the change in velocity imparted by the acceleration in the subinterval is approximately

$$a(t_k)\,\Delta t \text{ mph}.$$

The net change in velocity for $0 \le t \le 8$ is approximately

$$\sum a(t_k)\,\Delta t \text{ mph}.$$

Step 2: *Write a definite integral.* The limit of these sums as the norms of the partitions go to zero is

$$\int_0^8 a(t)\,dt.$$

continued

Step 3: *Evaluate the integral.* Using an antiderivative, we have

$$\text{Net velocity change} = \int_0^8 2.4t\, dt = 1.2t^2\Big]_0^8 = 76.8 \text{ mph.}$$

So, how fast is the car going when the 8 seconds are up? Its initial velocity is 5 mph and the acceleration adds another 76.8 mph for a total of 81.8 mph.

(b) There is nothing special about the upper limit 8 in the preceding calculation. Applying the acceleration for any length of time t adds

$$\int_0^t 2.4u\, du \text{ mph}$$

to the car's velocity, giving

$$v(t) = 5 + \int_0^t 2.4u\, du = 5 + 1.2t^2 \text{ mph.}$$

The distance traveled from $t = 0$ to $t = 8$ sec is

$$\int_0^8 |v(t)|\, dt = \int_0^8 (5 + 1.2t^2)\, dt$$
$$= \Big[5t + 0.4t^3\Big]_0^8$$
$$= 244.8 \text{ mph} \times \text{seconds.}$$

Miles-per-hour second is not a distance unit that we normally work with! To convert to miles we multiply by hours/second $= 1/3600$, obtaining

$$244.8 \times \frac{1}{3600} = 0.068 \text{ mile.}$$

The car traveled 0.068 mi during the 8 seconds of acceleration. ***Now try Exercise 9.***

Consumption Over Time

The integral is a natural tool to calculate net change and total accumulation of more quantities than just distance and velocity. Integrals can be used to calculate growth, decay, and, as in the next example, consumption. Whenever you want to find the cumulative effect of a varying rate of change, integrate it.

EXAMPLE 5 Potato Consumption

From 1970 to 1980, the rate of potato consumption in a particular country was $C(t) = 2.2 + 1.1^t$ millions of bushels per year, with t being years since the beginning of 1970. How many bushels were consumed from the beginning of 1972 to the end of 1973?

SOLUTION

We seek the cumulative effect of the consumption rate for $2 \le t \le 4$.

Step 1: *Riemann sum.* We partition $[2, 4]$ into subintervals of length Δt and let t_k be a time in the kth subinterval. The amount consumed during this interval is approximately

$$C(t_k)\,\Delta t \text{ million bushels.}$$

The consumption for $2 \le t \le 4$ is approximately

$$\sum C(t_k)\,\Delta t \text{ million bushels.}$$

continued

Step 2: *Definite integral.* The amount consumed from $t = 2$ to $t = 4$ is the limit of these sums as the norms of the partitions go to zero.

$$\int_2^4 C(t)\,dt = \int_2^4 (2.2 + 1.1^t)\,dt \text{ million bushels}$$

Step 3: *Evaluate.* Evaluating numerically, we obtain

$$\text{NINT}\,(2.2 + 1.1^t, t, 2, 4) \approx 7.066 \text{ million bushels.}$$

Now try Exercise 21.

Net Change from Data

Many real applications begin with data, not a fully modeled function. In the next example, we are given data on the rate at which a pump operates in consecutive 5-minute intervals and asked to find the total amount pumped.

Table 7.1 Pumping Rates

Time (min)	Rate (gal/min)
0	58
5	60
10	65
15	64
20	58
25	57
30	55
35	55
40	59
45	60
50	60
55	63
60	63

EXAMPLE 6 Finding Gallons Pumped from Rate Data

A pump connected to a generator operates at a varying rate, depending on how much power is being drawn from the generator to operate other machinery. The rate (gallons per minute) at which the pump operates is recorded at 5-minute intervals for one hour as shown in Table 7.1. How many gallons were pumped during that hour?

SOLUTION

Let $R(t)$, $0 \le t \le 60$, be the pumping rate as a continuous function of time for the hour. We can partition the hour into short subintervals of length Δt on which the rate is nearly constant and form the sum $\sum R(t_k)\,\Delta t$ as an approximation to the amount pumped during the hour. This reveals the integral formula for the number of gallons pumped to be

$$\text{Gallons pumped} = \int_0^{60} R(t)\,dt.$$

We have no formula for R in this instance, but the 13 equally spaced values in Table 7.1 enable us to estimate the integral with the Trapezoidal Rule:

$$\int_0^{60} R(t)\,dt \approx \frac{60}{2 \cdot 12}\left[58 + 2(60) + 2(65) + \cdots + 2(63) + 63\right]$$

$$= 3582.5.$$

The total amount pumped during the hour is about 3580 gal. *Now try Exercise 27.*

Joules

The joule, abbreviated J and pronounced "jewel," is named after the English physicist James Prescott Joule (1818–1889). The defining equation is

1 joule = (1 newton)(1 meter).

In symbols, 1 J = 1 N • m.

It takes a force of about 1 N to lift an apple from a table. If you lift it 1 m you have done about 1 J of work on the apple. If you eat the apple, you will have consumed about 80 food calories, the heat equivalent of nearly 335,000 joules. If this energy were directly useful for mechanical work (it's not), it would enable you to lift 335,000 more apples up 1 m.

Work

In everyday life, *work* means an activity that requires muscular or mental effort. In science, the term refers specifically to a force acting on a body and the body's subsequent displacement. When a body moves a distance d along a straight line as a result of the action of a force of constant magnitude F in the direction of motion, the **work** done by the force is

$$W = Fd.$$

The equation $W = Fd$ is the **constant-force formula** for work.

The units of work are force × distance. In the metric system, the unit is the newton-meter, which, for historical reasons, is called a joule (see margin note). In the U.S. customary system, the most common unit of work is the **foot-pound.**

Hooke's Law for springs says that the force it takes to stretch or compress a spring x units from its natural (unstressed) length is a constant times x. In symbols,

$$F = kx,$$

where k, measured in force units per unit length, is a characteristic of the spring called the **force constant.**

EXAMPLE 7 A Bit of Work

It takes a force of 10 N to stretch a spring 2 m beyond its natural length. How much work is done in stretching the spring 4 m from its natural length?

SOLUTION

We let $F(x)$ represent the force in newtons required to stretch the spring x meters from its natural length. By Hooke's Law, $F(x) = kx$ for some constant k. We are told that

$$F(2) = 10 = k \cdot 2,$$

so $k = 5$ N/m and $F(x) = 5x$ for this particular spring.

We construct an integral for the work done in applying F over the interval from $x = 0$ to $x = 4$.

Step 1: *Riemann sum.* We partition the interval into subintervals on each of which F is so nearly constant that we can apply the constant-force formula for work. If x_k is any point in the kth subinterval, the value of F throughout the interval is approximately $F(x_k) = 5x_k$. The work done by F across the interval is approximately $5x_k\,\Delta x$, where Δx is the length of the interval. The sum

$$\sum F(x_k)\,\Delta x = \sum 5x_k\,\Delta x$$

approximates the work done by F from $x = 0$ to $x = 4$.

Steps 2 and 3: *Integrate.* The limit of these sums as the norms of the partitions go to zero is

$$\int_0^4 F(x)\,dx = \int_0^4 5x\,dx = 5\frac{x^2}{2}\bigg]_0^4 = 40 \text{ N} \cdot \text{m}.$$

Numerically, work is the area under the force graph.

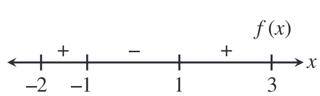
Now try Exercise 29.

We will revisit work in Section 7.5.

Quick Review 7.1 *(For help, go to Section 1.2.)*

In Exercises 1–10, find all values of x (if any) at which the function changes sign on the given interval. Sketch a number line graph of the interval, and indicate the sign of the function on each subinterval.

Example: $f(x) = x^2 - 1$ on $[-2, 3]$



Changes sign at $x = \pm 1$.

1. $\sin 2x$ on $[-3, 2]$
2. $x^2 - 3x + 2$ on $[-2, 4]$
3. $x^2 - 2x + 3$ on $[-4, 2]$
4. $2x^3 - 3x^2 + 1$ on $[-2, 2]$
5. $x \cos 2x$ on $[0, 4]$
6. xe^{-x} on $[0, \infty)$
7. $\dfrac{x}{x^2 + 1}$ on $[-5, 30]$
8. $\dfrac{x^2 - 2}{x^2 - 4}$ on $[-3, 3]$
9. $\sec\left(1 + \sqrt{1 - \sin^2 x}\right)$ on $(-\infty, \infty)$
10. $\sin(1/x)$ on $[0.1, 0.2]$

Section 7.1 Exercises

In Exercises 1–8, the function $v(t)$ is the velocity in m/sec of a particle moving along the x-axis. Use analytic methods to do each of the following:

(a) Determine when the particle is moving to the right, to the left, and stopped.

(b) Find the particle's displacement for the given time interval. If $s(0) = 3$, what is the particle's final position?

(c) Find the total distance traveled by the particle.

1. $v(t) = 5\cos t, \quad 0 \le t \le 2\pi$

2. $v(t) = 6\sin 3t, \quad 0 \le t \le \pi/2$

3. $v(t) = 49 - 9.8t, \quad 0 \le t \le 10$

4. $v(t) = 6t^2 - 18t + 12, \quad 0 \le t \le 2$

5. $v(t) = 5\sin^2 t\cos t, \quad 0 \le t \le 2\pi$

6. $v(t) = \sqrt{4 - t}, \quad 0 \le t \le 4$

7. $v(t) = e^{\sin t}\cos t, \quad 0 \le t \le 2\pi$

8. $v(t) = \dfrac{t}{1 + t^2}, \quad 0 \le t \le 3$

9. An automobile accelerates from rest at $1 + 3\sqrt{t}$ mph/sec for 9 seconds.

(a) What is its velocity after 9 seconds?

(b) How far does it travel in those 9 seconds?

10. A particle travels with velocity

$$v(t) = (t - 2)\sin t \ \text{ m/sec}$$

for $0 \le t \le 4$ sec.

(a) What is the particle's displacement?

(b) What is the total distance traveled?

11. ***Projectile*** Recall that the acceleration due to Earth's gravity is 32 ft/sec^2. From ground level, a projectile is fired straight upward with velocity 90 feet per second.

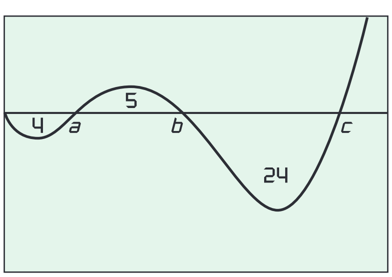
(a) What is its velocity after 3 seconds?

(b) When does it hit the ground?

(c) When it hits the ground, what is the net distance it has traveled?

(d) When it hits the ground, what is the total distance it has traveled?

In Exercises 12–16, a particle moves along the x-axis (units in cm). Its initial position at $t = 0$ sec is $x(0) = 15$. The figure shows the graph of the particle's velocity $v(t)$. The numbers are the *areas* of the enclosed regions.



12. What is the particle's displacement between $t = 0$ and $t = c$?

13. What is the total distance traveled by the particle in the same time period?

14. Give the positions of the particle at times a, b, and c.

15. Approximately where does the particle achieve its greatest positive acceleration on the interval $[0, b]$?

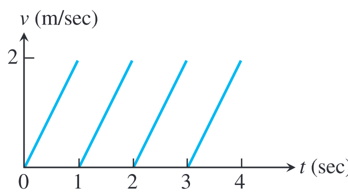
16. Approximately where does the particle achieve its greatest positive acceleration on the interval $[0, c]$?

In Exercises 17–20, the graph of the velocity of a particle moving on the x-axis is given. The particle starts at $x = 2$ when $t = 0$.

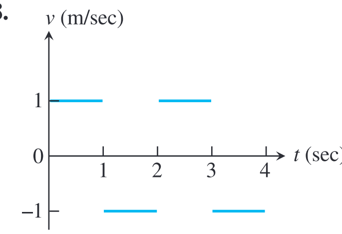
(a) Find where the particle is at the end of the trip.

(b) Find the total distance traveled by the particle.

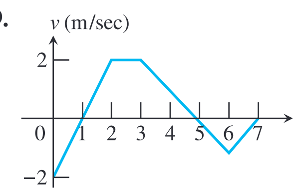
17.



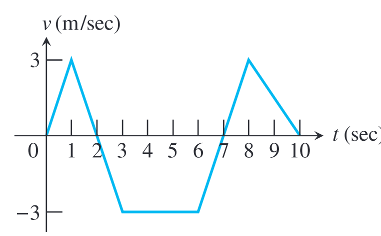
18.



19.



20.



21. ***U.S. Oil Consumption*** The rate of consumption of oil in the United States during the 1980s (in billions of barrels per year) is modeled by the function $C = 27.08 \cdot e^{t/25}$, where t is the number of years after January 1, 1980. Find the total consumption of oil in the United States from January 1, 1980 to January 1, 1990.

22. ***Home Electricity Use*** The rate at which your home consumes electricity is measured in kilowatts. If your home consumes electricity at the rate of 1 kilowatt for 1 hour, you will be charged for 1 "kilowatt-hour" of electricity. Suppose that the average consumption rate for a certain home is modeled by the function $C(t) = 3.9 - 2.4 \sin(\pi t/12)$, where $C(t)$ is measured in kilowatts and t is the number of hours past midnight. Find the average daily consumption for this home, measured in kilowatt-hours.

23. ***Population Density*** Population density measures the number of people per square mile inhabiting a given living area. Washerton's population density, which decreases as you move away from the city center, can be approximated by the function $10{,}000(2 - r)$ at a distance r miles from the city center.

(a) If the population density approaches zero at the edge of the city, what is the city's radius?

(b) A thin ring around the center of the city has thickness Δr and radius r. If you straighten it out, it suggests a rectangular strip. Approximately what is its area?

(c) **Writing to Learn** Explain why the population of the ring in part (b) is approximately

$$10{,}000(2 - r)(2\pi r)\,\Delta r.$$

(d) Estimate the total population of Washerton by setting up and evaluating a definite integral.

24. ***Oil Flow*** Oil flows through a cylindrical pipe of radius 3 inches, but friction from the pipe slows the flow toward the outer edge. The speed at which the oil flows at a distance r inches from the center is $8(10 - r^2)$ inches per second.

(a) In a plane cross section of the pipe, a thin ring with thickness Δr at a distance r inches from the center approximates a rectangular strip when you straighten it out. What is the area of the strip (and hence the approximate area of the ring)?

(b) Explain why we know that oil passes through this ring at approximately $8(10 - r^2)(2\pi r)\,\Delta r$ cubic inches per second.

(c) Set up and evaluate a definite integral that will give the rate (in cubic inches per second) at which oil is flowing through the pipe.

25. **Group Activity** ***Bagel Sales*** From 1995 to 2005, the Konigsberg Bakery noticed a consistent increase in annual sales of its bagels. The annual sales (in thousands of bagels) are shown below.

Year	Sales (thousands)
1995	5
1996	8.9
1997	16
1998	26.3
1999	39.8
2000	56.5
2001	76.4
2002	99.5
2003	125.8
2004	155.3
2005	188

(a) What was the total number of bagels sold over the 11-year period? (This is not a calculus question!)

(b) Use quadratic regression to model the annual bagel sales (in thousands) as a function $B(x)$, where x is the number of years after 1995.

(c) Integrate $B(x)$ over the interval $[0, 11]$ to find total bagel sales for the 11-year period.

(d) Explain graphically why the answer in part (a) is smaller than the answer in part (c).

26. **Group Activity** *(Continuation of Exercise 25)*

(a) Integrate $B(x)$ over the interval $[-0.5, 10.5]$ to find total bagel sales for the 11-year period.

(b) Explain graphically why the answer in part (a) is better than the answer in 25(c).

27. ***Filling Milk Cartons*** A machine fills milk cartons with milk at an approximately constant rate, but backups along the assembly line cause some variation. The rates (in cases per hour) are recorded at hourly intervals during a 10-hour period, from 8:00 A.M. to 6:00 P.M.

Time	Rate (cases/h)
8	120
9	110
10	115
11	115
12	119
1	120
2	120
3	115
4	112
5	110
6	121

Use the Trapezoidal Rule with $n = 10$ to determine approximately how many cases of milk were filled by the machine over the 10-hour period.

28. **Writing to Learn** As a school project, Anna accompanies her mother on a trip to the grocery store and keeps a log of the car's speed at 10-second intervals. Explain how she can use the data to estimate the distance from her home to the store. What is the connection between this process and the definite integral?

29. ***Hooke's Law*** A certain spring requires a force of 6 N to stretch it 3 cm beyond its natural length.

(a) What force would be required to stretch the string 9 cm beyond its natural length?

(b) What would be the work done in stretching the string 9 cm beyond its natural length?

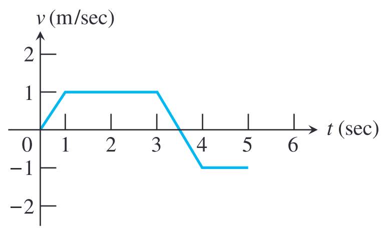
30. ***Hooke's Law*** Hooke's Law also applies to *compressing* springs; that is, it requires a force of kx to compress a spring a distance x from its natural length. Suppose a 10,000-lb force compressed a spring from its natural length of 12 inches to a length of 11 inches. How much work was done in compressing the spring

(a) the first half-inch? **(b)** the second half-inch?

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

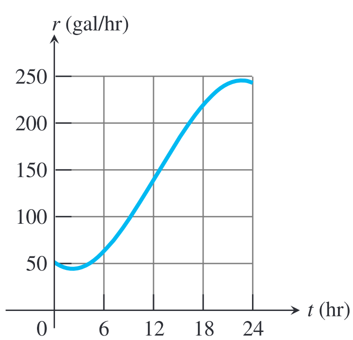
31. True or False The figure below shows the velocity for a particle moving along the x-axis. The displacement for this particle is negative. Justify your answer.



32. True or False If the velocity of a particle moving along the x-axis is always positive, then the displacement is equal to the total distance traveled. Justify your answer.

33. Multiple Choice The graph below shows the rate at which water is pumped from a storage tank. Approximate the total gallons of water pumped from the tank in 24 hours.

(A) 600 **(B)** 2400 **(C)** 3600 **(D)** 4200 **(E)** 4800



34. Multiple Choice The data for the acceleration $a(t)$ of a car from 0 to 15 seconds are given in the table below. If the velocity at $t = 0$ is 5 ft/sec, which of the following gives the approximate velocity at $t = 15$ using the Trapezoidal Rule?

(A) 47 ft/sec **(B)** 52 ft/sec **(C)** 120 ft/sec

(D) 125 ft/sec **(E)** 141 ft/sec

t (sec)	0	3	6	9	12	15
$a(t)$ (ft/sec^2)	4	8	6	9	10	10

35. Multiple Choice The rate at which customers arrive at a counter to be served is modeled by the function F defined by $F(t) = 12 + 6\cos\left(\frac{t}{\pi}\right)$ for $0 \le t \le 60$, where $F(t)$ is measured in customers per minute and t is measured in minutes. To the nearest whole number, how many customers arrive at the counter over the 60-minute period?

(A) 720 **(B)** 725 **(C)** 732 **(D)** 744 **(E)** 756

36. Multiple Choice Pollution is being removed from a lake at a rate modeled by the function $y = 20e^{-0.5t}$ tons/yr, where t is the number of years since 1995. Estimate the amount of pollution removed from the lake between 1995 and 2005. Round your answer to the nearest ton.

(A) 40 **(B)** 47 **(C)** 56 **(D)** 61 **(E)** 71

Extending the Ideas

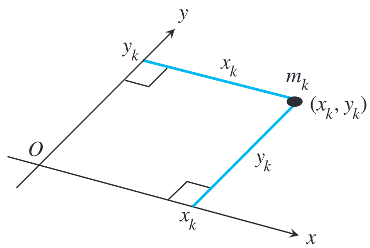
37. *Inflation* Although the economy is continuously changing, we analyze it with discrete measurements. The following table records the *annual* inflation rate as measured each month for 13 consecutive months. Use the Trapezoidal Rule with $n = 12$ to find the overall inflation rate for the year.

Month	Annual Rate
January	0.04
February	0.04
March	0.05
April	0.06
May	0.05
June	0.04
July	0.04
August	0.05
September	0.04
October	0.06
November	0.06
December	0.05
January	0.05

38. *Inflation Rate* The table below shows the *monthly* inflation rate (in *thousandths*) for energy prices for thirteen consecutive months. Use the Trapezoidal Rule with $n = 12$ to approximate the *annual* inflation rate for the 12-month period running from the middle of the first month to the middle of the last month.

Month	Monthly Rate (in thousandths)
January	3.6
February	4.0
March	3.1
April	2.8
May	2.8
June	3.2
July	3.3
August	3.1
September	3.2
October	3.4
November	3.4
December	3.9
January	4.0

39. ***Center of Mass*** Suppose we have a finite collection of masses in the coordinate plane, the mass m_k located at the point (x_k, y_k) as shown in the figure.



Each mass m_k has **moment $m_k y_k$ with respect to the x-axis** and **moment $m_k x_k$ about the y-axis.** The moments of the entire system with respect to the two axes are

$$\text{Moment about } x\text{-axis: } M_x = \sum m_k y_k,$$

$$\text{Moment about } y\text{-axis: } M_y = \sum m_k x_k.$$

The **center of mass** is $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{M_y}{M} = \frac{\sum m_k x_k}{\sum m_k} \quad \text{and} \quad \bar{y} = \frac{M_x}{M} = \frac{\sum m_k y_k}{\sum m_k}.$$

Suppose we have a thin, flat plate occupying a region in the plane.

(a) Imagine the region cut into thin strips parallel to the y-axis. Show that

$$\bar{x} = \frac{\int x\,dm}{\int dm},$$

where $dm = \delta\,dA$, $\delta =$ density (mass per unit area), and $A =$ area of the region.

(b) Imagine the region cut into thin strips parallel to the x-axis. Show that

$$\bar{y} = \frac{\int y\,dm}{\int dm},$$

where $dm = \delta\,dA$, $\delta =$ density, and $A =$ area of the region.

In Exercises 40 and 41, use Exercise 39 to find the center of mass of the region with given density.

40. the region bounded by the parabola $y = x^2$ and the line $y = 4$ with constant density δ

41. the region bounded by the lines $y = x$, $y = -x$, $x = 2$ with constant density δ

7.2 Areas in the Plane

What you'll learn about

- Area Between Curves
- Area Enclosed by Intersecting Curves
- Boundaries with Changing Functions
- Integrating with Respect to y
- Saving Time with Geometric Formulas

. . . and why

The techniques of this section allow us to compute areas of complex regions of the plane.

Area Between Curves

We know how to find the area of a region between a curve and the x-axis but many times we want to know the area of a region that is bounded above by one curve, $y = f(x)$, and below by another, $y = g(x)$ (Figure 7.3).

We find the area as an integral by applying the first two steps of the modeling strategy developed in Section 7.1.

1. We partition the region into vertical strips of equal width Δx and approximate each strip with a rectangle with base parallel to $[a, b]$ (Figure 7.4). Each rectangle has area

$$[f(c_k) - g(c_k)]\,\Delta x$$

for some c_k in its respective subinterval (Figure 7.5). This expression will be nonnegative even if the region lies below the x-axis. We approximate the area of the region with the Riemann sum

$$\sum [f(c_k) - g(c_k)]\,\Delta x.$$

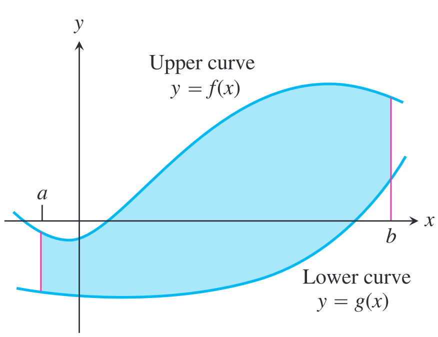


Figure 7.3 The region between $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$.

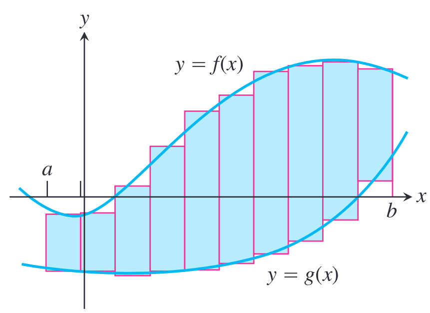


Figure 7.4 We approximate the region with rectangles perpendicular to the x-axis.

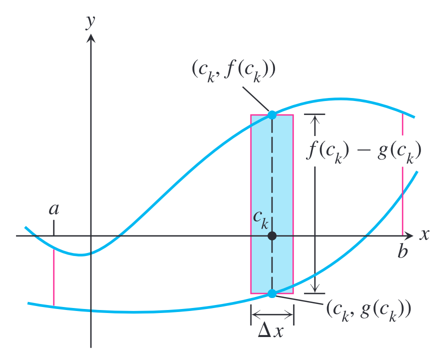


Figure 7.5 The area of a typical rectangle is $[f(c_k) - g(c_k)]\,\Delta x$.

2. The limit of these sums as $\Delta x \to 0$ is

$$\int_a^b [f(x) - g(x)]\,dx.$$

This approach to finding area captures the properties of area, so it can serve as a definition.

DEFINITION Area Between Curves

If f and g are continuous with $f(x) \ge g(x)$ throughout $[a, b]$, then the **area between the curves $y = f(x)$ and $y = g(x)$ from a to b** is the integral of $[f - g]$ from a to b,

$$A = \int_a^b [f(x) - g(x)]\,dx.$$

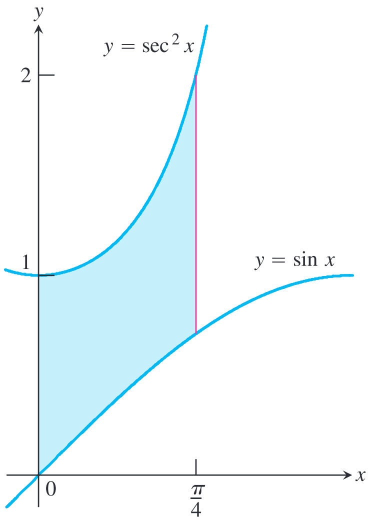


Figure 7.6 The region in Example 1.

EXAMPLE 1 Applying the Definition

Find the area of the region between $y = \sec^2 x$ and $y = \sin x$ from $x = 0$ to $x = \pi/4$.

SOLUTION

We graph the curves (Figure 7.6) to find their relative positions in the plane, and see that $y = \sec^2 x$ lies *above* $y = \sin x$ on $[0, \pi/4]$. The area is therefore

$$A = \int_0^{\pi/4} [\sec^2 x - \sin x]\, dx$$

$$= \Big[\tan x + \cos x\Big]_0^{\pi/4}$$

$$= \frac{\sqrt{2}}{2} \text{ units squared.}$$

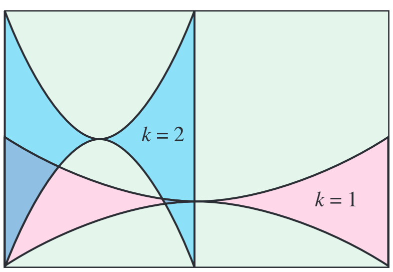
Now try Exercise 1.

EXPLORATION 1 A Family of Butterflies

For each positive integer k, let A_k denote the area of the butterfly-shaped region enclosed between the graphs of $y = k \sin kx$ and $y = 2k - k \sin kx$ on the interval $[0, \pi/k]$. The regions for $k = 1$ and $k = 2$ are shown in Figure 7.7.

1. Find the areas of the two regions in Figure 7.7.
2. Make a conjecture about the areas A_k for $k \geq 3$.
3. Set up a definite integral that gives the area A_k. Can you make a simple u-substitution that will transform this integral into the definite integral that gives the area A_1?
4. What is $\lim_{k\to\infty} A_k$?
5. If P_k denotes the perimeter of the kth butterfly-shaped region, what is $\lim_{k\to\infty} P_k$? (You can answer this question without an explicit formula for P_k.)

$y_1 = 2k - k \sin kx$
$y_2 = k \sin kx$



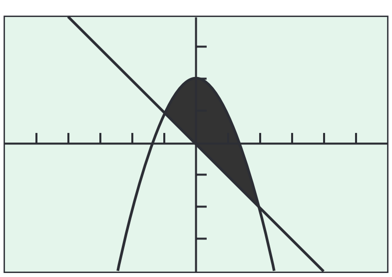
[0, π] by [0, 4]

Figure 7.7 Two members of the family of butterfly-shaped regions described in Exploration 1.

Area Enclosed by Intersecting Curves

When a region is enclosed by intersecting curves, the intersection points give the limits of integration.

$y_1 = 2 - x^2$
$y_2 = -x$



[–6, 6] by [–4, 4]

Figure 7.8 The region in Example 2.

EXAMPLE 2 Area of an Enclosed Region

Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

SOLUTION

We graph the curves to view the region (Figure 7.8).

The limits of integration are found by solving the equation

$$2 - x^2 = -x$$

either algebraically or by calculator. The solutions are $x = -1$ and $x = 2$.

continued

Since the parabola lies above the line on $[-1, 2]$, the area integrand is $2 - x^2 - (-x)$.

$$A = \int_{-1}^{2} [2 - x^2 - (-x)]\, dx$$

$$= \left[2x - \frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{2}$$

$$= \frac{9}{2} \text{ units squared}$$

Now try Exercise 5.

EXAMPLE 3 Using a Calculator

Find the area of the region enclosed by the graphs of $y = 2\cos x$ and $y = x^2 - 1$.

SOLUTION

The region is shown in Figure 7.9.

Using a calculator, we solve the equation

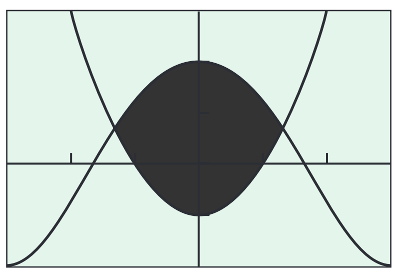
$$2\cos x = x^2 - 1$$

to find the x-coordinates of the points where the curves intersect. These are the limits of integration. The solutions are $x = \pm 1.265423706$. We store the negative value as A and the positive value as B. The area is

$$\text{NINT}\,(2\cos x - (x^2 - 1), x, A, B) \approx 4.994907788.$$

This is the final calculation, so we are now free to round. The area is about 4.99.

Now try Exercise 7.



[–3, 3] by [–2, 3]

Figure 7.9 The region in Example 3.

Boundaries with Changing Functions

If a boundary of a region is defined by more than one function, we can partition the region into subregions that correspond to the function changes and proceed as usual.

Finding Intersections by Calculator

The coordinates of the points of intersection of two curves are sometimes needed for other calculations. To take advantage of the accuracy provided by calculators, use them to solve for the values and *store* the ones you want.

EXAMPLE 4 Finding Area Using Subregions

Find the area of the region R in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.

SOLUTION

The region is shown in Figure 7.10.

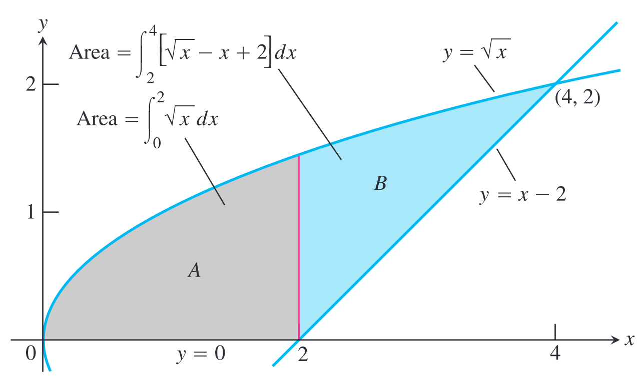


Figure 7.10 Region R split into subregions A and B. (Example 4)

continued

While it appears that no single integral can give the area of R (the bottom boundary is defined by two different curves), we can split the region at $x = 2$ into two regions A and B. The area of R can be found as the sum of the areas of A and B.

$$\text{Area of } R = \underbrace{\int_0^2 \sqrt{x}\,dx}_{} + \underbrace{\int_2^4 [\sqrt{x} - (x-2)]\,dx}_{}$$

$$= \frac{2}{3}x^{3/2}\Big]_0^2 + \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right]_2^4$$

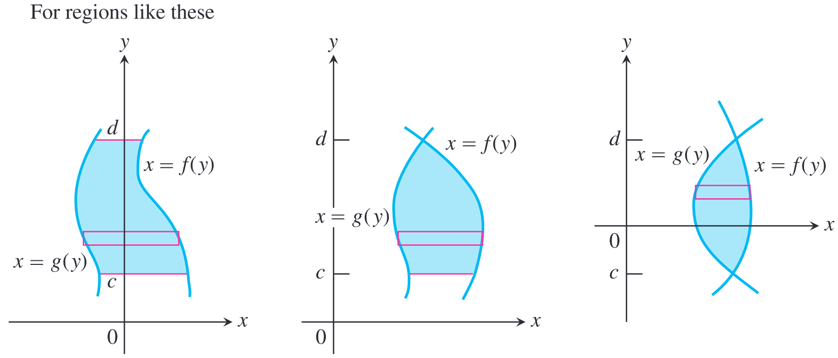
$$= \frac{10}{3} \text{ units squared}$$

Now try Exercise 9.

Integrating with Respect to *y*

Sometimes the boundaries of a region are more easily described by functions of y than by functions of x. We can use approximating rectangles that are horizontal rather than vertical and the resulting basic formula has y in place of x.

For regions like these



use this formula

$$A = \int_c^d [f(y) - g(y)]\,dy.$$

EXAMPLE 5 Integrating with Respect to *y*

Find the area of the region in Example 4 by integrating with respect to y.

SOLUTION

We remarked in solving Example 4 that "it appears that no single integral can give the area of R," but notice how appearances change when we think of our rectangles being summed over y. The interval of integration is $[0, 2]$, and the rectangles run between the same two curves on the entire interval. There is no need to split the region (Figure 7.11).

We need to solve for x in terms of y in both equations:

$$y = x - 2 \quad \text{becomes} \quad x = y + 2,$$

$$y = \sqrt{x} \quad \text{becomes} \quad x = y^2, \quad y \ge 0.$$

continued

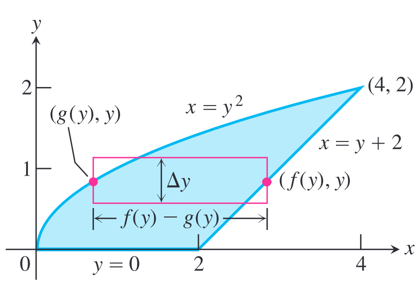
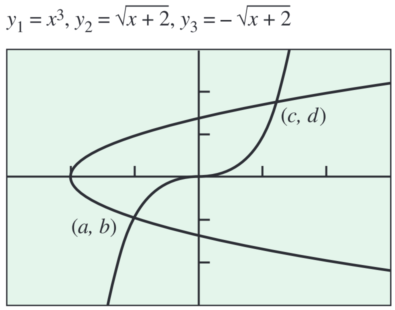


Figure 7.11 It takes two integrations to find the area of this region if we integrate with respect to x. It takes only one if we integrate with respect to y. (Example 5)

We must still be careful to subtract the lower number from the higher number when forming the integrand. In this case, the higher numbers are the higher x-values, which are on the line $x = y + 2$ because the line lies to the *right* of the parabola. So,

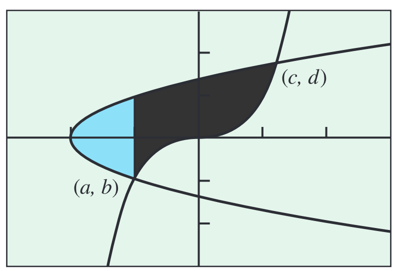
$$\text{Area of } R = \int_0^2 (y + 2 - y^2)\, dy = \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_0^2 = \frac{10}{3} \text{ units squared.}$$

Now try Exercise 11.



[–3, 3] by [–3, 3]

Figure 7.12 The region in Example 6.



[–3, 3] by [–3, 3]

Figure 7.13 If we integrate with respect to x in Example 6, we must split the region at $x = a$.

EXAMPLE 6 Making the Choice

Find the area of the region enclosed by the graphs of $y = x^3$ and $x = y^2 - 2$.

SOLUTION

We can produce a graph of the region on a calculator by graphing the three curves $y = x^3$, $y = \sqrt{x+2}$, and $y = -\sqrt{x+2}$ (Figure 7.12).

This conveniently gives us all of our bounding curves as functions of x. If we integrate in terms of x, however, we need to split the region at $x = a$ (Figure 7.13).

On the other hand, we can integrate from $y = b$ to $y = d$ and handle the entire region at once. We solve the cubic for x in terms of y:

$$y = x^3 \qquad \text{becomes} \qquad x = y^{1/3}.$$

To find the limits of integration, we solve $y^{1/3} = y^2 - 2$. It is easy to see that the lower limit is $b = -1$, but a calculator is needed to find that the upper limit $d = 1.793003715$. We store this value as D.

The cubic lies to the right of the parabola, so

$$\text{Area} = \text{NINT}\,(y^{1/3} - (y^2 - 2), y, -1, \text{D}) = 4.214939673.$$

The area is about 4.21. **Now try Exercise 27.**

Saving Time with Geometry Formulas

Here is yet another way to handle Example 4.

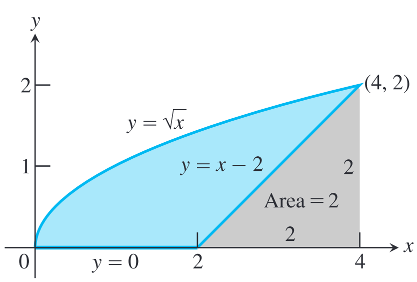


Figure 7.14 The area of the blue region is the area under the parabola $y = \sqrt{x}$ minus the area of the triangle. (Example 7)

EXAMPLE 7 Using Geometry

Find the area of the region in Example 4 by subtracting the area of the triangular region from the area under the square root curve.

SOLUTION

Figure 7.14 illustrates the strategy, which enables us to integrate with respect to x without splitting the region.

$$\text{Area} = \int_0^4 \sqrt{x}\, dx - \frac{1}{2}(2)(2) = \frac{2}{3}x^{3/2}\Big]_0^4 - 2 = \frac{10}{3} \text{ units squared}$$

Now try Exercise 35.

The moral behind Examples 4, 5, and 7 is that you often have options for finding the area of a region, some of which may be easier than others. You can integrate with respect to x or with respect to y, you can partition the region into subregions, and sometimes you can even use traditional geometry formulas. Sketch the region first and take a moment to determine the best way to proceed.

Quick Review 7.2 *(For help, go to Sections 1.2 and 5.1.)*

In Exercises 1–5, find the area between the x-axis and the graph of the given function over the given interval.

1. $y = \sin x$ over $[0, \pi]$

2. $y = e^{2x}$ over $[0, 1]$

3. $y = \sec^2 x$ over $[-\pi/4, \pi/4]$

4. $y = 4x - x^3$ over $[0, 2]$

5. $y = \sqrt{9 - x^2}$ over $[-3, 3]$

In Exercises 6–10, find the x- and y-coordinates of all points where the graphs of the given functions intersect. If the curves never intersect, write "NI."

6. $y = x^2 - 4x$ and $y = x + 6$

7. $y = e^x$ and $y = x + 1$

8. $y = x^2 - \pi x$ and $y = \sin x$

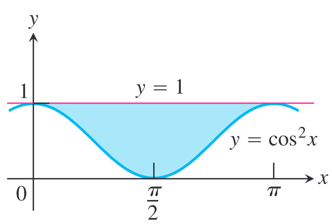
9. $y = \dfrac{2x}{x^2 + 1}$ and $y = x^3$

10. $y = \sin x$ and $y = x^3$

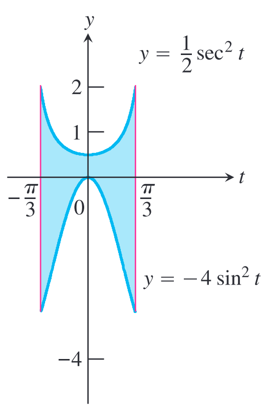
Section 7.2 Exercises

In Exercises 1–6, find the area of the shaded region analytically.

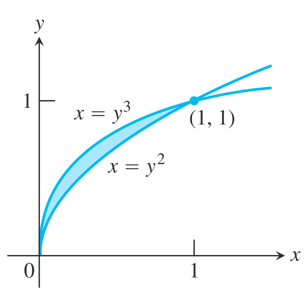
1.



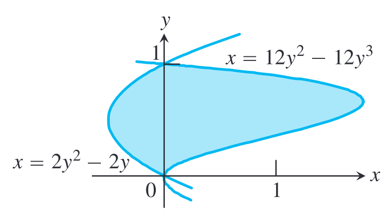
2.



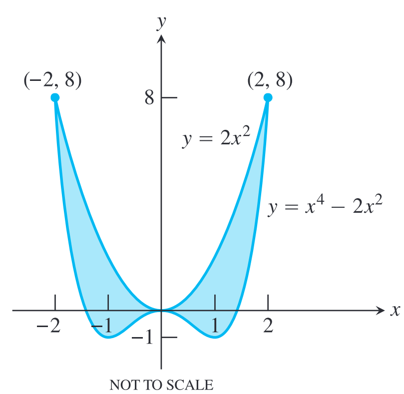
3.



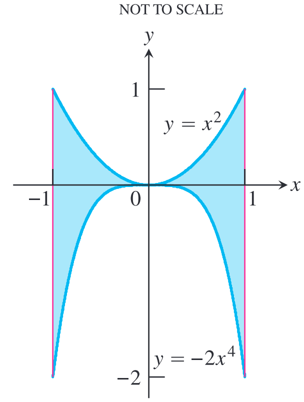
4.



5.



6.



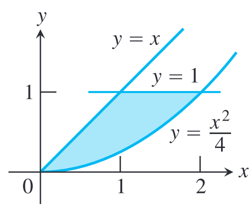
In Exercises 7 and 8, use a calculator to find the area of the region enclosed by the graphs of the two functions.

7. $y = \sin x,\ y = 1 - x^2$

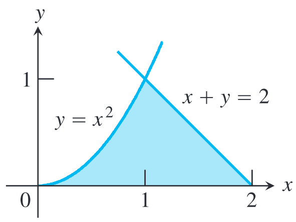
8. $y = \cos(2x),\ y = x^2 - 2$

In Exercises 9 and 10, find the area of the shaded region analytically.

9.



10.



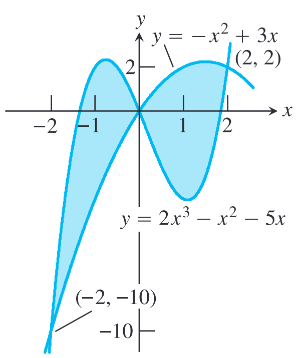
In Exercises 11 and 12, find the area enclosed by the graphs of the two curves by integrating with respect to y.

11. $y^2 = x + 1,\ y^2 = 3 - x$

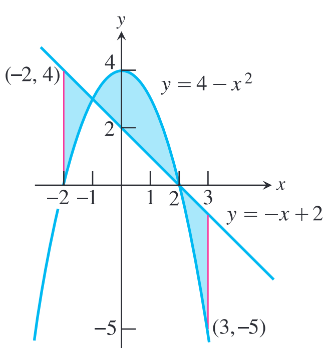
12. $y^2 = x + 3,\ y = 2x$

In Exercises 13 and 14, find the total shaded area.

13.



14.



In Exercises 15–34, find the area of the regions enclosed by the lines and curves.

15. $y = x^2 - 2$ and $y = 2$

16. $y = 2x - x^2$ and $y = -3$

17. $y = 7 - 2x^2$ and $y = x^2 + 4$

18. $y = x^4 - 4x^2 + 4$ and $y = x^2$

19. $y = x\sqrt{a^2 - x^2},\quad a > 0,$ and $y = 0$

20. $y = \sqrt{|x|}$ and $5y = x + 6$
(How many intersection points are there?)

21. $y = |x^2 - 4|$ and $y = (x^2/2) + 4$

22. $x = y^2$ and $x = y + 2$

23. $y^2 - 4x = 4$ and $4x - y = 16$

24. $x - y^2 = 0$ and $x + 2y^2 = 3$

25. $x + y^2 = 0$ and $x + 3y^2 = 2$

26. $4x^2 + y = 4$ and $x^4 - y = 1$

27. $x + y^2 = 3$ and $4x + y^2 = 0$

28. $y = 2 \sin x$ and $y = \sin 2x,\quad 0 \le x \le \pi$

29. $y = 8 \cos x$ and $y = \sec^2 x,\quad -\pi/3 \le x \le \pi/3$

30. $y = \cos(\pi x/2)$ and $y = 1 - x^2$

31. $y = \sin(\pi x/2)$ and $y = x$

32. $y = \sec^2 x,\quad y = \tan^2 x,\quad x = -\pi/4,\quad x = \pi/4$

33. $x = \tan^2 y$ and $x = -\tan^2 y,\quad -\pi/4 \le y \le \pi/4$

34. $x = 3 \sin y \sqrt{\cos y}$ and $x = 0,\quad 0 \le y \le \pi/2$

In Exercises 35 and 36, find the area of the region by subtracting the area of a triangular region from the area of a larger region.

35. The region on or above the x-axis bounded by the curves $y^2 = x + 3$ and $y = 2x$.

36. The region on or above the x-axis bounded by the curves $y = 4 - x^2$ and $y = 3x$.

37. Find the area of the propeller-shaped region enclosed by the curve $x - y^3 = 0$ and the line $x - y = 0$.

38. Find the area of the region in the first quadrant bounded by the line $y = x$, the line $x = 2$, the curve $y = 1/x^2$, and the x-axis.

39. Find the area of the "triangular" region in the first quadrant bounded on the left by the y-axis and on the right by the curves $y = \sin x$ and $y = \cos x$.

40. Find the area of the region between the curve $y = 3 - x^2$ and the line $y = -1$ by integrating with respect to **(a)** x, **(b)** y.

41. The region bounded below by the parabola $y = x^2$ and above by the line $y = 4$ is to be partitioned into two subsections of equal area by cutting across it with the horizontal line $y = c$.

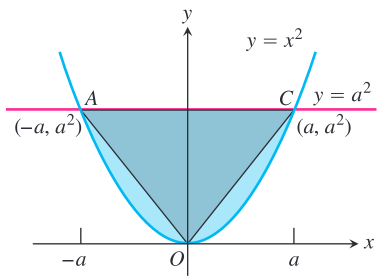
(a) Sketch the region and draw a line $y = c$ across it that looks about right. In terms of c, what are the coordinates of the points where the line and parabola intersect? Add them to your figure.

(b) Find c by integrating with respect to y. (This puts c in the limits of integration.)

(c) Find c by integrating with respect to x. (This puts c into the integrand as well.)

42. Find the area of the region in the first quadrant bounded on the left by the y-axis, below by the line $y = x/4$, above left by the curve $y = 1 + \sqrt{x}$, and above right by the curve $y = 2/\sqrt{x}$.

43. The figure here shows triangle AOC inscribed in the region cut from the parabola $y = x^2$ by the line $y = a^2$. Find the limit of the ratio of the area of the triangle to the area of the parabolic region as a approaches zero.

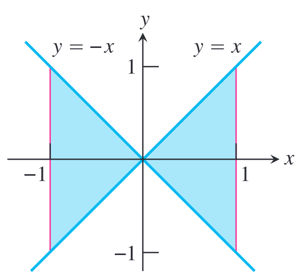


44. Suppose the area of the region between the graph of a positive continuous function f and the x-axis from $x = a$ to $x = b$ is 4 square units. Find the area between the curves $y = f(x)$ and $y = 2f(x)$ from $x = a$ to $x = b$.

45. Writing to Learn Which of the following integrals, if either, calculates the area of the shaded region shown here? Give reasons for your answer.

i. $\int_{-1}^{1} (x - (-x))\, dx = \int_{-1}^{1} 2x\, dx$

ii. $\int_{-1}^{1} (-x - (x))\, dx = \int_{-1}^{1} -2x\, dx$



46. Writing to Learn Is the following statement true, sometimes true, or never true? The area of the region between the graphs of the continuous functions $y = f(x)$ and $y = g(x)$ and the vertical lines $x = a$ and $x = b$ $(a < b)$ is

$$\int_a^b [f(x) - g(x)]\, dx.$$

Give reasons for your answer.

47. Find the area of the propeller-shaped region enclosed between the graphs of

$$y = \frac{2x}{x^2 + 1} \quad \text{and} \quad y = x^3.$$

48. Find the area of the propeller-shaped region enclosed between the graphs of $y = \sin x$ and $y = x^3$.

49. Find the positive value of k such that the area of the region enclosed between the graph of $y = k \cos x$ and the graph of $y = kx^2$ is 2.

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

50. True or False The area of the region enclosed by the graph of $y = x^2 + 1$ and the line $y = 10$ is 36. Justify your answer.

51. True or False The area of the region in the first quadrant enclosed by the graphs of $y = \cos x$, $y = x$, and the y-axis is given by the definite integral $\int_0^{0.739} (x - \cos x)\, dx$. Justify your answer.

52. Multiple Choice Let R be the region in the first quadrant bounded by the x-axis, the graph of $x = y^2 + 2$, and the line $x = 4$. Which of the following integrals gives the area of R?

(A) $\int_0^{\sqrt{2}} [4 - (y^2 + 2)]\, dy$ **(B)** $\int_0^{\sqrt{2}} [(y^2 + 2) - 4]\, dy$

(C) $\int_{-\sqrt{2}}^{\sqrt{2}} [4 - (y^2 + 2)]\, dy$ **(D)** $\int_{-\sqrt{2}}^{\sqrt{2}} [(y^2 + 2) - 4]\, dy$

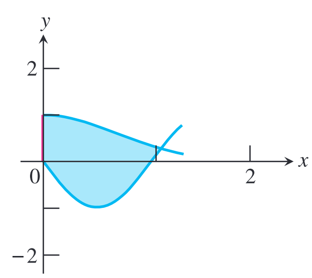
(E) $\int_2^4 [4 - (y^2 + 2)]\, dy$

53. Multiple Choice Which of the following gives the area of the region between the graphs of $y = x^2$ and $y = -x$ from $x = 0$ to $x = 3$?

(A) 2 **(B)** 9/2 **(C)** 13/2 **(D)** 13 **(E)** 27/2

54. Multiple Choice Let R be the shaded region enclosed by the graphs of $y = e^{-x^2}$, $y = -\sin(3x)$, and the y-axis as shown in the figure below. Which of the following gives the approximate area of the region R?

(A) 1.139 **(B)** 1.445 **(C)** 1.869 **(D)** 2.114 **(E)** 2.340



55. Multiple Choice Let f and g be the functions given by $f(x) = e^x$ and $g(x) = 1/x$. Which of the following gives the area of the region enclosed by the graphs of f and g between $x = 1$ and $x = 2$?

(A) $e^2 - e - \ln 2$

(B) $\ln 2 - e^2 + e$

(C) $e^2 - \frac{1}{2}$

(D) $e^2 - e - \frac{1}{2}$

(E) $\frac{1}{e} - \ln 2$

Exploration

56. Group Activity ***Area of Ellipse***

An ellipse with major axis of length $2a$ and minor axis of length $2b$ can be coordinatized with its center at the origin and its major axis horizontal, in which case it is defined by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

(a) Find the equations that define the upper and lower semiellipses as functions of x.

(b) Write an integral expression that gives the area of the ellipse.

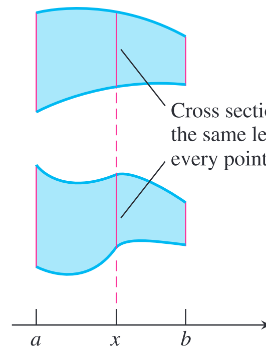
(c) With your group, use NINT to find the areas of ellipses for various lengths of a and b.

(d) There is a simple formula for the area of an ellipse with major axis of length $2a$ and minor axis of length $2b$. Can you tell what it is from the areas you and your group have found?

(e) Work with your group to write a *proof* of this area formula by showing that it is the exact value of the integral expression in part (b).

Extending the Ideas

57. ***Cavalieri's Theorem*** Bonaventura Cavalieri (1598–1647) discovered that if two plane regions can be arranged to lie over the same interval of the x-axis in such a way that they have identical vertical cross sections at every point (see figure), then the regions have the same area. Show that this theorem is true.



58. Find the area of the region enclosed by the curves

$$y = \frac{x}{x^2 + 1} \quad \text{and} \quad y = mx, \quad 0 < m < 1.$$

7.3 Volumes

What you'll learn about

- Volume As an Integral
- Square Cross Sections
- Circular Cross Sections
- Cylindrical Shells
- Other Cross Sections

. . . and why

The techniques of this section allow us to compute volumes of certain solids in three dimensions.

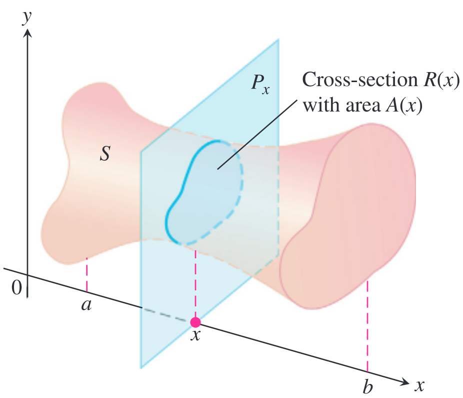


Figure 7.15 The cross section of an arbitrary solid at point x.

Volume As an Integral

In Section 5.1, Example 3, we estimated the volume of a sphere by partitioning it into thin slices that were nearly cylindrical and summing the cylinders' volumes using MRAM. MRAM sums are Riemann sums, and had we known how at the time, we could have continued on to express the volume of the sphere as a definite integral.

Starting the same way, we can now find the volumes of a great many solids by integration. Suppose we want to find the volume of a solid like the one in Figure 7.15. The cross section of the solid at each point x in the interval $[a, b]$ is a region $R(x)$ of area $A(x)$. If A is a continuous function of x, we can use it to define and calculate the volume of the solid as an integral in the following way.

We partition $[a, b]$ into subintervals of length Δx and slice the solid, as we would a loaf of bread, by planes perpendicular to the x-axis at the partition points. The kth slice, the one between the planes at x_{k-1} and x_k, has approximately the same volume as the cylinder between the two planes based on the region $R(x_k)$ (Figure 7.16).

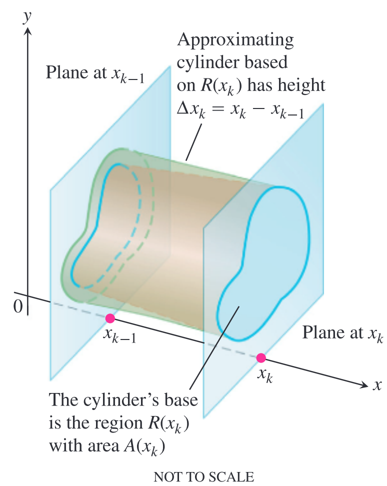


Figure 7.16 Enlarged view of the slice of the solid between the planes at x_{k-1} and x_k.

The volume of the cylinder is

$$V_k = \text{base area} \times \text{height} = A(x_k) \times \Delta x.$$

The sum

$$\sum V_k = \sum A(x_k) \times \Delta x$$

approximates the volume of the solid.

This is a Riemann sum for $A(x)$ on $[a, b]$. We expect the approximations to improve as the norms of the partitions go to zero, so we define their limiting integral to be the *volume of the solid.*

DEFINITION Volume of a Solid

The **volume of a solid** of known integrable cross section area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b,

$$V = \int_a^b A(x)\,dx.$$

To apply the formula in the previous definition, we proceed as follows.

How to Find Volume by the Method of Slicing

1. Sketch the solid and a typical cross section.
2. Find a formula for $A(x)$.
3. Find the limits of integration.
4. Integrate $A(x)$ to find the volume.

Square Cross Sections

Let us apply the volume formula to a solid with square cross sections.

EXAMPLE 1 A Square-Based Pyramid

A pyramid 3 m high has congruent triangular sides and a square base that is 3 m on each side. Each cross section of the pyramid parallel to the base is a square. Find the volume of the pyramid.

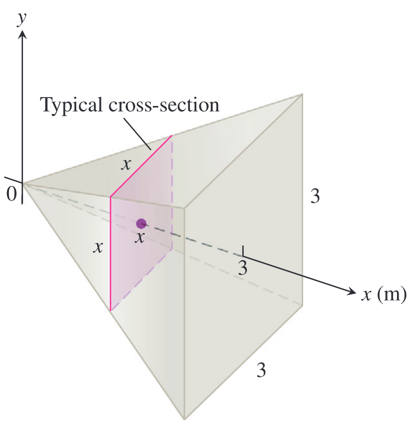


Figure 7.17 A cross section of the pyramid in Example 1.

SOLUTION

We follow the steps for the method of slicing.

1. *Sketch.* We draw the pyramid with its vertex at the origin and its altitude along the interval $0 \le x \le 3$. We sketch a typical cross section at a point x between 0 and 3 (Figure 7.17).
2. *Find a formula for $A(x)$.* The cross section at x is a square x meters on a side, so
$$A(x) = x^2.$$
3. *Find the limits of integration.* The squares go from $x = 0$ to $x = 3$.
4. *Integrate to find the volume.*
$$V = \int_0^3 A(x)\,dx = \int_0^3 x^2 = \frac{x^3}{3}\Big]_0^3 = 9 \text{ m}^3$$

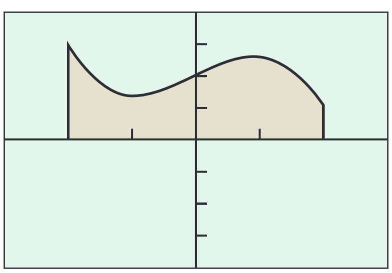
Now try Exercise 3.

Circular Cross Sections

The only thing that changes when the cross sections of a solid are circular is the formula for $A(x)$. Many such solids are **solids of revolution,** as in the next example.

EXAMPLE 2 A Solid of Revolution

The region between the graph of $f(x) = 2 + x\cos x$ and the x-axis over the interval $[-2, 2]$ is revolved about the x-axis to generate a solid. Find the volume of the solid.



[–3, 3] by [–4, 4]

Figure 7.18 The region in Example 2.

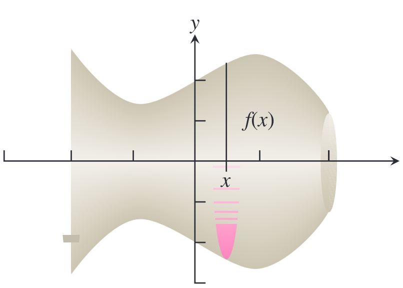


Figure 7.19 The region in Figure 7.18 is revolved about the x-axis to generate a solid. A typical cross section is circular, with radius $f(x) = 2 + x\cos x$. (Example 2)

SOLUTION

Revolving the region (Figure 7.18) about the x-axis generates the vase-shaped solid in Figure 7.19. The cross section at a typical point x is circular, with radius equal to $f(x)$. Its area is
$$A(x) = \pi(f(x))^2.$$

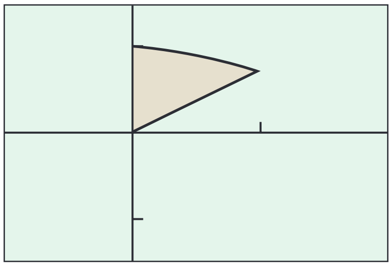
continued

The volume of the solid is

$$V = \int_{-2}^{2} A(x)\,dx$$

$$\approx \text{NINT}\,(\pi(2 + x\cos x)^2, x, -2, 2) \approx 52.43 \text{ units cubed.}$$

Now try Exercise 7.



[–π/4, π/2] by [–1.5, 1.5]

Figure 7.20 The region in Example 3.

EXAMPLE 3 Washer Cross Sections

The region in the first quadrant enclosed by the y-axis and the graphs of $y = \cos x$ and $y = \sin x$ is revolved about the x-axis to form a solid. Find its volume.

SOLUTION

The region is shown in Figure 7.20.

We revolve it about the x-axis to generate a solid with a cone-shaped cavity in its center (Figure 7.21).

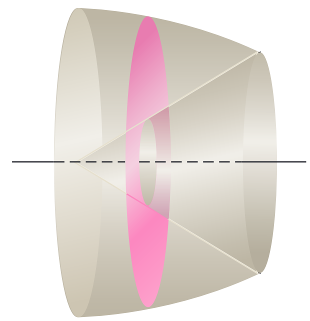


Figure 7.21 The solid generated by revolving the region in Figure 7.20 about the x-axis. A typical cross section is a washer: a circular region with a circular region cut out of its center. (Example 3)

This time each cross section perpendicular to the *axis of revolution* is a *washer,* a circular region with a circular region cut from its center. The area of a washer can be found by subtracting the inner area from the outer area (Figure 7.22).

In our region the cosine curve defines the outer radius, and the curves intersect at $x = \pi/4$. The volume is

$$V = \int_{0}^{\pi/4} \pi(\cos^2 x - \sin^2 x)\,dx$$

$$= \pi\int_{0}^{\pi/4} \cos 2x\,dx$$

$$= \pi\left[\frac{\sin 2x}{2}\right]_{0}^{\pi/4} = \frac{\pi}{2} \text{ units cubed.}$$

Now try Exercise 17.

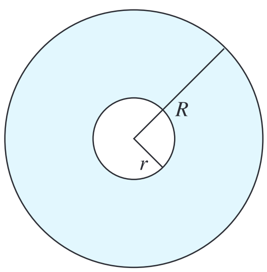


Figure 7.22 The area of a washer is $\pi R^2 - \pi r^2$. (Example 3)

CAUTION!

The area of a washer is $\pi R^2 - \pi r^2$, which you can simplify to $\pi(R^2 - r^2)$, but *not* to $\pi(R - r)^2$. No matter how tempting it is to make the latter simplification, it's wrong. Don't do it.

We could have done the integration in Example 3 with NINT, but we wanted to demonstrate how a trigonometric identity can be useful under unexpected circumstances in calculus. The double-angle identity turned a difficult integrand into an easy one and enabled us to get an exact answer by antidifferentiation.

Cylindrical Shells

There is another way to find volumes of solids of rotation that can be useful when the axis of revolution is perpendicular to the axis containing the natural interval of integration. Instead of summing volumes of thin slices, we sum volumes of thin cylindrical shells that grow outward from the axis of revolution like tree rings.

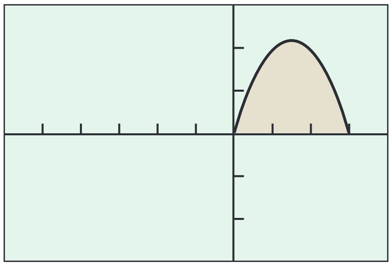


Figure 7.23 The graph of the region in Exploration 1, before revolution.

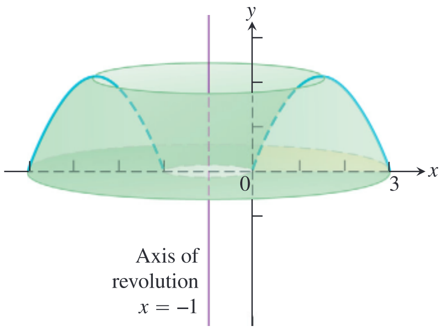


Figure 7.24 The region in Figure 7.23 is revolved about the line $x = -1$ to form a solid cake. The natural interval of integration is along the x-axis, perpendicular to the axis of revolution. (Exploration 1)

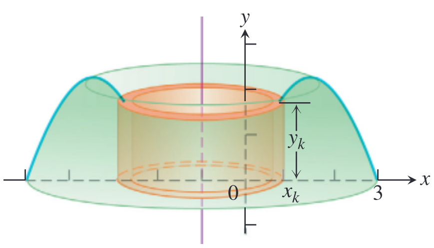


Figure 7.25 Cutting the cake into thin cylindrical slices, working from the inside out. Each slice occurs at some x_k between 0 and 3 and has thickness Δx. (Exploration 1)

EXPLORATION 1 Volume by Cylindrical Shells

The region enclosed by the x-axis and the parabola $y = f(x) = 3x - x^2$ is revolved about the line $x = -1$ to generate the shape of a cake (Figures 7.23, 7.24). (Such a cake is often called a bundt cake.) What is the volume of the cake?

Integrating with respect to y would be awkward here, as it is not easy to get the original parabola in terms of y. (Try finding the volume by washers and you will soon see what we mean.) To integrate with respect to x, you can do the problem by *cylindrical shells,* which requires that you cut the cake in a rather unusual way.

1. Instead of cutting the usual wedge shape, cut a *cylindrical* slice by cutting straight down all the way around close to the inside hole. Then cut another cylindrical slice around the enlarged hole, then another, and so on. The radii of the cylinders gradually increase, and the heights of the cylinders follow the contour of the parabola: smaller to larger, then back to smaller (Figure 7.25). Each slice is sitting over a subinterval of the x-axis of length Δx. Its radius is approximately $(1 + x_k)$. What is its height?
2. If you unroll the cylinder at x_k and flatten it out, it becomes (essentially) a rectangular slab with thickness Δx. Show that the volume of the slab is approximately $2\pi(x_k + 1)(3x_k - x_k^2)\Delta x$.
3. $\sum 2\pi(x_k + 1)(3x_k - x_k^2)\Delta x$ is a Riemann sum. What is the limit of these Riemann sums as $\Delta x \to 0$?
4. Evaluate the integral you found in step 3 to find the volume of the cake!

EXAMPLE 4 Finding Volumes Using Cylindrical Shells

The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

SOLUTION

1. Sketch the region and draw a line segment across it parallel to the axis of revolution (Figure 7.26). Label the segment's length (shell height) and distance from the axis of revolution (shell radius). The width of the segment is the shell thickness dy. (We drew the shell in Figure 7.27, but you need not do that.)

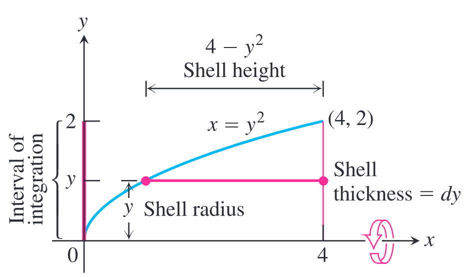


Figure 7.26 The region, shell dimensions, and interval of integration in Example 4.

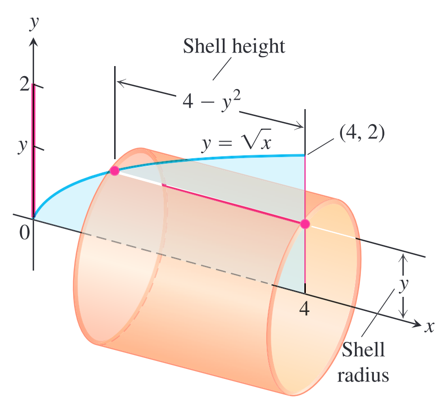


Figure 7.27 The shell swept out by the line segment in Figure 7.26.

2. Identify the limits of integration: y runs from 0 to 2.

3. Integrate to find the volume.

$$V = \int_0^2 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix} dy$$

$$= \int_0^2 2\pi(y)(4 - y^2)\, dy = 8\pi$$

Now try Exercise 33(a).

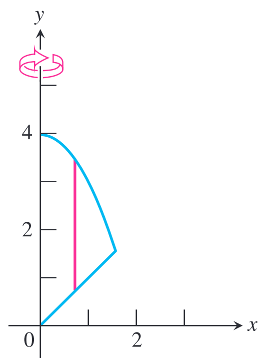


Figure 7.28 The region and the height of a typical shell in Example 5.

EXAMPLE 5 Finding Volumes Using Cylindrical Shells

The region bounded by the curves $y = 4 - x^2$, $y = x$, and $x = 0$ is revolved about the y-axis to form a solid. Use cylindrical shells to find the volume of the solid.

SOLUTION

1. Sketch the region and draw a line segment across it parallel to the y-axis (Figure 7.28). The segment's length (shell height) is $4 - x^2 - x$. The distance of the segment from the axis of revolution (shell radius) is x.

2. Identify the limits of integration: The x-coordinate of the point of intersection of the curves $y = 4 - x^2$ and $y = x$ in the first quadrant is about 1.562. So x runs from 0 to 1.562.

3. Integrate to find the volume.

$$V = \int_0^{1.562} 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix} dx$$

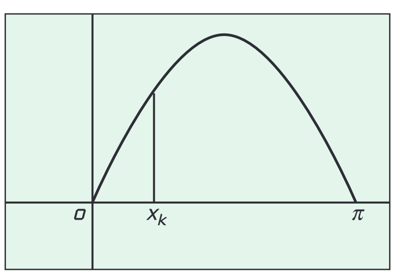
$$= \int_0^{1.562} 2\pi(x)(4 - x^2 - x)\, dx$$

$$\approx 13.327$$

Now try Exercise 35.

Other Cross Sections

The method of cross-section slicing can be used to find volumes of a wide variety of unusually shaped solids, so long as the cross sections have areas that we can describe with some formula. Admittedly, it does take a special artistic talent to *draw* some of these solids, but a crude picture is usually enough to suggest how to set up the integral.



[–1, 3.5] by [–0.8, 2.2]

Figure 7.29 The base of the paperweight in Example 6. The segment perpendicular to the x-axis at x_k is the diameter of a semicircle that is perpendicular to the base.

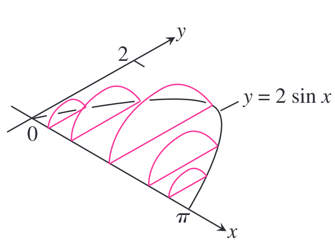


Figure 7.30 Cross sections perpendicular to the region in Figure 7.29 are semicircular. (Example 6)

EXAMPLE 6 A Mathematician's Paperweight

A mathematician has a paperweight made so that its base is the shape of the region between the x-axis and one arch of the curve $y = 2 \sin x$ (linear units in inches). Each cross section cut perpendicular to the x-axis (and hence to the xy-plane) is a semicircle whose diameter runs from the x-axis to the curve. (Think of the cross section as a semicircular fin sticking up out of the plane.) Find the volume of the paperweight.

SOLUTION

The paperweight is not easily drawn, but we know what it looks like. Its base is the region in Figure 7.29, and the cross sections perpendicular to the base are semicircular fins like those in Figure 7.30.

The semicircle at each point x has

$$\text{radius} = \frac{2 \sin x}{2} = \sin x \quad \text{and area} \quad A(x) = \frac{1}{2}\pi(\sin x)^2.$$

continued

The volume of the paperweight is

$$V = \int_0^{\pi} A(x)\,dx$$

$$= \frac{\pi}{2}\int_0^{\pi} (\sin x)^2\,dx$$

$$\approx \frac{\pi}{2}\,\text{NINT}\,((\sin x)^2, x, 0, \pi)$$

$$\approx \frac{\pi}{2}(1.570796327).$$

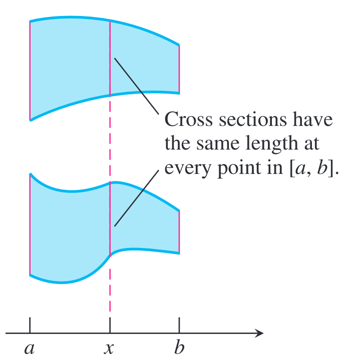
The number in parentheses looks like half of π, an observation that can be confirmed analytically, and which we support numerically by dividing by π to get 0.5. The volume of the paperweight is

$$\frac{\pi}{2}\cdot\frac{\pi}{2} = \frac{\pi^2}{4} \approx 2.47 \text{ in}^3.$$

Now try Exercise 39(a).

Bonaventura Cavalieri
(1598–1647)

Cavalieri, a student of Galileo, discovered that if two plane regions can be arranged to lie over the same interval of the *x*-axis in such a way that they have identical vertical cross sections at every point, then the regions have the same area. This theorem and a letter of recommendation from Galileo were enough to win Cavalieri a chair at the University of Bologna in 1629. The solid geometry version in Example 7, which Cavalieri never proved, was named after him by later geometers.



EXAMPLE 7 Cavalieri's Volume Theorem

Cavalieri's volume theorem says that solids with equal altitudes and identical cross section areas at each height have the same volume (Figure 7.31). This follows immediately from the definition of volume, because the cross section area function $A(x)$ and the interval $[a, b]$ are the same for both solids.

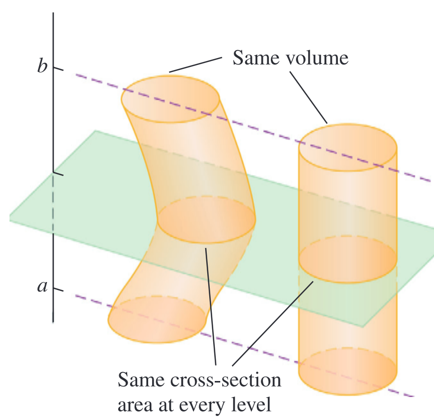
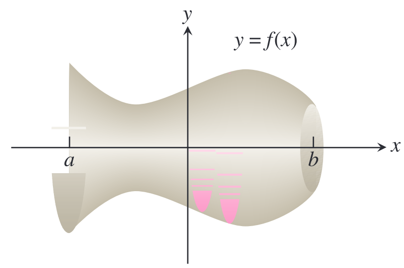


Figure 7.31 *Cavalieri's volume theorem:* These solids have the same volume. You can illustrate this yourself with stacks of coins. (Example 7)

Now try Exercise 43.

EXPLORATION 2 Surface Area

We know how to find the volume of a solid of revolution, but how would we find the *surface area*? As before, we partition the solid into thin slices, but now we wish to form a Riemann sum of approximations to *surface areas of slices* (rather than of volumes of slices).



A typical slice has a surface area that can be approximated by $2\pi \cdot f(x) \cdot \Delta s$, where Δs is the tiny *slant height* of the slice. We will see in Section 7.4, when we study arc length, that $\Delta s = \sqrt{\Delta x^2 + \Delta y^2}$, and that this can be written as $\Delta s = \sqrt{1 + (f'(x_k))^2}\,\Delta x$.

Thus, the surface area is approximated by the Riemann sum

$$\sum_{k=1}^{n} 2\pi f(x_k)\sqrt{1 + (f'(x_k))^2}\,\Delta x.$$

1. Write the limit of the Riemann sums as a definite integral from a to b. When will the limit exist?
2. Use the formula from part 1 to find the surface area of the solid generated by revolving a single arch of the curve $y = \sin x$ about the x-axis.
3. The region enclosed by the graphs of $y^2 = x$ and $x = 4$ is revolved about the x-axis to form a solid. Find the surface area of the solid.

Quick Review 7.3 *(For help, go to Section 1.2.)*

In Exercises 1–10, give a formula for the area of the plane region in terms of the single variable x.

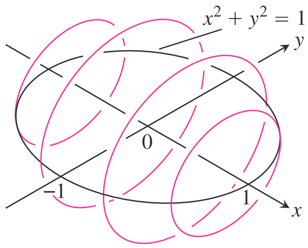
1. a square with sides of length x
2. a square with diagonals of length x
3. a semicircle of radius x
4. a semicircle of diameter x
5. an equilateral triangle with sides of length x
6. an isosceles right triangle with legs of length x
7. an isosceles right triangle with hypotenuse x
8. an isosceles triangle with two sides of length $2x$ and one side of length x
9. a triangle with sides $3x$, $4x$, and $5x$
10. a regular hexagon with sides of length x

Section 7.3 Exercises

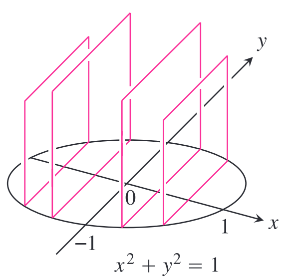
In Exercises 1 and 2, find a formula for the area $A(x)$ of the cross sections of the solid that are perpendicular to the x-axis.

1. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross sections perpendicular to the x-axis between these planes run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$.

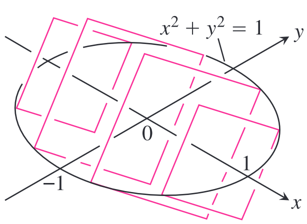
(a) The cross sections are circular disks with diameters in the xy-plane.



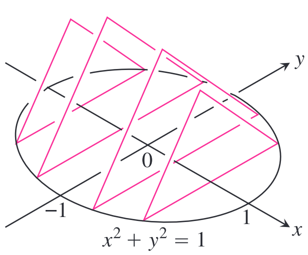
(b) The cross sections are squares with bases in the xy-plane.



(c) The cross sections are squares with diagonals in the xy-plane. (The length of a square's diagonal is $\sqrt{2}$ times the length of its sides.)

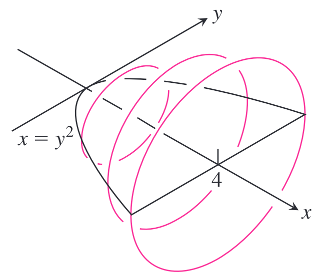


(d) The cross sections are equilateral triangles with bases in the xy-plane.

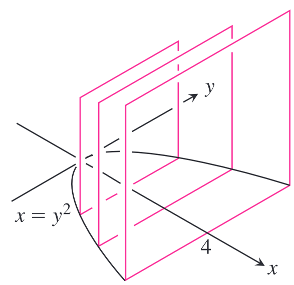


2. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross sections perpendicular to the x-axis between these planes run from $y = -\sqrt{x}$ to $y = \sqrt{x}$.

(a) The cross sections are circular disks with diameters in the xy-plane.



(b) The cross sections are squares with bases in the xy-plane.



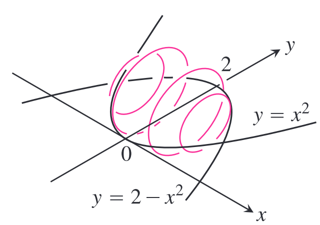
(c) The cross sections are squares with diagonals in the xy-plane.

(d) The cross sections are equilateral triangles with bases in the xy-plane.

In Exercises 3–6, find the volume of the solid analytically.

3. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross sections perpendicular to the axis on the interval $0 \le x \le 4$ are squares whose diagonals run from $y = -\sqrt{x}$ to $y = \sqrt{x}$.

4. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross sections perpendicular to the x-axis are circular disks whose diameters run from the parabola $y = x^2$ to the parabola $y = 2 - x^2$.

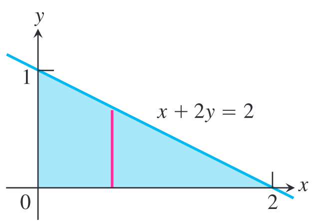


5. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross sections perpendicular to the x-axis between these planes are squares whose bases run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$.

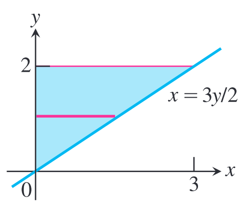
6. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross sections perpendicular to the x-axis between these planes are squares whose diagonals run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$.

In Exercises 7–10, find the volume of the solid generated by revolving the shaded region about the given axis.

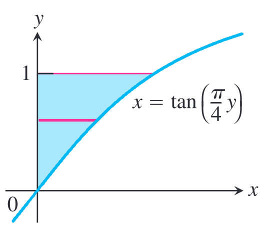
7. about the x-axis



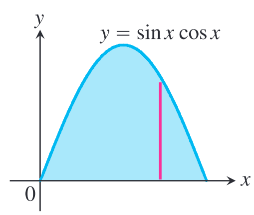
8. about the y-axis



9. about the y-axis



10. about the x-axis



In Exercises 11–20, find the volume of the solid generated by revolving the region bounded by the lines and curves about the x-axis.

11. $y = x^2, \quad y = 0, \quad x = 2$

12. $y = x^3, \quad y = 0, \quad x = 2$

13. $y = \sqrt{9 - x^2}, \quad y = 0$

14. $y = x - x^2, \quad y = 0$

15. $y = x, \quad y = 1, \quad x = 0$

16. $y = 2x, \quad y = x, \quad x = 1$

17. $y = x^2 + 1, \quad y = x + 3$

18. $y = 4 - x^2, \quad y = 2 - x$

19. $y = \sec x, \quad y = \sqrt{2}, \quad -\pi/4 \le x \le \pi/4$

20. $y = -\sqrt{x}, \quad y = -2, \quad x = 0$

In Exercises 21 and 22, find the volume of the solid generated by revolving the region about the given line.

21. the region in the first quadrant bounded above by the line $y = \sqrt{2}$, below by the curve $y = \sec x \tan x$, and on the left by the y-axis, about the line $y = \sqrt{2}$

22. the region in the first quadrant bounded above by the line $y = 2$, below by the curve $y = 2 \sin x$, $0 \le x \le \pi/2$, and on the left by the y-axis, about the line $y = 2$

In Exercises 23–28, find the volume of the solid generated by revolving the region about the y-axis.

23. the region enclosed by $x = \sqrt{5}y^2, \; x = 0, \; y = -1, \; y = 1$

24. the region enclosed by $x = y^{3/2}, \; x = 0, \; y = 2$

25. the region enclosed by the triangle with vertices (1, 0), (2, 1), and (1, 1)

26. the region enclosed by the triangle with vertices (0, 1), (1, 0), and (1, 1)

27. the region in the first quadrant bounded above by the parabola $y = x^2$, below by the x-axis, and on the right by the line $x = 2$

28. the region bounded above by the curve $y = \sqrt{x}$ and below by the line $y = x$

Group Activity In Exercises 29–32, find the volume of the solid described.

29. Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 2$ and $x = 0$ about

(a) the x-axis. **(b)** the y-axis.

(c) the line $y = 2$. **(d)** the line $x = 4$.

30. Find the volume of the solid generated by revolving the triangular region bounded by the lines $y = 2x$, $y = 0$, and $x = 1$ about

(a) the line $x = 1$. **(b)** the line $x = 2$.

31. Find the volume of the solid generated by revolving the region bounded by the parabola $y = x^2$ and the line $y = 1$ about

(a) the line $y = 1$. **(b)** the line $y = 2$.

(c) the line $y = -1$.

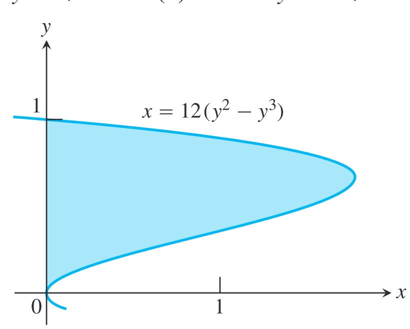
32. By integration, find the volume of the solid generated by revolving the triangular region with vertices (0, 0), (b, 0), (0, h) about

(a) the x-axis. **(b)** the y-axis.

In Exercises 33 and 34, use the cylindrical shell method to find the volume of the solid generated by revolving the shaded region about the indicated axis.

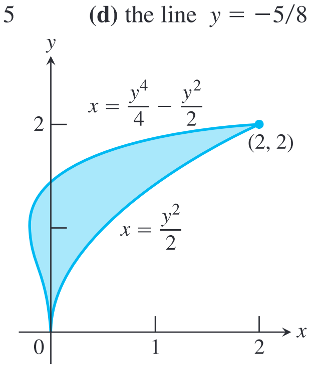
33. (a) the x-axis **(b)** the line $y = 1$

(c) the line $y = 8/5$ **(d)** the line $y = -2/5$



34. (a) the x-axis **(b)** the line $y = 2$

(c) the line $y = 5$ **(d)** the line $y = -5/8$



In Exercises 35–38, use the cylindrical shell method to find the volume of the solid generated by revolving the region bounded by the curves about the y-axis.

35. $y = x, \quad y = -x/2, \quad x = 2$

36. $y = x^2, \quad y = 2 - x, \quad x = 0, \quad \text{for } x \geq 0$

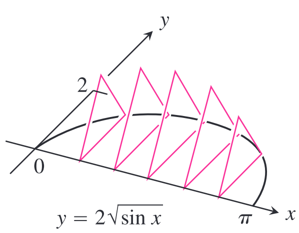
37. $y = \sqrt{x}, \quad y = 0, \quad x = 4$

38. $y = 2x - 1, \quad y = \sqrt{x}, \quad x = 0$

In Exercises 39–42, find the volume of the solid analytically.

39. The base of a solid is the region between the curve $y = 2\sqrt{\sin x}$ and the interval $[0, \pi]$ on the x-axis. The cross sections perpendicular to the x-axis are

(a) equilateral triangles with bases running from the x-axis to the curve as shown in the figure.



(b) squares with bases running from the x-axis to the curve.

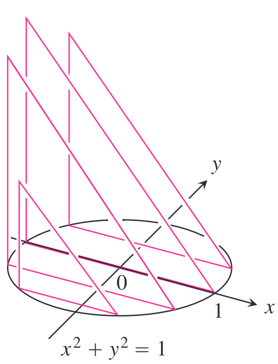
40. The solid lies between planes perpendicular to the x-axis at $x = -\pi/3$ and $x = \pi/3$. The cross sections perpendicular to the x-axis are

(a) circular disks with diameters running from the curve $y = \tan x$ to the curve $y = \sec x$.

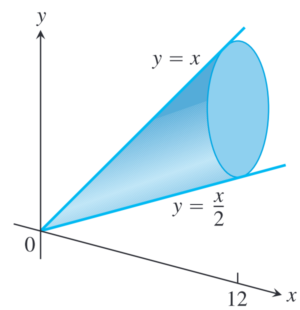
(b) squares whose bases run from the curve $y = \tan x$ to the curve $y = \sec x$.

41. The solid lies between planes perpendicular to the y-axis at $y = 0$ and $y = 2$. The cross sections perpendicular to the y-axis are circular disks with diameters running from the y-axis to the parabola $x = \sqrt{5}y^2$.

42. The base of the solid is the disk $x^2 + y^2 \leq 1$. The cross sections by planes perpendicular to the y-axis between $y = -1$ and $y = 1$ are isosceles right triangles with one leg in the disk.



43. **Writing to Learn** A solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 12$. The cross sections by planes perpendicular to the x-axis are circular disks whose diameters run from the line $y = x/2$ to the line $y = x$ as shown in the figure. Explain why the solid has the same volume as a right circular cone with base radius 3 and height 12.



44. ***A Twisted Solid*** A square of side length s lies in a plane perpendicular to a line L. One vertex of the square lies on L. As this square moves a distance h along L, the square turns one revolution about L to generate a corkscrew-like column with square cross sections.

(a) Find the volume of the column.

(b) **Writing to Learn** What will the volume be if the square turns twice instead of once? Give reasons for your answer.

45. Find the volume of the solid generated by revolving the region in the first quadrant bounded by $y = x^3$ and $y = 4x$ about

(a) the x-axis,
(b) the line $y = 8$.

46. Find the volume of the solid generated by revolving the region bounded by $y = 2x - x^2$ and $y = x$ about

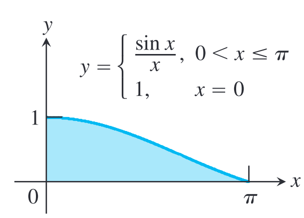
(a) the y-axis,
(b) the line $x = 1$.

47. The region in the first quadrant that is bounded above by the curve $y = 1/\sqrt{x}$, on the left by the line $x = 1/4$, and below by the line $y = 1$ is revolved about the y-axis to generate a solid. Find the volume of the solid by **(a)** the washer method and **(b)** the cylindrical shell method.

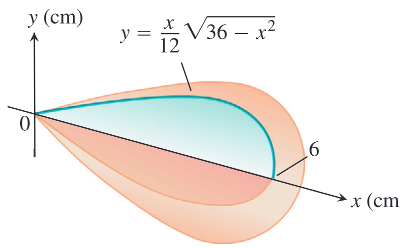
48. Let $f(x) = \begin{cases} (\sin x)/x, & 0 < x \leq \pi \\ 1, & x = 0. \end{cases}$

(a) Show that $x\,f(x) = \sin x, \quad 0 \leq x \leq \pi$.

(b) Find the volume of the solid generated by revolving the shaded region about the y-axis.



49. *Designing a Plumb Bob* Having been asked to design a brass plumb bob that will weigh in the neighborhood of 190 g, you decide to shape it like the solid of revolution shown here.



(a) Find the plumb bob's volume.

(b) If you specify a brass that weighs 8.5 g/cm^3, how much will the plumb bob weigh to the nearest gram?

50. *Volume of a Bowl* A bowl has a shape that can be generated by revolving the graph of $y = x^2/2$ between $y = 0$ and $y = 5$ about the y-axis.

(a) Find the volume of the bowl.

(b) If we fill the bowl with water at a constant rate of 3 cubic units per second, how fast will the water level in the bowl be rising when the water is 4 units deep?

51. *The Classical Bead Problem* A round hole is drilled through the center of a spherical solid of radius r. The resulting cylindrical hole has height 4 cm.

(a) What is the volume of the solid that remains?

(b) What is unusual about the answer?

52. Writing to Learn Explain how you could estimate the volume of a solid of revolution by measuring the shadow cast on a table parallel to its axis of revolution by a light shining directly above it.

53. *Same Volume about Each Axis* The region in the first quadrant enclosed between the graph of $y = ax - x^2$ and the x-axis generates the same volume whether it is revolved about the x-axis or the y-axis. Find the value of a.

54. *(Continuation of Exploration 2)* Let $x = g(y) > 0$ have a continuous first derivative on $[c, d]$. Show that the area of the surface generated by revolving the curve $x = g(y)$ about the y-axis is

$$S = \int_c^d 2\pi\, g(y)\sqrt{1 + (g'(y))^2}\, dy.$$

In Exercises 55–62, find the area of the surface generated by revolving the curve about the indicated axis.

55. $x = \sqrt{y}, \quad 0 \le y \le 2; \quad y\text{-axis}$

56. $x = y^3/3, \quad 0 \le y \le 1; \quad y\text{-axis}$

57. $x = y^{1/2} - (1/3)^{3/2}, \quad 1 \le y \le 3; \quad y\text{-axis}$

58. $x = \sqrt{2y - 1}, \quad (5/8) \le y \le 1; \quad y\text{-axis}$

59. $y = x^2, \quad 0 \le x \le 2; \quad x\text{-axis}$

60. $y = 3x - x^2, \quad 0 \le x \le 3; \quad x\text{-axis}$

61. $y = \sqrt{2x - x^2}, \quad 0.5 \le x \le 1.5; \quad x\text{-axis}$

62. $y = \sqrt{x + 1}, \quad 1 \le x \le 5; \quad x\text{-axis}$

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

63. True or False The volume of a solid of a known integrable cross section area $A(x)$ from $x = a$ to $x = b$ is $\int_a^b A(x)\, dx$. Justify your answer.

64. True or False If the region enclosed by the y-axis, the line $y = 2$, and the curve $y = \sqrt{x}$ is revolved about the y-axis, the volume of the solid is given by the definite integral $\int_0^2 \pi y^2\, dy$. Justify your answer.

65. Multiple Choice The base of a solid S is the region enclosed by the graph of $y = \ln x$, the line $x = e$, and the x-axis. If the cross sections of S perpendicular to the x-axis are squares, which of the following gives the best approximation of the volume of S?

(A) 0.718 **(B)** 1.718 **(C)** 2.718 **(D)** 3.171 **(E)** 7.388

66. Multiple Choice Let R be the region in the first quadrant bounded by the graph of $y = 8 - x^{3/2}$, the x-axis, and the y-axis. Which of the following gives the best approximation of the volume of the solid generated when R is revolved about the x-axis?

(A) 60.3 **(B)** 115.2 **(C)** 225.4 **(D)** 319.7 **(E)** 361.9

67. Multiple Choice Let R be the region enclosed by the graph of $y = x^2$, the line $x = 4$, and the x-axis. Which of the following gives the best approximation of the volume of the solid generated when R is revolved about the y-axis?

(A) 64π **(B)** 128π **(C)** 256π **(D)** 360 **(E)** 512

68. Multiple Choice Let R be the region enclosed by the graphs of $y = e^{-x}$, $y = e^x$, and $x = 1$. Which of the following gives the volume of the solid generated when R is revolved about the x-axis?

(A) $\int_0^1 (e^x - e^{-x})\, dx$

(B) $\int_0^1 (e^{2x} - e^{-2x})\, dx$

(C) $\int_0^1 (e^x - e^{-x})^2\, dx$

(D) $\pi\int_0^1 (e^{2x} - e^{-2x})\, dx$

(E) $\pi\int_0^1 (e^x - e^{-x})^2\, dx$

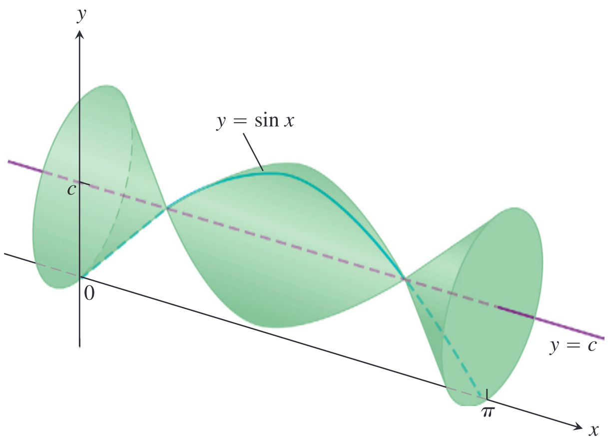
Explorations

69. ***Max-Min*** The arch $y = \sin x$, $0 \le x \le \pi$, is revolved about the line $y = c$, $0 \le c \le 1$, to generate the solid in the figure.

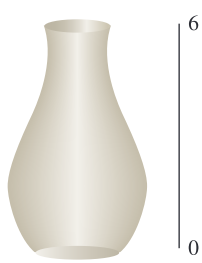
(a) Find the value of c that minimizes the volume of the solid. What is the minimum volume?

(b) What value of c in $[0, 1]$ maximizes the volume of the solid?

(c) **Writing to Learn** Graph the solid's volume as a function of c, first for $0 \le c \le 1$ and then on a larger domain. What happens to the volume of the solid as c moves away from $[0, 1]$? Does this make sense physically? Give reasons for your answers.



70. ***A Vase*** We wish to estimate the volume of a flower vase using only a calculator, a string, and a ruler. We measure the height of the vase to be 6 inches. We then use the string and the ruler to find circumferences of the vase (in inches) at half-inch intervals. (We list them from the top down to correspond with the picture of the vase.)



Circumferences	
5.4	10.8
4.5	11.6
4.4	11.6
5.1	10.8
6.3	9.0
7.8	6.3
9.4	

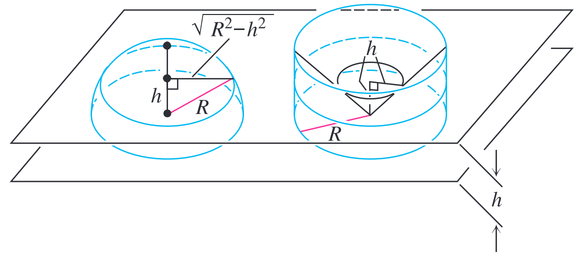
(a) Find the areas of the cross sections that correspond to the given circumferences.

(b) Express the volume of the vase as an integral with respect to y over the interval $[0, 6]$.

(c) Approximate the integral using the Trapezoidal Rule with $n = 12$.

Extending the Ideas

71. ***Volume of a Hemisphere*** Derive the formula $V = (2/3)\,\pi R^3$ for the volume of a hemisphere of radius R by comparing its cross sections with the cross sections of a solid right circular cylinder of radius R and height R from which a solid right circular cone of base radius R and height R has been removed as suggested by the figure.



72. ***Volume of a Torus*** The disk $x^2 + y^2 \le a^2$ is revolved about the line $x = b$ $(b > a)$ to generate a solid shaped like a doughnut, called a *torus*. Find its volume. (*Hint:* $\int_{-a}^{a} \sqrt{a^2 - y^2}\, dy = \pi a^2/2$, since it is the area of a semicircle of radius a.)

73. ***Filling a Bowl***

(a) ***Volume*** A hemispherical bowl of radius a contains water to a depth h. Find the volume of water in the bowl.

(b) ***Related Rates*** Water runs into a sunken concrete hemispherical bowl of radius 5 m at a rate of 0.2 m^3/sec. How fast is the water level in the bowl rising when the water is 4 m deep?

74. ***Consistency of Volume Definitions*** The volume formulas in calculus are consistent with the standard formulas from geometry in the sense that they agree on objects to which both apply.

(a) As a case in point, show that if you revolve the region enclosed by the semicircle $y = \sqrt{a^2 - x^2}$ and the x-axis about the x-axis to generate a solid sphere, the calculus formula for volume at the beginning of the section will give $(4/3)\pi a^3$ for the volume just as it should.

(b) Use calculus to find the volume of a right circular cone of height h and base radius r.

Quick Quiz for AP* Preparation: Sections 7.1–7.3

You may use a graphing calculator to solve the following problems.

1. **Multiple Choice** The base of a solid is the region in the first quadrant bounded by the x-axis, the graph of $y = \sin^{-1} x$, and the vertical line $x = 1$. For this solid, each cross section perpendicular to the x-axis is a square. What is the volume?

   (A) 0.117 **(B)** 0.285 **(C)** 0.467 **(D)** 0.571 **(E)** 1.571

2. **Multiple Choice** Let R be the region in the first quadrant bounded by the graph of $y = 3x - x^2$ and the x-axis. A solid is generated when R is revolved about the vertical line $x = -1$. Set up, but do not evaluate, the definite integral that gives the volume of this solid.

   (A) $\int_0^3 2\pi(x+1)(3x - x^2)\,dx$

   (B) $\int_{-1}^3 2\pi(x+1)(3x - x^2)\,dx$

   (C) $\int_0^3 2\pi(x)(3x - x^2)\,dx$

   (D) $\int_0^3 2\pi(3x - x^2)^2\,dx$

   (E) $\int_0^3 (3x - x^2)\,dx$

3. **Multiple Choice** A developing country consumes oil at a rate given by $r(t) = 20e^{0.2t}$ million barrels per year, where t is time measured in years, for $0 \le t \le 10$. Which of the following expressions gives the amount of oil consumed by the country during the time interval $0 \le t \le 10$?

   (A) $r(10)$

   (B) $r(10) - r(0)$

   (C) $\int_0^{10} r'(t)\,dt$

   (D) $\int_0^{10} r(t)\,dt$

   (E) $10 \cdot r(10)$

4. **Free Response** Let R be the region bounded by the graphs of $y = \sqrt{x}$, $y = e^{-x}$, and the y-axis.

   (a) Find the area of R.

   (b) Find the volume of the solid generated when R is revolved about the horizontal line $y = -1$.

   (c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a semicircle whose diameter runs from the graph of $y = \sqrt{x}$ to the graph of $y = e^{-x}$. Find the volume of this solid.

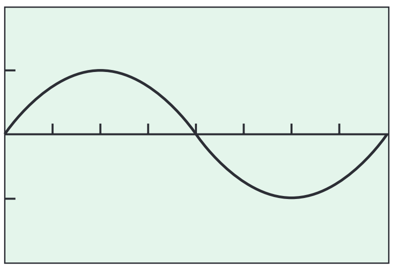
7.4 Lengths of Curves

What you'll learn about

- A Sine Wave
- Length of a Smooth Curve
- Vertical Tangents, Corners, and Cusps

. . . and why

The length of a smooth curve can be found using a definite integral.



[0, 2π] by [–2, 2]

Figure 7.32 One wave of a sine curve has to be longer than 2π.

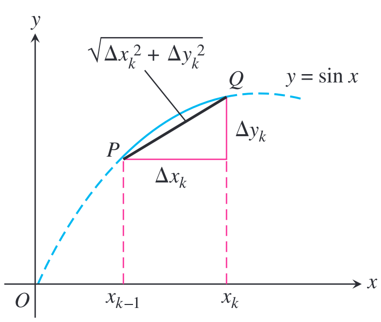


Figure 7.33 The line segment approximating the arc PQ of the sine curve above the subinterval $[x_{k-1}, x_k]$. (Example 1)

Group Exploration

Later in this section we will use an integral to find the length of the sine wave with great precision. But there are ways to get good approximations without integrating. Take five minutes to come up with a written estimate of the curve's length. No fair looking ahead.

A Sine Wave

How long is a sine wave (Figure 7.32)?

The usual meaning of *wavelength* refers to the fundamental period, which for $y = \sin x$ is 2π. But how long is the curve itself? If you straightened it out like a piece of string along the positive x-axis with one end at 0, where would the other end be?

EXAMPLE 1 The Length of a Sine Wave

What is the length of the curve $y = \sin x$ from $x = 0$ to $x = 2\pi$?

SOLUTION

We answer this question with integration, following our usual plan of breaking the whole into measurable parts. We partition $[0, 2\pi]$ into intervals so short that the pieces of curve (call them "arcs") lying directly above the intervals are nearly straight. That way, each arc is nearly the same as the line segment joining its two ends and we can take the length of the segment as an approximation to the length of the arc.

Figure 7.33 shows the segment approximating the arc above the subinterval $[x_{k-1}, x_k]$. The length of the segment is $\sqrt{\Delta x_k^2 + \Delta y_k^2}$. The sum

$$\sum \sqrt{\Delta x_k^2 + \Delta y_k^2}$$

over the entire partition approximates the length of the curve. All we need now is to find the limit of this sum as the norms of the partitions go to zero. That's the usual plan, but this time there is a problem. Do you see it?

The problem is that the sums as written are not Riemann sums. They do not have the form $\sum f(c_k)\,\Delta x$. We can rewrite them as Riemann sums if we multiply and divide each square root by Δx_k.

$$\sum \sqrt{\Delta x_k^2 + \Delta y_k^2} = \sum \frac{\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}}{\Delta x_k}\,\Delta x_k$$

$$= \sum \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2}\,\Delta x_k$$

This is better, but we still need to write the last square root as a function evaluated at some c_k in the kth subinterval. For this, we call on the Mean Value Theorem for differentiable functions (Section 4.2), which says that since $\sin x$ is continuous on $[x_{k-1}, x_k]$ and is differentiable on (x_{k-1}, x_k) there is a point c_k in $[x_{k-1}, x_k]$ at which $\Delta y_k/\Delta x_k = \sin' c_k$ (Figure 7.34). That gives us

$$\sum \sqrt{1 + (\sin' c_k)^2}\,\Delta x_k,$$

which *is* a Riemann sum.

Now we take the limit as the norms of the subdivisions go to zero and find that the length of one wave of the sine function is

$$\int_0^{2\pi} \sqrt{1 + (\sin' x)^2}\,dx = \int_0^{2\pi} \sqrt{1 + \cos^2 x}\,dx \approx 7.64.$$

How close was your estimate? *Now try Exercise 9.*

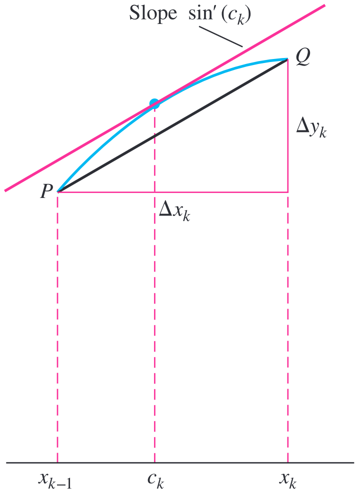


Figure 7.34 The portion of the sine curve above $[x_{k-1}, x_k]$. At some c_k in the interval, $\sin'(c_k) = \Delta y_k/\Delta x_k$, the slope of segment PQ. (Example 1)

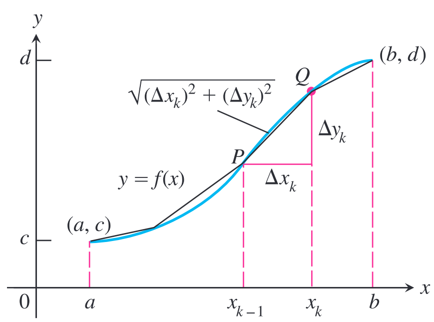


Figure 7.35 The graph of f, approximated by line segments.

Length of a Smooth Curve

We are almost ready to define the length of a curve as a definite integral, using the procedure of Example 1. We first call attention to two properties of the sine function that came into play along the way.

We obviously used *differentiability* when we invoked the Mean Value Theorem to replace $\Delta y_k/\Delta x_k$ by $\sin'(c_k)$ for some c_k in the interval $[x_{k-1}, x_k]$. Less obviously, we used the continuity of the derivative of sine in passing from $\sum \sqrt{1 + (\sin'(c_k))^2}\,\Delta x_k$ to the Riemann integral. The requirement for finding the length of a curve by this method, then, is that the function have a continuous first derivative. We call this property **smoothness.** A function with a continuous first derivative is **smooth** and its graph is a **smooth curve.**

Let us review the process, this time with a general smooth function $f(x)$. Suppose the graph of f begins at the point (a, c) and ends at (b, d), as shown in Figure 7.35. We partition the interval $a \le x \le b$ into subintervals so short that the arcs of the curve above them are nearly straight. The length of the segment approximating the arc above the subinterval $[x_{k-1}, x_k]$ is $\sqrt{\Delta x_k^2 + \Delta y_k^2}$. The sum $\sum \sqrt{\Delta x_k^2 + \Delta y_k^2}$ approximates the length of the curve. We apply the Mean Value Theorem to f on each subinterval to rewrite the sum as a Riemann sum,

$$\sum \sqrt{\Delta x_k^2 + \Delta y_k^2} = \sum \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2}\,\Delta x_k$$

$$= \sum \sqrt{1 + (f'(c_k))^2}\,\Delta x_k.$$

Passing to the limit as the norms of the subdivisions go to zero gives the length of the curve as

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\,dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.$$

We could as easily have transformed $\sum \sqrt{\Delta x_k^2 + \Delta y_k^2}$ into a Riemann sum by dividing and multiplying by Δy_k, giving a formula that involves x as a function of y (say, $x = g(y)$) on the interval $[c, d]$:

$$L \approx \sum \frac{\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}}{\Delta y_k}\,\Delta y_k = \sum \sqrt{1 + \left(\frac{\Delta x_k}{\Delta y_k}\right)^2}\,\Delta y_k$$

$$= \sum \sqrt{1 + (g'(c_k))^2}\,\Delta y_k.$$

The limit of these sums, as the norms of the subdivisions go to zero, gives another reasonable way to calculate the curve's length,

$$L = \int_c^d \sqrt{1 + (g'(y))^2}\,dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy.$$

Putting these two formulas together, we have the following definition for the length of a smooth curve.

DEFINITION Arc Length: Length of a Smooth Curve

If a smooth curve begins at (a, c) and ends at (b, d), $a < b$, $c < d$, then the **length (arc length) of the curve** is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \qquad \text{if } y \text{ is a smooth function of } x \text{ on } [a, b];$$

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy \qquad \text{if } x \text{ is a smooth function of } y \text{ on } [c, d].$$

EXAMPLE 2 Applying the Definition

Find the *exact* length of the curve

$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1 \quad \text{for} \quad 0 \le x \le 1.$$

SOLUTION

$$\frac{dy}{dx} = \frac{4\sqrt{2}}{3} \cdot \frac{3}{2}x^{1/2} = 2\sqrt{2}\,x^{1/2},$$

which is continuous on $[0, 1]$. Therefore,

$$\begin{aligned} L &= \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \\ &= \int_0^1 \sqrt{1 + \left(2\sqrt{2}x^{1/2}\right)^2}\,dx \\ &= \int_0^1 \sqrt{1 + 8x}\,dx \\ &= \frac{2}{3} \cdot \frac{1}{8}(1 + 8x)^{3/2}\Big]_0^1 \\ &= \frac{13}{6}. \end{aligned}$$

Now try Exercise 11.

We asked for an exact length in Example 2 to take advantage of the rare opportunity it afforded of taking the antiderivative of an arc length integrand. When you add 1 to the square of the derivative of an arbitrary smooth function and then take the square root of that sum, the result is rarely antidifferentiable by reasonable methods. We know a few more functions that give "nice" integrands, but we are saving those for the exercises.

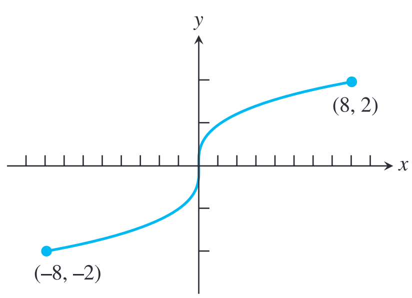


Figure 7.36 The graph of $y = x^{1/3}$ has a vertical tangent line at the origin where dy/dx does not exist. (Example 3)

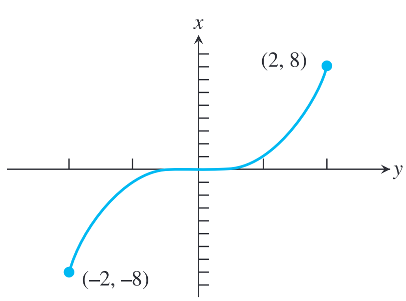


Figure 7.37 The curve in Figure 7.36 plotted with x as a function of y. The tangent at the origin is now horizontal. (Example 3)

Vertical Tangents, Corners, and Cusps

Sometimes a curve has a vertical tangent, corner, or cusp where the derivative we need to work with is undefined. We can sometimes get around such a difficulty in ways illustrated by the following examples.

EXAMPLE 3 A Vertical Tangent

Find the length of the curve $y = x^{1/3}$ between $(-8, -2)$ and $(8, 2)$.

SOLUTION

The derivative

$$\frac{dy}{dx} = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$$

is not defined at $x = 0$. Graphically, there is a vertical tangent at $x = 0$ where the derivative becomes infinite (Figure 7.36). If we change to x as a function of y, the tangent at the origin will be horizontal (Figure 7.37) and the derivative will be zero instead of undefined. Solving $y = x^{1/3}$ for x gives $x = y^3$, and we have

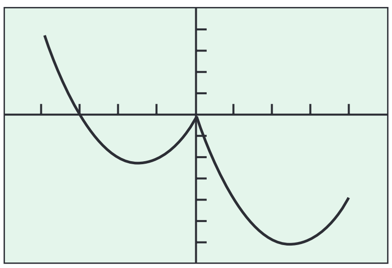
$$L = \int_{-2}^{2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \int_{-2}^{2} \sqrt{1 + (3y^2)^2}\,dy \approx 17.26.$$

Now try Exercise 25.

What happens if you fail to notice that dy/dx is undefined at $x = 0$ and ask your calculator to compute

$$\text{NINT}\left(\sqrt{1 + \left((1/3)\, x^{-2/3}\right)^2},\ x,\ -8,\ 8\right)?$$

This actually depends on your calculator. If, in the process of its calculations, it tries to evaluate the function at $x = 0$, then some sort of domain error will result. If it tries to find convergent Riemann sums near $x = 0$, it might get into a long, futile loop of computations that you will have to interrupt. Or it might actually produce an answer—in which case you hope it would be sufficiently bizarre for you to realize that it should not be trusted.



[–5, 5] by [–7, 5]

Figure 7.38 The graph of

$$y = x^2 - 4|x| - x,\ -4 \le x \le 4,$$

has a corner at $x = 0$ where neither dy/dx nor dx/dy exists. We find the lengths of the two smooth pieces and add them together. (Example 4)

EXAMPLE 4 Getting Around a Corner

Find the length of the curve $y = x^2 - 4|x| - x$ from $x = -4$ to $x = 4$.

SOLUTION

We should always be alert for abrupt slope changes when absolute value is involved. We graph the function to check (Figure 7.38).

There is clearly a corner at $x = 0$ where neither dy/dx nor dx/dy can exist. To find the length, we split the curve at $x = 0$ to write the function *without* absolute values:

$$x^2 - 4|x| - x = \begin{cases} x^2 + 3x & \text{if} \quad x < 0, \\ x^2 - 5x & \text{if} \quad x \ge 0. \end{cases}$$

Then,

$$L = \int_{-4}^{0} \sqrt{1 + (2x + 3)^2}\, dx + \int_{0}^{4} \sqrt{1 + (2x - 5)^2}\, dx$$

$$\approx 19.56.$$

Now try Exercise 27.

Finally, cusps are handled the same way corners are: split the curve into smooth pieces and add the lengths of those pieces.

Quick Review 7.4 *(For help, go to Sections 1.3 and 3.2.)*

In Exercises 1–5, simplify the function.

1. $\sqrt{1 + 2x + x^2}$ on $[1, 5]$
2. $\sqrt{1 - x + \dfrac{x^2}{4}}$ on $[-3, -1]$
3. $\sqrt{1 + (\tan x)^2}$ on $[0, \pi/3]$
4. $\sqrt{1 + (x/4 - 1/x)^2}$ on $[4, 12]$
5. $\sqrt{1 + \cos 2x}$ on $[0, \pi/2]$

In Exercises 6–10, identify all values of x for which the function fails to be differentiable.

6. $f(x) = |x - 4|$
7. $f(x) = 5x^{2/3}$
8. $f(x) = \sqrt[5]{x + 3}$
9. $f(x) = \sqrt{x^2 - 4x + 4}$
10. $f(x) = 1 + \sqrt[3]{\sin x}$

Section 7.4 Exercises

In Exercises 1–10,

(a) set up an integral for the length of the curve;

(b) graph the curve to see what it looks like;

(c) use NINT to find the length of the curve.

1. $y = x^2, \quad -1 \le x \le 2$

2. $y = \tan x, \quad -\pi/3 \le x \le 0$

3. $x = \sin y, \quad 0 \le y \le \pi$

4. $x = \sqrt{1 - y^2}, \quad -1/2 \le y \le 1/2$

5. $y^2 + 2y = 2x + 1$, from $(-1, -1)$ to $(7, 3)$

6. $y = \sin x - x \cos x, \quad 0 \le x \le \pi$

7. $y = \int_0^x \tan t \, dt, \quad 0 \le x \le \pi/6$

8. $x = \int_0^y \sqrt{\sec^2 t - 1} \, dt, \quad -\pi/3 \le y \le \pi/4$

9. $y = \sec x, \quad -\pi/3 \le x \le \pi/3$

10. $y = (e^x + e^{-x})/2, \quad -3 \le x \le 3$

In Exercises 11–18, find the exact length of the curve analytically by antidifferentiation. You will need to simplify the integrand algebraically before finding an antiderivative.

11. $y = (1/3)(x^2 + 2)^{3/2}$ from $x = 0$ to $x = 3$

12. $y = x^{3/2}$ from $x = 0$ to $x = 4$

13. $x = (y^3/3) + 1/(4y)$ from $y = 1$ to $y = 3$
[*Hint:* $1 + (dx/dy)^2$ is a perfect square.]

14. $x = (y^4/4) + 1/(8y^2)$ from $y = 1$ to $y = 2$
[*Hint:* $1 + (dx/dy)^2$ is a perfect square.]

15. $x = (y^3/6) + 1/(2y)$ from $y = 1$ to $y = 2$
[*Hint:* $1 + (dx/dy)^2$ is a perfect square.]

16. $y = (x^3/3) + x^2 + x + 1/(4x + 4), \quad 0 \le x \le 2$

17. $x = \int_0^y \sqrt{\sec^4 t - 1} \, dt, \quad -\pi/4 \le y \le \pi/4$

18. $y = \int_{-2}^x \sqrt{3t^4 - 1} \, dt, \quad -2 \le x \le -1$

19. **(a)** **Group Activity** Find a curve through the point (1, 1) whose length integral is

$$L = \int_1^4 \sqrt{1 + \frac{1}{4x}} \, dx.$$

(b) **Writing to Learn** How many such curves are there? Give reasons for your answer.

20. **(a)** **Group Activity** Find a curve through the point (0, 1) whose length integral is

$$L = \int_1^2 \sqrt{1 + \frac{1}{y^4}} \, dy.$$

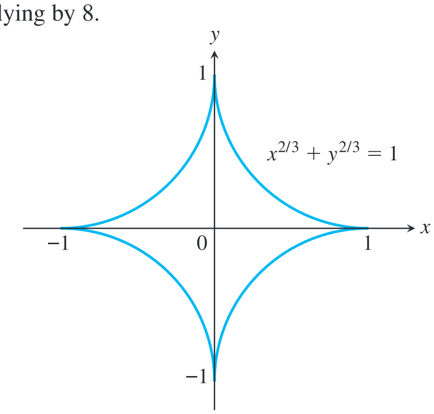
(b) **Writing to Learn** How many such curves are there? Give reasons for your answer.

21. Find the length of the curve

$$y = \int_0^x \sqrt{\cos 2t} \, dt$$

from $x = 0$ to $x = \pi/4$.

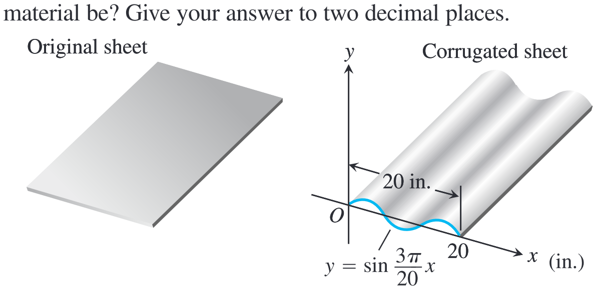
22. ***The Length of an Astroid*** The graph of the equation $x^{2/3} + y^{2/3} = 1$ is one of the family of curves called *astroids* (not "asteroids") because of their starlike appearance (see figure). Find the length of this particular astroid by finding the length of half the first quadrant portion, $y = (1 - x^{2/3})^{3/2}$, $\sqrt{2}/4 \le x \le 1$, and multiplying by 8.



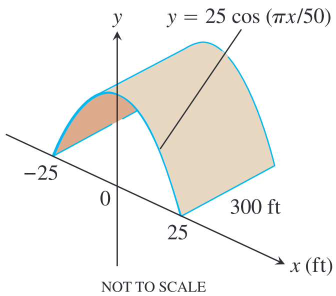
23. ***Fabricating Metal Sheets*** Your metal fabrication company is bidding for a contract to make sheets of corrugated steel roofing like the one shown here. The cross sections of the corrugated sheets are to conform to the curve

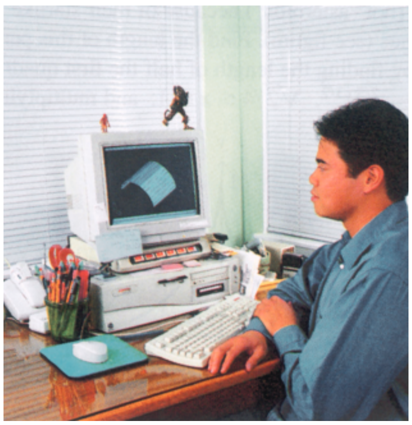
$$y = \sin\left(\frac{3\pi}{20}x\right), \qquad 0 \le x \le 20 \text{ in.}$$

If the roofing is to be stamped from flat sheets by a process that does not stretch the material, how wide should the original material be? Give your answer to two decimal places.



24. ***Tunnel Construction*** Your engineering firm is bidding for the contract to construct the tunnel shown on the next page. The tunnel is 300 ft long and 50 ft wide at the base. The cross section is shaped like one arch of the curve $y = 25 \cos(\pi x/50)$. Upon completion, the tunnel's inside surface (excluding the roadway) will be treated with a waterproof sealer that costs \$1.75 per square foot to apply. How much will it cost to apply the sealer?





In Exercises 25 and 26, find the length of the curve.

25. $f(x) = x^{1/3} + x^{2/3}, \quad 0 \le x \le 2$

26. $f(x) = \dfrac{x-1}{4x^2+1}, \quad -\dfrac{1}{2} \le x \le 1$

In Exercises 27–29, find the length of the nonsmooth curve.

27. $y = x^3 + 5|x|$ from $x = -2$ to $x = 1$

28. $\sqrt{x} + \sqrt{y} = 1$

29. $y = \sqrt[4]{x}$ from $x = 0$ to $x = 16$

30. Writing to Learn Explain geometrically why it does not work to use short *horizontal* line segments to approximate the lengths of small arcs when we search for a Riemann sum that leads to the formula for arc length.

31. Writing to Learn A curve is totally contained inside the square with vertices (0, 0), (1, 0), (1, 1), and (0, 1). Is there any limit to the possible length of the curve? Explain.

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

32. True or False If a function $y = f(x)$ is continuous on an interval $[a, b]$, then the length of its curve is given by $\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$. Justify your answer.

33. True or False If a function $y = f(x)$ is differentiable on an interval $[a, b]$, then the length of its curve is given by $\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$. Justify your answer.

34. Multiple Choice Which of the following gives the best approximation of the length of the arc of $y = \cos(2x)$ from $x = 0$ to $x = \pi/4$?

(A) 0.785 **(B)** 0.955 **(C)** 1.0 **(D)** 1.318 **(E)** 1.977

35. Multiple Choice Which of the following expressions gives the length of the graph of $x = y^3$ from $y = -2$ to $y = 2$?

(A) $\int_{-2}^{2} (1 + y^6)\, dy$ **(B)** $\int_{-2}^{2} \sqrt{1 + y^6}\, dy$

(C) $\int_{-2}^{2} \sqrt{1 + 9y^4}\, dy$ **(D)** $\int_{-2}^{2} \sqrt{1 + x^2}\, dx$

(E) $\int_{-2}^{2} \sqrt{1 + x^4}\, dx$

36. Multiple Choice Find the length of the curve described by $y = \frac{2}{3}x^{3/2}$ from $x = 0$ to $x = 8$.

(A) $\dfrac{26}{3}$ **(B)** $\dfrac{52}{3}$ **(C)** $\dfrac{512\sqrt{2}}{15}$

(D) $\dfrac{512\sqrt{2}}{15} + 8$ **(E)** 96

37. Multiple Choice Which of the following expressions should be used to find the length of the curve $y = x^{2/3}$ from $x = -1$ to $x = 1$?

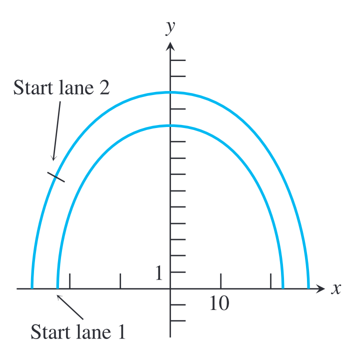
(A) $2\int_{0}^{1} \sqrt{1 + \frac{9}{4}y}\, dy$ **(B)** $\int_{-1}^{1} \sqrt{1 + \frac{9}{4}y}\, dy$

(C) $\int_{0}^{1} \sqrt{1 + y^3}\, dy$ **(D)** $\int_{0}^{1} \sqrt{1 + y^6}\, dy$

(E) $\int_{0}^{1} \sqrt{1 + y^{9/4}}\, dy$

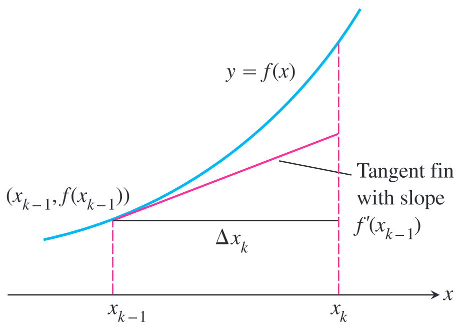
Exploration

38. *Modeling Running Tracks* Two lanes of a running track are modeled by the semiellipses as shown. The equation for lane 1 is $y = \sqrt{100 - 0.2x^2}$, and the equation for lane 2 is $y = \sqrt{150 - 0.2x^2}$. The starting point for lane 1 is at the negative x-intercept $(-\sqrt{500}, 0)$. The finish points for both lanes are the positive x-intercepts. Where should the starting point be placed on lane 2 so that the two lane lengths will be equal (running clockwise)?



Extending the Ideas

39. ***Using Tangent Fins to Find Arc Length*** Assume f is smooth on $[a, b]$ and partition the interval $[a, b]$ in the usual way. In each subinterval $[x_{k-1}, x_k]$ construct the *tangent fin* at the point $(x_{k-1}, f(x_{k-1}))$ as shown in the figure.



(a) Show that the length of the kth tangent fin over the interval $[x_{k-1}, x_k]$ equals

$$\sqrt{(\Delta x_k)^2 + (f'(x_{k-1})\Delta x_k)^2}.$$

(b) Show that

$$\lim_{n\to\infty} \sum_{k=1}^{n} (\text{length of } k\text{th tangent fin}) = \int_a^b \sqrt{1 + (f'(x))^2}\, dx,$$

which is the length L of the curve $y = f(x)$ from $x = a$ to $x = b$.

40. Is there a smooth curve $y = f(x)$ whose length over the interval $0 \le x \le a$ is always $a\sqrt{2}$? Give reasons for your answer.

7.5 Applications from Science and Statistics

What you'll learn about

- Work Revisited
- Fluid Force and Fluid Pressure
- Normal Probabilities

. . . and why

It is important to see applications of integrals as various accumulation functions.

Our goal in this section is to hint at the diversity of ways in which the definite integral can be used. The contexts may be new to you, but we will explain what you need to know as we go along.

Work Revisited

Recall from Section 7.1 that *work* is defined as force (in the direction of motion) times displacement. A familiar example is to move against the force of gravity to lift an object. The object has to move, incidentally, before "work" is done, no matter how tired you get *trying*.

If the force $F(x)$ is not constant, then the work done in moving an object from $x = a$ to $x = b$ is the definite integral $W = \int_a^b F(x)dx$.

4.4 newtons ≈ 1 lb

(1 newton)(1 meter) = 1 N • m = 1 Joule

EXAMPLE 1 Finding the Work Done by a Force

Find the work done by the force $F(x) = \cos(\pi x)$ newtons along the x-axis from $x = 0$ meters to $x = 1/2$ meter.

SOLUTION

$$W = \int_0^{1/2} \cos(\pi x)\, dx$$

$$= \frac{1}{\pi}\sin(\pi x)\Big|_0^{1/2}$$

$$= \frac{1}{\pi}\left(\sin\left(\frac{\pi}{2}\right) - \sin(0)\right)$$

$$= \frac{1}{\pi} \approx 0.318$$

Now try Exercise 1.

EXAMPLE 2 Work Done Lifting

A leaky bucket weighs 22 newtons (N) empty. It is lifted from the ground at a constant rate to a point 20 m above the ground by a rope weighing 0.4 N/m. The bucket starts with 70 N (approximately 7.1 liters) of water, but it leaks at a constant rate and just finishes draining as the bucket reaches the top. Find the amount of work done

(a) lifting the bucket alone;

(b) lifting the water alone;

(c) lifting the rope alone;

(d) lifting the bucket, water, and rope together.

SOLUTION

(a) *The bucket alone.* This is easy because the bucket's weight is constant. To lift it, you must exert a force of 22 N through the entire 20-meter interval.

$$\text{Work} = (22\text{ N}) \times (20\text{ m}) = 440\text{ N} \cdot \text{m} = 440\text{ J}$$

Figure 7.39 shows the graph of force vs. distance applied. The work corresponds to the area under the force graph.

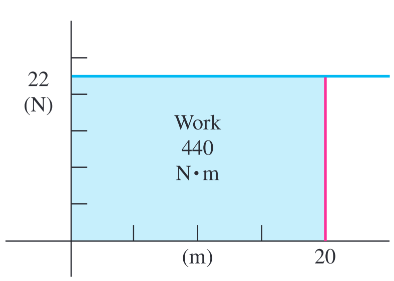


Figure 7.39 The work done by a constant 22-N force lifting a bucket 20 m is 440 N • m. (Example 2)

continued

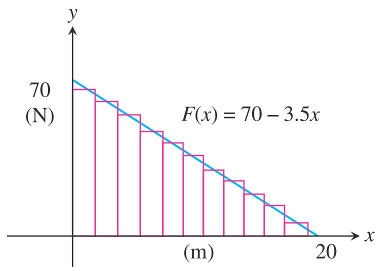


Figure 7.40 The force required to lift the water varies with distance but the work still corresponds to the area under the force graph. (Example 2)

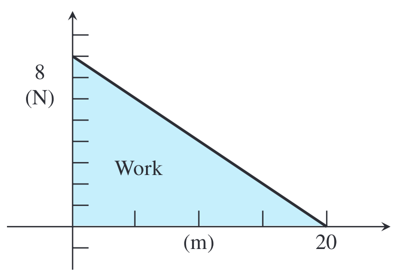


Figure 7.41 The work done lifting the rope to the top corresponds to the area of another triangle. (Example 2)

(b) *The water alone.* The force needed to lift the water is equal to the water's weight, which decreases steadily from 70 N to 0 N over the 20-m lift. When the bucket is x m off the ground, the water weighs

$$F(x) = 70\left(\frac{20 - x}{20}\right) = 70\left(1 - \frac{x}{20}\right) = 70 - 3.5x \text{ N.}$$

The work done is (Figure 7.40)

$$W = \int_a^b F(x)\,dx$$

$$= \int_0^{20} (70 - 3.5x)\,dx = \Big[70x - 1.75x^2\Big]_0^{20} = 1400 - 700 = 700 \text{ J.}$$

(c) *The rope alone.* The force needed to lift the rope is also variable, starting at $(0.4)(20) = 8$ N when the bucket is on the ground and ending at 0 N when the bucket and rope are all at the top. As with the leaky bucket, the rate of decrease is constant. At elevation x meters, the $(20 - x)$ meters of rope still there to lift weigh $F(x) = (0.4)(20 - x)$ N. Figure 7.41 shows the graph of F. The work done lifting the rope is

$$\int_0^{20} F(x)\,dx = \int_0^{20} (0.4)(20 - x)\,dx$$

$$= \Big[8x - 0.2x^2\Big]_0^{20} = 160 - 80 = 80 \text{ N}\cdot\text{m} = 80 \text{ J.}$$

(d) *The bucket, water, and rope together.* The total work is

$$440 + 700 + 80 = 1220 \text{ J.}$$

Now try Exercise 5.

EXAMPLE 3 Work Done Pumping

The conical tank in Figure 7.42 is filled to within 2 ft of the top with olive oil weighing 57 lb/ft^3. How much work does it take to pump the oil to the rim of the tank?

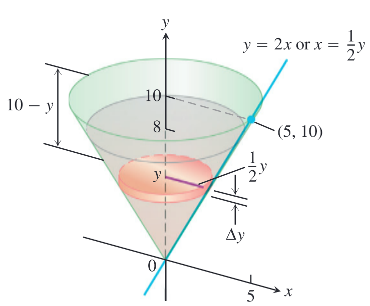


Figure 7.42 The conical tank in Example 3.

SOLUTION

We imagine the oil partitioned into thin slabs by planes perpendicular to the y-axis at the points of a partition of the interval $[0, 8]$. (The 8 represents the top of the oil, not the top of the tank.)

The typical slab between the planes at y and $y + \Delta y$ has a volume of about

$$\Delta V = \pi(\text{radius})^2(\text{thickness}) = \pi\left(\frac{1}{2}y\right)^2 \Delta y = \frac{\pi}{4}y^2\,\Delta y \text{ ft}^3.$$

The force $F(y)$ required to lift this slab is equal to its weight,

$$F(y) = 57\,\Delta V = \frac{57\pi}{4}y^2\,\Delta y \text{ lb.}$$

The distance through which $F(y)$ must act to lift this slab to the level of the rim of the cone is about $(10 - y)$ ft, so the work done lifting the slab is about

$$\Delta W = \frac{57\pi}{4}(10 - y)y^2\,\Delta y \text{ ft}\cdot\text{lb.}$$

The work done lifting all the slabs from $y = 0$ to $y = 8$ to the rim is approximately

$$W \approx \sum \frac{57\pi}{4}(10 - y)\,y^2\,\Delta y \text{ ft}\cdot\text{lb.}$$

continued

This is a Riemann sum for the function $(57\pi/4)(10 - y)y^2$ on the interval from $y = 0$ to $y = 8$. The work of pumping the oil to the rim is the limit of these sums as the norms of the partitions go to zero.

$$W = \int_0^8 \frac{57\pi}{4}(10 - y)y^2\,dy = \frac{57\pi}{4}\int_0^8 (10y^2 - y^3)\,dy$$

$$= \frac{57\pi}{4}\left[\frac{10y^3}{3} - \frac{y^4}{4}\right]_0^8 \approx 30{,}561 \text{ ft} \cdot \text{lb}$$

Now try Exercise 17.

Fluid Force and Fluid Pressure

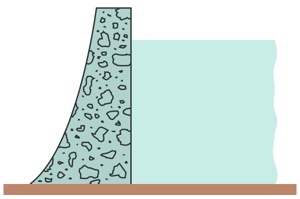


Figure 7.43 To withstand the increasing pressure, dams are built thicker toward the bottom.

We make dams thicker at the bottom than at the top (Figure 7.43) because the pressure against them increases with depth. It is a remarkable fact that the pressure at any point on a dam depends only on how far below the surface the point lies and not on how much water the dam is holding back. In any liquid, the **fluid pressure** p (force per unit area) at depth h is

$$p = wh,$$

where w is the *weight-density* (weight per unit volume) of the liquid.

Typical Weight-densities (lb/ft³)

Gasoline	42
Mercury	849
Milk	64.5
Molasses	100
Seawater	64
Water	62.4

EXAMPLE 4 The Great Molasses Flood of 1919

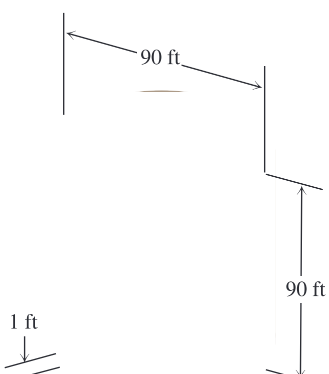


Figure 7.44 The molasses tank of Example 4.

At 1:00 P.M. on January 15, 1919 (an unseasonably warm day), a 90-ft-high, 90-foot-diameter cylindrical metal tank in which the Puritan Distilling Company stored molasses at the corner of Foster and Commercial streets in Boston's North End exploded. Molasses flooded the streets 30 feet deep, trapping pedestrians and horses, knocking down buildings, and oozing into homes. It was eventually tracked all over town and even made its way into the suburbs via trolley cars and people's shoes. It took weeks to clean up.

(a) Given that the tank was full of molasses weighing 100 lb/ft³, what was the total force exerted by the molasses on the bottom of the tank at the time it ruptured?

(b) What was the total force against the bottom foot-wide band of the tank wall (Figure 7.44)?

continued

Figure 7.45 The 1-ft band at the bottom of the tank wall can be partitioned into thin strips on which the pressure is approximately constant. (Example 4)

SOLUTION

(a) At the bottom of the tank, the molasses exerted a constant pressure of

$$p = wh = \left(100\,\frac{\text{lb}}{\text{ft}^3}\right)(90\text{ ft}) = 9000\,\frac{\text{lb}}{\text{ft}^2}.$$

Since the area of the base was $\pi(45)^2$, the total force on the base was

$$\left(9000\,\frac{\text{lb}}{\text{ft}^2}\right)(2025\,\pi\text{ ft}^2) \approx 57{,}225{,}526\text{ lb}.$$

(b) We partition the band from depth 89 ft to depth 90 ft into narrower bands of width Δy and choose a depth y_k in each one. The pressure at this depth y_k is $p = wh = 100\,y_k$ lb/ft^2 (Figure 7.45). The force against each narrow band is approximately

$$\text{pressure} \times \text{area} = (100y_k)(90\pi\,\Delta y) = 9000\pi\,y_k\,\Delta y\text{ lb}.$$

Adding the forces against all the bands in the partition and passing to the limit as the norms go to zero, we arrive at

$$F = \int_{89}^{90} 9000\pi y\,dy = 9000\pi \int_{89}^{90} y\,dy \approx 2{,}530{,}553\text{ lb}$$

for the force against the bottom foot of tank wall. ***Now try Exercise 25.***

Normal Probabilities

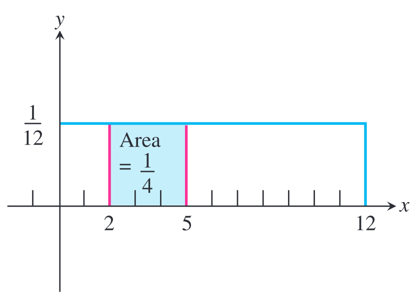


Figure 7.46 The probability that the clock has stopped between 2:00 and 5:00 can be represented as an area of 1/4. The rectangle over the entire interval has area 1.

Suppose you find an old clock in the attic. What is the probability that it has stopped somewhere between 2:00 and 5:00?

If you imagine time being measured continuously over a 12-hour interval, it is easy to conclude that the answer is 1/4 (since the interval from 2:00 to 5:00 contains one-fourth of the time), and that is correct. Mathematically, however, the situation is not quite that clear because both the 12-hour interval and the 3-hour interval contain an *infinite* number of times. In what sense does the ratio of one infinity to another infinity equal 1/4?

The easiest way to resolve that question is to look at area. We represent the total probability of the 12-hour interval as a rectangle of area 1 sitting above the interval (Figure 7.46).

Not only does it make perfect sense to say that the rectangle over the time interval [2, 5] has an area that is one-fourth the area of the total rectangle, the area actually *equals* 1/4, since the total rectangle has area 1. That is why mathematicians represent probabilities as areas, and that is where definite integrals enter the picture.

Improper Integrals

More information about improper integrals like $\int_{-\infty}^{\infty} f(x)\,dx$ can be found in Section 8.3. (You will not need that information here.)

DEFINITION Probability Density Function (pdf)

A **probability density function** is a function $f(x)$ with domain all reals such that

$$f(x) \ge 0 \quad \text{for all } x \qquad \text{and} \qquad \int_{-\infty}^{\infty} f(x)\,dx = 1.$$

Then the probability associated with an interval $[a, b]$ is

$$\int_a^b f(x)\,dx.$$

Probabilities of events, such as the clock stopping between 2:00 and 5:00, are integrals of an appropriate pdf.

EXAMPLE 5 Probability of the Clock Stopping

Find the probability that the clock stopped between 2:00 and 5:00.

SOLUTION

The pdf of the clock is

$$f(t) = \begin{cases} 1/12, & 0 \le t \le 12 \\ 0, & \text{otherwise.} \end{cases}$$

The probability that the clock stopped at some time t with $2 \le t \le 5$ is

$$\int_2^5 f(t)\, dt = \frac{1}{4}.$$

Now try Exercise 27.

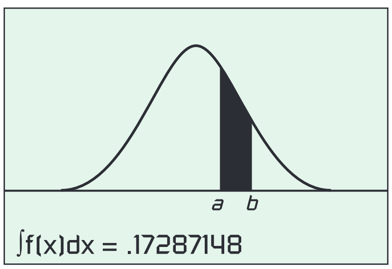


Figure 7.47 A normal probability density function. The probability associated with the interval $[a, b]$ is the area under the curve, as shown.

By far the most useful kind of pdf is the *normal* kind. ("Normal" here is a technical term, referring to a curve with the shape in Figure 7.47.) The **normal curve,** often called the "bell curve," is one of the most significant curves in applied mathematics because it enables us to describe entire populations based on the statistical measurements taken from a reasonably-sized sample. The measurements needed are the *mean* (μ) and the *standard deviation* (σ), which your calculators will approximate for you from the data. The symbols on the calculator will probably be $\overline{x}$ and s (see your *Owner's Manual*), but go ahead and use them as μ and σ, respectively. Once you have the numbers, you can find the curve by using the following remarkable formula discovered by Karl Friedrich Gauss.

DEFINITION Normal Probability Density Function (pdf)

The **normal probability density function (Gaussian curve)** for a population with mean μ and standard deviation σ is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}.$$

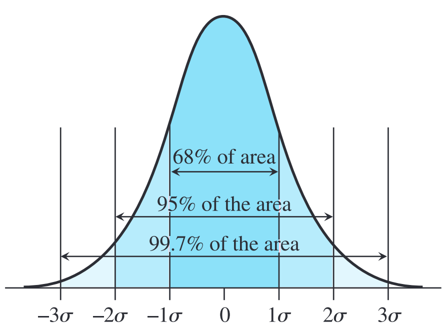


Figure 7.48 The 68-95-99.7 rule for normal distributions.

The mean μ represents the average value of the variable x. The standard deviation σ measures the "scatter" around the mean. For a normal curve, the mean and standard deviation tell you where most of the probability lies. The rule of thumb, illustrated in Figure 7.48, is this:

The 68-95-99.7 Rule for Normal Distributions

Given a normal curve,

- 68% of the area will lie within σ of the mean μ,
- 95% of the area will lie within 2σ of the mean μ,
- 99.7% of the area will lie within 3σ of the mean μ.

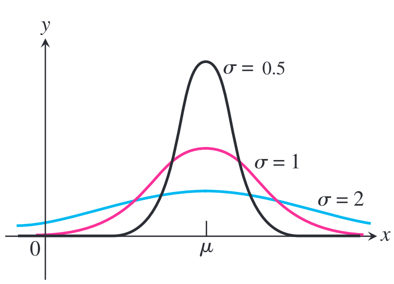


Figure 7.49 Normal pdf curves with mean $\mu = 2$ and $\sigma = 0.5$, 1, and 2.

Even with the 68-95-99.7 rule, the area under the curve can spread quite a bit, depending on the size of σ. Figure 7.49 shows three normal pdfs with mean $\mu = 2$ and standard deviations equal to 0.5, 1, and 2.

EXAMPLE 6 A Telephone Help Line

Suppose a telephone help line takes a mean of 2 minutes to answer calls. If the standard deviation is $\sigma = 0.5$, then 68% of the calls are answered in the range of 1.5 to 2.5 minutes and 99.7% of the calls are answered in the range of 0.5 to 3.5 minutes.

Now try Exercise 29.

EXAMPLE 7 Weights of Spinach Boxes

Suppose that frozen spinach boxes marked as "10 ounces" of spinach have a mean weight of 10.3 ounces and a standard deviation of 0.2 ounce.

(a) What percentage of *all* such spinach boxes can be expected to weigh between 10 and 11 ounces?

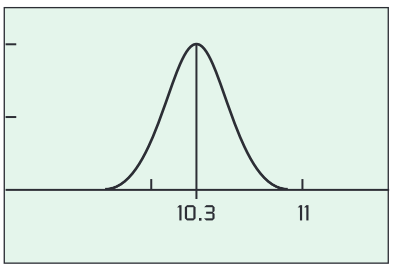
(b) What percentage would we expect to weigh less than 10 ounces?

(c) What is the probability that a box weighs *exactly* 10 ounces?

SOLUTION

Assuming that some person or machine is *trying* to pack 10 ounces of spinach into these boxes, we expect that most of the weights will be around 10, with probabilities tailing off for boxes being heavier or lighter. We expect, in other words, that a normal pdf will model these probabilities. First, we define $f(x)$ using the formula:

$$f(x) = \frac{1}{0.2\sqrt{2\pi}} e^{-(x-10.3)^2/(0.08)}.$$



[9, 11.5] by [–1, 2.5]

Figure 7.50 The normal pdf for the spinach weights in Example 7. The mean is at the center.

The graph (Figure 7.50) has the look we are expecting.

(a) For an arbitrary box of this spinach, the probability that it weighs between 10 and 11 ounces is the area under the curve from 10 to 11, which is

$$\text{NINT}\,(f(x), x, 10, 11) \approx 0.933.$$

So without doing any more measuring, we can predict that about 93.3% of all such spinach boxes will weigh between 10 and 11 ounces.

(b) For the probability that a box weighs less than 10 ounces, we use the entire area under the curve to the left of $x = 10$. The curve actually approaches the x-axis as an asymptote, but you can see from the graph (Figure 7.50) that $f(x)$ approaches zero quite quickly. Indeed, $f(9)$ is only slightly larger than a billionth. So getting the area from 9 to 10 should do it:

$$\text{NINT}\,(f(x), x, 9, 10) \approx 0.067.$$

We would expect only about 6.7% of the boxes to weigh less than 10 ounces.

(c) This would be the integral from 10 to 10, which is zero. This zero probability might seem strange at first, but remember that we are assuming a continuous, unbroken interval of possible spinach weights, and 10 is but one of an infinite number of them.

Now try Exercise 31.

Quick Review 7.5 *(For help, go to Section 5.2.)*

In Exercises 1–5, find the definite integral by **(a)** antiderivatives and **(b)** using NINT.

1. $\int_0^1 e^{-x}\,dx$

2. $\int_0^1 e^x\,dx$

3. $\int_{\pi/4}^{\pi/2} \sin x\,dx$

4. $\int_0^3 (x^2 + 2)\,dx$

5. $\int_1^2 \frac{x^2}{x^3 + 1}\,dx$

In Exercises 6–10 find, but do not evaluate, the definite integral that is the limit as the norms of the partitions go to zero of the Riemann sums on the closed interval [0, 7].

6. $\sum 2\pi(x_k + 2)(\sin x_k)\Delta x$

7. $\sum (1 - x_k^2)(2\pi x_k)\Delta x$

8. $\sum \pi(\cos x_k)^2 \Delta x$

9. $\sum \pi\left(\frac{y_k}{2}\right)^2 (10 - y_k)\Delta y$

10. $\sum \frac{\sqrt{3}}{4}(\sin^2 x_k)\Delta x$

Section 7.5 Exercises

In Exercises 1–4, find the work done by the force of $F(x)$ newtons along the x-axis from $x = a$ meters to $x = b$ meters.

1. $F(x) = xe^{-x/3}, \quad a = 0, \quad b = 5$
2. $F(x) = x\sin(\pi x/4), \quad a = 0, \quad b = 3$
3. $F(x) = x\sqrt{9 - x^2}, \quad a = 0, \quad b = 3$
4. $F(x) = e^{\sin x}\cos x + 2, \quad a = 0, \quad b = 10$

5. ***Leaky Bucket*** The workers in Example 2 changed to a larger bucket that held 50 L (490 N) of water, but the new bucket had an even larger leak so that it too was empty by the time it reached the top. Assuming the water leaked out at a steady rate, how much work was done lifting the water to a point 20 meters above the ground? (Do not include the rope and bucket.)

6. ***Leaky Bucket*** The bucket in Exercise 5 is hauled up more quickly so that there is still 10 L (98 N) of water left when the bucket reaches the top. How much work is done lifting the water this time? (Do not include the rope and bucket.)

7. ***Leaky Sand Bag*** A bag of sand originally weighing 144 lb was lifted at a constant rate. As it rose, sand leaked out at a constant rate. The sand was half gone by the time the bag had been lifted 18 ft. How much work was done lifting the sand this far? (Neglect the weights of the bag and lifting equipment.)

8. ***Stretching a Spring*** A spring has a natural length of 10 in. An 800-lb force stretches the spring to 14 in.

   (a) Find the force constant.

   (b) How much work is done in stretching the spring from 10 in. to 12 in.?

   (c) How far beyond its natural length will a 1600-lb force stretch the spring?

9. ***Subway Car Springs*** It takes a force of 21,714 lb to compress a coil spring assembly on a New York City Transit Authority subway car from its free height of 8 in. to its fully compressed height of 5 in.

   (a) What is the assembly's force constant?

   (b) How much work does it take to compress the assembly the first half inch? the second half inch? Answer to the nearest inch-pound.

   (*Source:* Data courtesy of Bombardier, Inc., Mass Transit Division, for spring assemblies in subway cars delivered to the New York City Transit Authority from 1985 to 1987.)

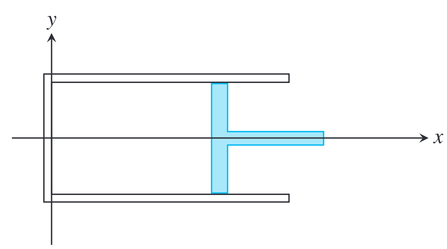
10. ***Bathroom Scale*** A bathroom scale is compressed 1/16 in. when a 150-lb person stands on it. Assuming the scale behaves like a spring that obeys Hooke's Law,

    (a) how much does someone who compresses the scale 1/8 in. weigh?

    (b) how much work is done in compressing the scale 1/8 in.?

11. ***Hauling a Rope*** A mountain climber is about to haul up a 50-m length of hanging rope. How much work will it take if the rope weighs 0.624 N/m?

12. ***Compressing Gas*** Suppose that gas in a circular cylinder of cross section area A is being compressed by a piston (see figure).



    (a) If p is the pressure of the gas in pounds per square inch and V is the volume in cubic inches, show that the work done in compressing the gas from state (p_1, V_1) to state (p_2, V_2) is given by the equation

$$\text{Work} = \int_{(p_1, V_1)}^{(p_2, V_2)} p\, dV \quad \text{in.} \cdot \text{lb},$$

    where the force against the piston is pA.

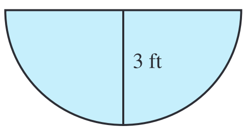
    (b) Find the work done in compressing the gas from $V_1 = 243\ \text{in}^3$ to $V_2 = 32\ \text{in}^3$ if $p_1 = 50\ \text{lb/in}^3$ and p and V obey the gas law $pV^{1.4} = \text{constant}$ (for adiabatic processes).

Group Activity In Exercises 13–16, the vertical end of a tank containing water (blue shading) weighing 62.4 lb/ft^3 has the given shape.

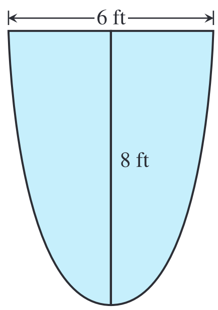
(a) **Writing to Learn** Explain how to approximate the force against the end of the tank by a Riemann sum.

(b) Find the force as an integral and evaluate it.

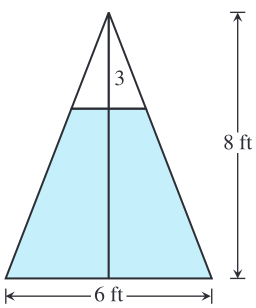
13. semicircle



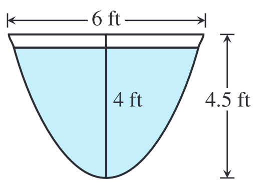
14. semiellipse



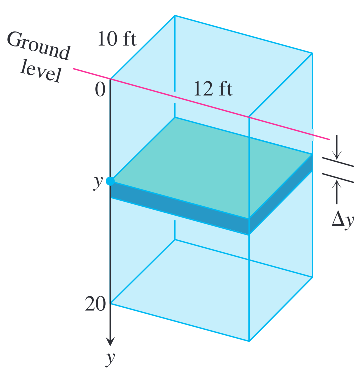
15. triangle



16. parabola



17. ***Pumping Water*** The rectangular tank shown here, with its top at ground level, is used to catch runoff water. Assume that the water weighs 62.4 lb/ft^3.



(a) How much work does it take to empty the tank by pumping the water back to ground level once the tank is full?

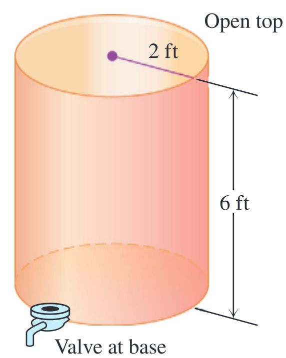
(b) If the water is pumped to ground level with a (5/11)-horsepower motor (work output 250 ft • lb/sec), how long will it take to empty the full tank (to the nearest minute)?

(c) Show that the pump in part (b) will lower the water level 10 ft (halfway) during the first 25 min of pumping.

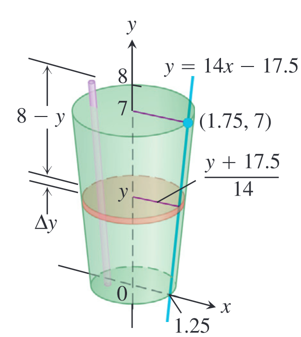
(d) ***The Weight of Water*** Because of differences in the strength of Earth's gravitational field, the weight of a cubic foot of water at sea level can vary from as little as 62.26 lb at the equator to as much as 62.59 lb near the poles, a variation of about 0.5%. A cubic foot of water that weighs 62.4 lb in Melbourne or New York City will weigh 62.5 lb in Juneau or Stockholm. What are the answers to parts (a) and (b) in a location where water weighs 62.26 lb/ft^3? 62.5 lb/ft^3?

18. ***Emptying a Tank*** A vertical right cylindrical tank measures 30 ft high and 20 ft in diameter. It is full of kerosene weighing 51.2 lb/ft^3. How much work does it take to pump the kerosene to the level of the top of the tank?

19. **Writing to Learn** The cylindrical tank shown here is to be filled by pumping water from a lake 15 ft below the bottom of the tank. There are two ways to go about this. One is to pump the water through a hose attached to a valve in the bottom of the tank. The other is to attach the hose to the rim of the tank and let the water pour in. Which way will require less work? Give reasons for your answer.



20. ***Drinking a Milkshake*** The truncated conical container shown here is full of strawberry milkshake that weighs (4/9) oz/in^3. As you can see, the container is 7 in. deep, 2.5 in. across at the base, and 3.5 in. across at the top (a standard size at Brigham's in Boston). The straw sticks up an inch above the top. About how much work does it take to drink the milkshake through the straw (neglecting friction)? Answer in inch-ounces.

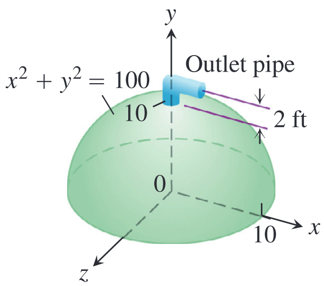


Dimensions in inches

21. ***Revisiting Example 3*** How much work will it take to pump the oil in Example 3 to a level 3 ft above the cone's rim?

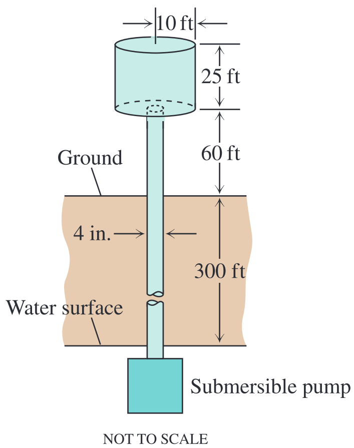
22. ***Pumping Milk*** Suppose the conical tank in Example 3 contains milk weighing 64.5 lb/ft^3 instead of olive oil. How much work will it take to pump the contents to the rim?

23. **Writing to Learn** You are in charge of the evacuation and repair of the storage tank shown here. The tank is a hemisphere of radius 10 ft and is full of benzene weighing 56 lb/ft^3.



A firm you contacted says it can empty the tank for 1/2 cent per foot-pound of work. Find the work required to empty the tank by pumping the benzene to an outlet 2 ft above the tank. If you have budgeted $5000 for the job, can you afford to hire the firm?

24. ***Water Tower*** Your town has decided to drill a well to increase its water supply. As the town engineer, you have determined that a water tower will be necessary to provide the pressure needed for distribution, and you have designed the system shown here. The water is to be pumped from a 300-ft well through a vertical 4-in. pipe into the base of a cylindrical tank 20 ft in diameter and 25 ft high. The base of the tank will be 60 ft above ground. The pump is a 3-hp pump, rated at 1650 ft • lb/sec. To the nearest hour, how long will it take to fill the tank the first time? (Include the time it takes to fill the pipe.) Assume water weighs 62.4 lb/ft^3.



25. ***Fish Tank*** A rectangular freshwater fish tank with base 2×4 ft and height 2 ft (interior dimensions) is filled to within 2 in. of the top.

(a) Find the fluid force against each end of the tank.

(b) Suppose the tank is sealed and stood on end (without spilling) so that one of the square ends is the base. What does that do to the fluid forces on the rectangular sides?

26. ***Milk Carton*** A rectangular milk carton measures 3.75 in. by 3.75 in. at the base and is 7.75 in. tall. Find the force of the milk (weighing 64.5 lb/ft^3) on one side when the carton is full.

27. Find the probability that a clock stopped between 1:00 and 5:00.

28. Find the probability that a clock stopped between 3:00 and 6:00.

29. Suppose a telephone help line takes a mean of 2 minutes to answer calls. If the standard deviation is $\sigma = 2$, what percentage of the calls are answered in the range of 0 to 4 minutes?

30. ***Test Scores*** The mean score on a national aptitude test is 498 with a standard deviation of 100 points.

(a) What percentage of the population has scores between 400 and 500?

(b) If we sample 300 test-takers at random, about how many should have scores above 700?

31. ***Heights of Females*** The mean height of an adult female in New York City is estimated to be 63.4 inches with a standard deviation of 3.2 inches. What proportion of the adult females in New York City are

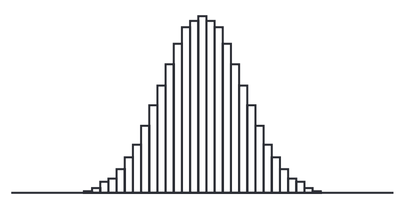
(a) less than 63.4 inches tall?

(b) between 63 and 65 inches tall?

(c) taller than 6 feet?

(d) exactly 5 feet tall?

32. **Writing to Learn** Exercises 30 and 31 are subtly different, in that the heights in Exercise 31 are measured *continuously* and the scores in Exercise 30 are measured *discretely*. The discrete probabilities determine rectangles above the individual test scores, so that there actually is a nonzero probability of scoring, say, 560. The rectangles would look like the figure below, and would have total area 1.



Explain why integration gives a good estimate for the probability, even in the discrete case.

33. **Writing to Learn** Suppose that $f(t)$ is the probability density function for the lifetime of a certain type of lightbulb where t is in hours. What is the meaning of the integral

$$\int_{100}^{800} f(t)\, dt?$$

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

34. **True or False** A force is applied to compress a spring several inches. Assume the spring obeys Hooke's Law. Twice as much work is required to compress the spring the second inch than is required to compress the spring the first inch. Justify your answer.

35. True or False An aquarium contains water weighing 62.4 lb/ft^3. The aquarium is in the shape of a cube where the length of each edge is 3 ft. Each side of the aquarium is engineered to withstand 1000 pounds of force. This should be sufficient to withstand the force from water pressure. Justify your answer.

36. Multiple Choice A force of $F(x) = 350x$ newtons moves a particle along a line from $x = 0$ m to $x = 5$ m. Which of the following gives the best approximation of the work done by the force?

(A) 1750 J **(B)** 2187.5 J **(C)** 2916.67 J
(D) 3281.25 J **(E)** 4375 J

37. Multiple Choice A leaky bag of sand weighs 50 n. It is lifted from the ground at a constant rate, to a height of 20 m above the ground. The sand leaks at a constant rate and just finishes draining as the bag reaches the top. Which of the following gives the work done to lift the sand to the top? (Neglect the bag.)

(A) 50 J **(B)** 100 J **(C)** 250 J **(D)** 500 J **(E)** 1000 J

38. Multiple Choice A spring has a natural length of 0.10 m. A 200-n force stretches the spring to a length of 0.15 m. Which of the following gives the work done in stretching the spring from 0.10 m to 0.15 m?

(A) 0.05 J **(B)** 5 J **(C)** 10 J **(D)** 200 J **(E)** 4000 J

39. Multiple Choice A vertical right cylindrical tank measures 12 ft high and 16 ft in diameter. It is full of water weighing 62.4 lb/ft^3. How much work does it take to pump the water to the level of the top of the tank? Round your answer to the nearest ft-lb.

(A) 149,490 ft-lb
(B) 285,696 ft-lb
(C) 360,240 ft-lb
(D) 448,776 ft-lb
(E) 903,331 ft-lb

Extending the Ideas

40. *Putting a Satellite into Orbit* The strength of Earth's gravitational field varies with the distance r from Earth's center, and the magnitude of the gravitational force experienced by a satellite of mass m during and after launch is

$$F(r) = \frac{mMG}{r^2}.$$

Here, $M = 5.975 \times 10^{24}$ kg is Earth's mass, $G = 6.6726 \times 10^{-11}\ \text{N} \cdot \text{m}^2\text{kg}^{-2}$ is the *universal gravitational constant,* and r is measured in meters. The work it takes to lift a 1000-kg satellite from Earth's surface to a circular orbit 35,780 km above Earth's center is therefore given by the integral

$$\text{Work} = \int_{6,370,000}^{35,780,000} \frac{1000MG}{r^2}\, dr \text{ joules.}$$

The lower limit of integration is Earth's radius in meters at the launch site. Evaluate the integral. (This calculation does not take into account energy spent lifting the launch vehicle or energy spent bringing the satellite to orbit velocity.)

41. *Forcing Electrons Together* Two electrons r meters apart repel each other with a force of

$$F = \frac{23 \times 10^{-29}}{r^2} \text{ newton.}$$

(a) Suppose one electron is held fixed at the point (1, 0) on the x-axis (units in meters). How much work does it take to move a second electron along the x-axis from the point $(-1, 0)$ to the origin?

(b) Suppose an electron is held fixed at each of the points $(-1, 0)$ and $(1, 0)$. How much work does it take to move a third electron along the x-axis from (5, 0) to (3, 0)?

42. *Kinetic Energy* If a variable force of magnitude $F(x)$ moves a body of mass m along the x-axis from x_1 to x_2, the body's velocity v can be written as dx/dt (where t represents time). Use Newton's second law of motion, $F = m(dv/dt)$, and the Chain Rule

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

to show that the net work done by the force in moving the body from x_1 to x_2 is

$$W = \int_{x_1}^{x_2} F(x)\, dx = \frac{1}{2}mv_2^{\,2} - \frac{1}{2}mv_1^{\,2}, \qquad (1)$$

where v_1 and v_2 are the body's velocities at x_1 and x_2. In physics the expression $(1/2)mv^2$ is the *kinetic energy* of the body moving with velocity v. Therefore, *the work done by the force equals the change in the body's kinetic energy,* and we can find the work by calculating this change.

> **Weight vs. Mass**
>
> Weight is the force that results from gravity pulling on a mass. The two are related by the equation in Newton's second law,
>
> $$\text{weight} = \text{mass} \times \text{acceleration.}$$
>
> Thus,
>
> $$\text{newtons} = \text{kilograms} \times \text{m/sec}^2,$$
>
> $$\text{pounds} = \text{slugs} \times \text{ft/sec}^2.$$
>
> To convert mass to weight, multiply by the acceleration of gravity. To convert weight to mass, divide by the acceleration of gravity.

In Exercises 43–49, use Equation 1 from Exercise 42.

43. *Tennis* A 2-oz tennis ball was served at 160 ft/sec (about 109 mph). How much work was done on the ball to make it go this fast?

44. *Baseball* How many foot-pounds of work does it take to throw a baseball 90 mph? A baseball weighs 5 oz = 0.3125 lb.

45. *Golf* A 1.6-oz golf ball is driven off the tee at a speed of 280 ft/sec (about 191 mph). How many foot-pounds of work are done getting the ball into the air?

46. *Tennis* During the match in which Pete Sampras won the 1990 U.S. Open men's tennis championship, Sampras hit a serve that was clocked at a phenomenal 124 mph. How much work did Sampras have to do on the 2-oz ball to get it to that speed?

47. ***Football*** A quarterback threw a 14.5-oz football 88 ft/sec (60 mph). How many foot-pounds of work were done on the ball to get it to that speed?

48. ***Softball*** How much work has to be performed on a 6.5-oz softball to pitch it at 132 ft/sec (90 mph)?

49. ***A Ball Bearing*** A 2-oz steel ball bearing is placed on a vertical spring whose force constant is $k = 18$ lb/ft. The spring is compressed 3 in. and released. About how high does the ball bearing go? (*Hint:* The kinetic (compression) energy, mgh, of a spring is $\frac{1}{2}ks^2$, where s is the distance the spring is compressed, m is the mass, g is the acceleration of gravity, and h is the height.)

Quick Quiz for AP* Preparation: Sections 7.4 and 7.5

You should solve the following problems without using a graphing calculator.

1. Multiple Choice The length of a curve from $x = 0$ to $x = 1$ is given by $\int_0^1 \sqrt{1 + 16x^6}\, dx$. If the curve contains the point (1, 4), which of the following could be an equation for this curve?

(A) $y = x^4 + 3$

(B) $y = x^4 + 1$

(C) $y = 1 + 16x^6$

(D) $y = \sqrt{1 + 16x^6}$

(E) $y = x + \frac{x^7}{7}$

2. Multiple Choice Which of the following gives the length of the path described by the parametric equations $x = \frac{1}{4}t^4$ and $y = t^3$, where $0 \le t \le 2$?

(A) $\int_0^2 t^6 + 9t^4\, dt$

(B) $\int_0^2 \sqrt{t^6 + 1}\, dt$

(C) $\int_0^2 \sqrt{1 + 9t^4}\, dt$

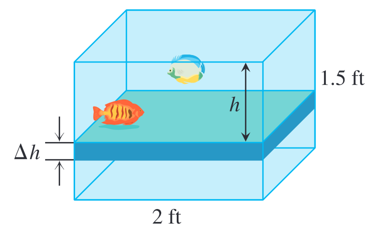
(D) $\int_0^2 \sqrt{t^6 + 9t^4}\, dt$

(E) $\int_0^2 \sqrt{t^3 + 3t^2}\, dt$

3. Multiple Choice The base of a solid is a circle of radius 2 inches. Each cross section perpendicular to a certain diameter is a square with one side lying in the circle. The volume of the solid in cubic inches is

(A) 16 **(B)** 16π **(C)** $\frac{128}{3}$ **(D)** $\frac{128\pi}{3}$ **(E)** 32π

4. Free Response The front of a fish tank is rectangular in shape and measures 2 ft wide by 1.5 ft tall. The water in the tank exerts pressure on the front of the tank. The pressure at any point on the front of the tank depends only on how far below the surface the point lies and is given by the equation $p = 62.4h$, where h is depth below the surface measured in feet and p is pressure measured in pounds/ft^2.



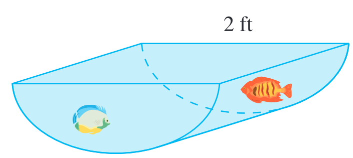
The front of the tank can be partitioned into narrow horizontal bands of height Δh. The force exerted by the water on a band at depth h_i is approximately

$$\text{pressure} \cdot \text{area} = 62.4h_i \cdot 2\Delta h.$$

(a) Write the Riemann sum that approximates the force exerted on the entire front of the tank.

(b) Use the Riemann sum from part (a) to write and evaluate a definite integral that gives the force exerted on the front of the tank. Include correct units.

(c) Find the total force exerted on the front of the tank if the front (and back) are semicircles with diameter 2 ft. Include correct units.



Calculus at Work

I am working toward becoming an archeaoastronomer and ethnoastronomer of Africa. I have a Bachelor's degree in Physics, a Master's degree in Astronomy, and a Ph.D. in Astronomy and Astrophysics. From 1988 to 1990 I was a member of the Peace Corps, and I taught mathematics to high school students in the Fiji Islands. Calculus is a required course in high schools there.

For my Ph.D. dissertation, I investigated the possibility of the birthrate of stars being related to the composition of star formation clouds. I collected data on the absorption of electromagnetic emissions emanating from these regions. The intensity of emissions graphed versus wavelength produces a flat curve with downward spikes at the characteristic wavelengths of the elements present. An estimate of the area between a spike and the flat curve results in a concentration in molecules/cm^3 of an element. This area is the difference in the integrals of the flat and spike curves. In particular, I was looking for a large concentration of water-ice, which increases the probability of planets forming in a region.

Currently, I am applying for two research grants. One will allow me to use the NASA infrared telescope on Mauna Kea to search for C_3S_2 in comets. The other will help me study the history of astronomy in Tunisia.

Jarita Holbrook

Chapter 7 Key Terms

arc length (p. 413)
area between curves (p. 390)
Cavalieri's theorems (p. 404)
center of mass (p. 389)
constant-force formula (p. 384)
cylindrical shells (p. 402)
displacement (p. 380)
fluid force (p. 421)
fluid pressure (p. 421)
foot-pound (p. 384)
force constant (p. 385)
Gaussian curve (p. 423)
Hooke's Law (p. 385)
inflation rate (p. 388)
joule (p. 384)
length of a curve (p. 413)
mean (p. 423)
moment (p. 389)
net change (p. 379)
newton (p. 384)
normal curve (p. 423)
normal pdf (p. 423)
probability density function (pdf) (p. 422)
68-95-99.7 rule (p. 423)
smooth curve (p. 413)
smooth function (p. 413)
solid of revolution (p. 400)
standard deviation (p. 423)
surface area (p. 405)
total distance traveled (p. 381)
universal gravitational constant (p. 428)
volume by cylindrical shells (p. 402)
volume by slicing (p. 400)
volume of a solid (p. 399)
weight-density (p. 421)
work (p. 384)

Chapter 7 Review Exercises

The collection of exercises marked in red could be used as a chapter test.

In Exercises 1–5, the application involves the accumulation of small changes over an interval to give the net change over that entire interval. Set up an integral to model the accumulation and evaluate it to answer the question.

1. A toy car slides down a ramp and coasts to a stop after 5 sec. Its velocity from $t = 0$ to $t = 5$ is modeled by $v(t) = t^2 - 0.2t^3$ ft/sec. How far does it travel?

2. The fuel consumption of a diesel motor between weekly maintenance periods is modeled by the function $c(t) = 4 + 0.001t^4$ gal/day, $0 \le t \le 7$. How many gallons does it consume in a week?

3. The number of billboards per mile along a 100-mile stretch of an interstate highway approaching a certain city is modeled by the function $B(x) = 21 - e^{0.03x}$, where x is the distance from the city in miles. About how many billboards are along that stretch of highway?

4. A 2-meter rod has a variable density modeled by the function $\rho(x) = 11 - 4x$ g/m, where x is the distance in meters from the base of the rod. What is the total mass of the rod?

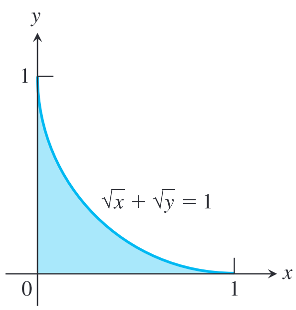
5. The electrical power consumption (measured in kilowatts) at a factory t hours after midnight during a typical day is modeled by $E(t) = 300(2 - \cos(\pi t/12))$. How many kilowatt-hours of electrical energy does the company consume in a typical day?

In Exercises 6–19, find the area of the region enclosed by the lines and curves.

6. $y = x, \quad y = 1/x^2, \quad x = 2$

7. $y = x + 1, \quad y = 3 - x^2$

8. $\sqrt{x} + \sqrt{y} = 1, \quad x = 0, \quad y = 0$



9. $x = 2y^2, \quad x = 0, \quad y = 3$

10. $4x = y^2 - 4, \quad 4x = y + 16$

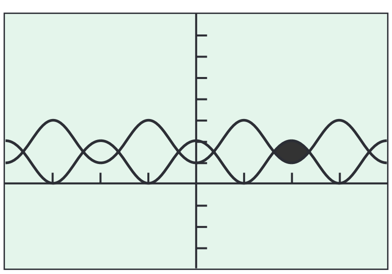
11. $y = \sin x, \quad y = x, \quad x = \pi/4$

12. $y = 2\sin x, \quad y = \sin 2x, \quad 0 \le x \le \pi$

13. $y = \cos x, \quad y = 4 - x^2$

14. $y = \sec^2 x, \quad y = 3 - |x|$

15. ***The Necklace*** one of the smaller bead-shaped regions enclosed by the graphs of $y = 1 + \cos x$ and $y = 2 - \cos x$



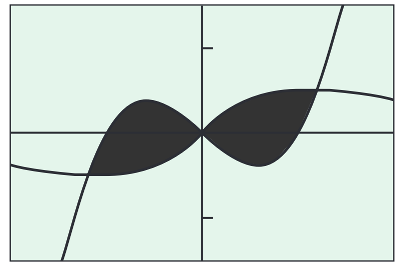
$[-4\pi, 4\pi]$ by $[-4, 8]$

16. one of the larger bead-shaped regions enclosed by the curves in Exercise 15

17. ***The Bow Tie*** the region enclosed by the graphs of

$$y = x^3 - x \quad \text{and} \quad y = \frac{x}{x^2 + 1}$$

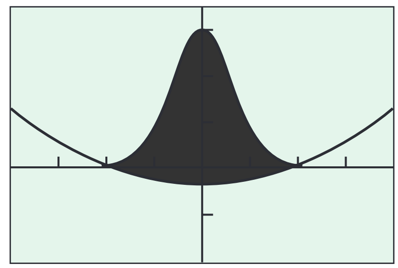
(shown in the next column).



$[-2, 2]$ by $[-1.5, 1.5]$

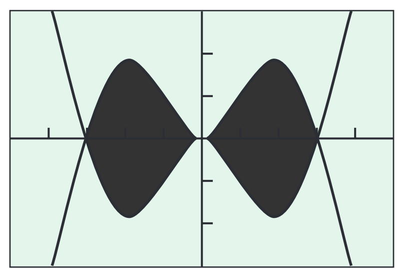
18. ***The Bell*** the region enclosed by the graphs of

$$y = 3^{1-x^2} \quad \text{and} \quad y = \frac{x^2 - 3}{10}$$



$[-4, 4]$ by $[-2, 3.5]$

19. ***The Kissing Fish*** the region enclosed between the graphs of $y = x \sin x$ and $y = -x \sin x$ over the interval $[-\pi, \pi]$



$[-5, 5]$ by $[-3, 3]$

20. Find the volume of the solid generated by revolving the region bounded by the x-axis, the curve $y = 3x^4$, and the lines $x = -1$ and $x = 1$ about the x-axis.

21. Find the volume of the solid generated by revolving the region enclosed by the parabola $y^2 = 4x$ and the line $y = x$ about

(a) the x-axis. **(b)** the y-axis.

(c) the line $x = 4$. **(d)** the line $y = 4$.

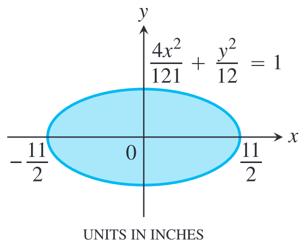
22. The section of the parabola $y = x^2/2$ from $y = 0$ to $y = 2$ is revolved about the y-axis to form a bowl.

(a) Find the volume of the bowl.

(b) Find how much the bowl is holding when it is filled to a depth of k units $(0 < k < 2)$.

(c) If the bowl is filled at a rate of 2 cubic units per second, how fast is the depth k increasing when $k = 1$?

23. The profile of a football resembles the ellipse shown here (all dimensions in inches). Find the volume of the football to the nearest cubic inch.



24. The base of a solid is the region enclosed between the graphs of $y = \sin x$ and $y = -\sin x$ from $x = 0$ to $x = \pi$. Each cross section perpendicular to the x-axis is a semicircle with diameter connecting the two graphs. Find the volume of the solid.

25. The region enclosed by the graphs of $y = e^{x/2}$, $y = 1$, and $x = \ln 3$ is revolved about the x-axis. Find the volume of the solid generated.

26. A round hole of radius $\sqrt{3}$ feet is bored through the center of a sphere of radius 2 feet. Find the volume of the piece cut out.

27. Find the length of the arch of the parabola $y = 9 - x^2$ that lies above the x-axis.

28. Find the *perimeter* of the bow-tie-shaped region enclosed between the graphs of $y = x^3 - x$ and $y = x - x^3$.

29. A particle travels at 2 units per second along the curve $y = x^3 - 3x^2 + 2$. How long does it take to travel from the local maximum to the local minimum?

30. Group Activity One of the following statements is true for all $k > 0$ and one is false. Which is which? Explain.

(a) The graphs of $y = k \sin x$ and $y = \sin kx$ have the same length on the interval $[0, 2\pi]$.

(b) The graph of $y = k \sin x$ is k times as long as the graph of $y = \sin x$ on the interval $[0, 2\pi]$.

31. Let $F(x) = \int_1^x \sqrt{t^4 - 1}\, dt$. Find the *exact* length of the graph of F from $x = 2$ to $x = 5$ without using a calculator.

32. *Rock Climbing* A rock climber is about to haul up 100 N (about 22.5 lb) of equipment that has been hanging beneath her on 40 m of rope weighing 0.8 N/m. How much work will it take to lift

(a) the equipment? **(b)** the rope?

(c) the rope and equipment together?

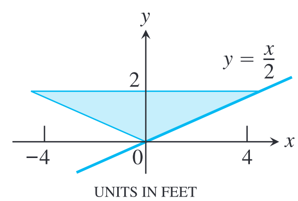
33. *Hauling Water* You drove an 800-gallon tank truck from the base of Mt. Washington to the summit and discovered on arrival that the tank was only half full. You had started out with a full tank of water, had climbed at a steady rate, and had taken 50 minutes to accomplish the 4750-ft elevation change. Assuming that the water leaked out at a steady rate, how much work was spent in carrying the water to the summit? Water weighs 8 lb/gal. (Do not count the work done getting you and the truck to the top.)

34. *Stretching a Spring* If a force of 80 N is required to hold a spring 0.3 m beyond its unstressed length, how much work does it take to stretch the spring this far? How much work does it take to stretch the spring an additional meter?

35. Writing to Learn It takes a lot more effort to roll a stone up a hill than to roll the stone down the hill, but the weight of the stone and the distance it covers are the same. Does this mean that the same amount of work is done? Explain.

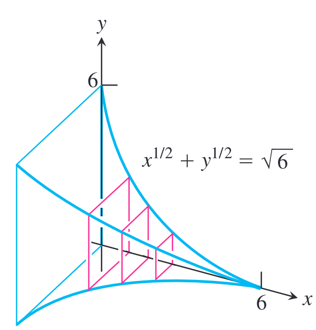
36. *Emptying a Bowl* A hemispherical bowl with radius 8 inches is filled with punch (weighing 0.04 pound per cubic inch) to within 2 inches of the top. How much work is done emptying the bowl if the contents are pumped just high enough to get over the rim?

37. *Fluid Force* The vertical triangular plate shown below is the end plate of a feeding trough full of hog slop, weighing 80 pounds per cubic foot. What is the force against the plate?



38. *Fluid Force* A standard olive oil can measures 5.75 in. by 3.5 in. by 10 in. Find the fluid force against the base and each side of the can when it is full. (Olive oil has a weight-density of 57 pounds per cubic foot.)

39. *Volume* A solid lies between planes perpendicular to the x-axis at $x = 0$ and at $x = 6$. The cross sections between the planes are squares whose bases run from the x-axis up to the curve $\sqrt{x} + \sqrt{y} = \sqrt{6}$. Find the volume of the solid.



40. ***Yellow Perch*** A researcher measures the lengths of 3-year-old yellow perch in a fish hatchery and finds that they have a mean length of 17.2 cm with a standard deviation of 3.4 cm. What proportion of 3-year-old yellow perch raised under similar conditions can be expected to reach a length of 20 cm or more?

41. **Group Activity** Using as large a sample of classmates as possible, measure the span of each person's fully stretched hand, from the tip of the pinky finger to the tip of the thumb. Based on the mean and standard deviation of your sample, what percentage of students your age would have a finger span of more than 10 inches?

42. ***The 68-95-99.7 Rule*** **(a)** Verify that for every normal pdf, the proportion of the population lying within one standard deviation of the mean is close to 68%. (*Hint:* Since it is the same for every pdf, you can simplify the function by assuming that $\mu = 0$ and $\sigma = 1$. Then integrate from -1 to 1.)

(b) Verify the two remaining parts of the rule.

43. **Writing to Learn** Explain why the area under the graph of a probability density function has to equal 1.

In Exercises 44–48, use the cylindrical shell method to find the volume of the solid generated by revolving the region bounded by the curves about the y-axis.

44. $y = 2x, \quad y = x/2, \quad x = 1$

45. $y = 1/x, \quad y = 0, \quad x = 1/2, \quad x = 2$

46. $y = \sin x, \quad y = 0, \quad 0 \le x \le \pi$

47. $y = x - 3, \quad y = x^2 - 3x$

48. the bell-shaped region in Exercise 18

49. ***Bundt Cake*** A bundt cake (see Exploration 1, Section 7.3) has a hole of radius 2 inches and an outer radius of 6 inches at the base. It is 5 inches high, and each cross-sectional slice is parabolic.

(a) Model a typical slice by finding the equation of the parabola with y-intercept 5 and x-intercepts ± 2.

(b) Revolve the parabolic region about an appropriate line to generate the bundt cake and find its volume.

50. ***Finding a Function*** Find a function f that has a continuous derivative on $(0, \infty)$ and that has both of the following properties.

i. The graph of f goes through the point (1, 1).

ii. The length L of the curve from (1, 1) to any point $(x, f(x))$ is given by the formula $L = \ln x + f(x) - 1$.

In Exercises 51 and 52, find the area of the surface generated by revolving the curve about the indicated axis.

51. $y = \tan x, \quad 0 \le x \le \pi/4; \quad x$-axis

52. $xy = 1, \quad 1 \le y \le 2; \quad y$-axis

AP* Examination Preparation

You may use a graphing calculator to solve the following problems.

53. Let R be the region in the first quadrant enclosed by the y-axis and the graphs of $y = 2 + \sin x$ and $y = \sec x$.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is revolved about the x-axis.

(c) Find the volume of the solid whose base is R and whose cross sections cut by planes perpendicular to the x-axis are squares.

54. The temperature outside a house during a 24-hour period is given by

$$F(t) = 80 - 10\cos\left(\frac{\pi t}{12}\right), \quad 0 \le t \le 24,$$

where $F(t)$ is measured in degrees Fahrenheit and t is measured in hours.

(a) Find the average temperature, to the nearest degree Fahrenheit, between $t = 6$ and $t = 14$.

(b) An air conditioner cooled the house whenever the outside temperature was at or above 78 degrees Fahrenheit. For what values of t was the air conditioner cooling the house?

(c) The cost of cooling the house accumulates at the rate of \$0.05 per hour for each degree the outside temperature exceeds 78 degrees Fahrenheit. What was the total cost, to the nearest cent, to cool the house for this 24-hour period?

55. The rate at which people enter an amusement park on a given day is modeled by the function E defined by

$$E(t) = \frac{15600}{t^2 - 24t + 160}.$$

The rate at which people leave the same amusement park on the same day is modeled by the function L defined by

$$L(t) = \frac{9890}{t^2 - 38t + 370}.$$

Both $E(t)$ and $L(t)$ are measured in people per hour, and time t is measured in hours after midnight. These functions are valid for $9 \le t \le 23$, which are the hours that the park is open. At time $t = 9$, there are no people in the park.

(a) How many people have entered the park by 5:00 P.M. ($t = 17$)? Round your answer to the nearest whole number.

(b) The price of admission to the park is \$15 until 5:00 P.M. ($t = 17$). After 5:00 P.M., the price of admission to the park is \$11. How many dollars are collected from admissions to the park on the given day? Round your answer to the nearest whole number.

(c) Let $H(t) = \int_9^t (E(x) - L(x))dx$ for $9 \le t \le 23$. The value of $H(17)$ to the nearest whole number is 3725. Find the value of $H'(17)$ and explain the meaning of $H(17)$ and $H'(17)$ in the context of the park.

(d) At what time t, for $9 \le t \le 23$, does the model predict that the number of people in the park is a maximum?